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On filial and left filial rings

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Abstract. The prime radical of filial and left filial is studied. It is proved that prime radical filial rings are left filial and that there exist nilpotent left filial rings which are not filial. Filial and left filial matrix rings, polynomial rings and rings being direct sums of copies of a ring are described.

1. Introduction

All rings in this paper are associative but we do not assume that each ring has an identity element. We denote by $R^\star$ the usual extension with identity of the ring R.

The ring of integers is denoted by $\mathbb{Z}$ and for a given positive integer n, $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}/n\mathbb{Z}$. Throughout the paper β denotes the prime radical.

We use the notation $I \lhd R$ (respectively, $I <_l R$) to mean that I is an ideal (respectively, a left ideal) of the ring R.

It is well known that $A \lhd B \lhd C$ (respectively, $A <_l B <_l C$) does not imply $A \lhd C$ (respectively, $A <_l C$). Nevertheless in some cases (e.g. for specific A, B or C) the implication does hold. Systematic studies of this subject were started by Sands [7] and continued by Veldsman [11].

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In particular Sands characterized the following classes of rings:

- $\{C \mid \text{ if } A \lhd B \lhd C \text{ then } A \lhd C\}$
- $\{C \mid \text{ if } A <_l B <_l C \text{ then } A <_l C\}$.

Rings satisfying the former condition were studied earlier in [3] (mainly in the commutative case) and next in [1], [2], [9]. Following the terminology of [3], we call rings from the first class *filial* and from the second *left filial*. Some results on left filial rings were obtained in [10]. However the studies of [9], [10] concerned rather the upper radicals determined by the class of filial and the class of left filial rings than rings of these classes themselves. Systematic studies of left filial rings were started in [4]. In particular a structure theorem describing semiprime left filial was obtained there and it was shown that semiprime left filial rings are filial.

In this paper we concentrate on studying the prime radical of filial and left filial rings. We show that the twosided annihilator of nonzero β-radical filial and left filial rings is nonzero. Moreover β-radical filial rings are sums of their nilpotent ideals. This in particular implies that β-radical filial rings are left filial. We give an example of a nilpotent left filial ring which is not filial. Applying some results on the prime radical we describe filial and left filial matrix rings, polynomial rings and direct sums of rings.

The following very useful characterization of filial and left filial rings will be used many times in the paper.

Theorem 1 ([2], [7], [9], [10]). *A ring R is*

(i) *filial if and only if for every $a \in R$, $\mathbb{Z}a + (a)^2 = (a)$;*

(ii) *left filial if and only if for every $a \in R$, $\mathbb{Z}a + R^\star a^2 = R^\star a$.*

2. On the prime radical of filial and left filial rings

The following results on the β-radical of filial and left filial rings were obtained in [4].

Theorem 2 ([4]). (i) *Rings which are sums of nilpotent ideals are filial if and only if their subrings are ideals. In particular filial rings which are sums of nilpotent ideals are left filial.*

(ii) *For every left filial ring R, $\beta(R)$ coincides with the set of nilpotent elements of R as well as with the sum of nilpotent ideals of R. Moreover β-radical rings are left filial if and only if their subrings are left ideals.*

In [4], Theorem 5, it was proved that if a left filial ring R satisfies the condition

$$\textit{if for some } x \in R \textit{ and a positive integer } n,\ n^2x = 0,\ \textit{then } nx = 0 \qquad (\star)$$

then $(\beta(R))^3 = 0$.

The condition $(\star)$ is equivalent to say that the additive group of the torsion part of R is elementary. Thus it is satisfied when R is a torsion-free ring or an algebra over a field F. In [4], Theorem 18, it was proved that if for every prime p, F is not isomorphic to $\mathbb{Z}_p$, and R is an F-algebra, which is a left filial ring, then $R\beta(R) = 0$. In particular if $R \in \beta$, then $R^2 = 0$. This property holds also for β-radical left filial torsion-free rings.

Proposition 3. *If $R \in \beta$ is a torsion-free left filial ring, then $R^2 = 0$.*

Proof. Take any $a \in R$. Since $R^3 = 0$, $T = \mathbb{Z}a + \mathbb{Z}a^2$ is a subring of R. Hence by Theorem 2 (ii), $T <_l R$. Note that $\mathbb{Z}(2a + 2a^2) + 4T <_l 2T <_l R$, so $\mathbb{Z}(2a + 2a^2) + 4T <_l R$. In particular $2a^2 = a(2a + 2a^2) = k(2a + 2a^2) + 4la + 4ma^2$ for some integers k, l, m. Multiplying this equality by a we get that $0 = 2ka^2 + 4la^2$. Hence, since R is torsion-free, $a^2 = 0$ or $2k + 4l = 0$. In the latter case we get that $2a^2 = 2ka^2 + 4ma^2 = -4la^2 + 4ma^2$. Hence $(2 + 4l - 4m)a^2 = 0$. Obviously $2 + 4l - 4m \neq 0$, so, since R is torsion-free, $a^2 = 0$. Now $\mathbb{Z}a$ is a subring of R, so $\mathbb{Z}a <_l R$. Consequently $Ra \subseteq \mathbb{Z}a$. Hence for every $b \in R$ there exists $n \in \mathbb{Z}$ such that $ba = na$. However $b^2 = 0$, so $nba = 0$. Since R is torsion-free, these imply that $ba = 0$. Thus $R^2 = 0$. □

One can easily check that every subgroup of the additive group of the matrix ring $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$ is a left ideal of R. Hence R is a torsion-free left filial ring. Obviously $\beta(R) = \left(\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$, so $R\beta(R) \neq 0$.

By Theorem 2 (i) every nilpotent filial ring is left filial. Hence if R is a filial ring satisfying $(\star)$, then for every nilpotent ideal I of R, $I^3 = 0$. This, Proposition 3 and Theorem 18 from [4] give the following corollary.

Corollary 4. *If a filial ring R satisfies $(\star)$, then $(\beta(R))^3 = 0$. If R is torsion-free or an algebra over a field F such that for every prime p, F is not isomorphic to $\mathbb{Z}_p$, then $(\beta(R))^2 = 0$.*

For every prime p, the $\mathbb{Z}_p$-algebra, $x\mathbb{Z}_p[x]/x^3\mathbb{Z}_p[x]$ is a filial ring which is nilpotent of index 3. Not all β-radical filial and left filial rings are nilpotent (as an example one can take the ring $\bigoplus_{p-\text{prime}}(p\mathbb{Z}_{p^p})$) but they are S-nilpotent.

A ring R will be called *S-nilpotent* (cf. [8]) if for every nonzero homomorphic image R' of R, $\operatorname{Ann}(R') = \{x \in R' \mid R'x = xR' = 0\} \neq 0$.

Theorem 5. *If $R \in \beta$ is filial or left filial, then R is S-nilpotent.*

Proof. Clearly one can reduce the proof to show that if $R \neq 0$, then $Ra = aR = 0$ for some $0 \neq a \in R$. If R is torsion-free then this a consequence of Proposition 3 and Corollary 4. Thus we can assume that there is a prime p and $0 \neq x \in R$ such that $px = 0$. Then $pRx = xpR = 0$. Now $R/pR \in \beta$ is a filial or left filial $\mathbb{Z}_p$-algebra. Hence by the above quoted Theorem 5 from [4], $R^3 \subseteq pR$. Consequently $R^3x = xR^3 = 0$.

Clearly if i, j are non-negative integers such that $i + j \geq 5$, then $R^i x R^j = 0$ (we follow the convention that $R^0 = R^\star$). Let k be the minimal positive integer such that if for some non-negative integers i and j, $i+j = k$, then $R^i x R^j = 0$. Then for some non-negative integers n and m with $n + m = k - 1$, $T = R^n x R^m \neq 0$. Clearly $RT = TR = 0$. The result follows. $\square$

Note that if an S-nilpotent ring R is idempotent then $R = 0$. Indeed, if $R = R^2$, then $\{x \in R \mid Rx + xR \subseteq \operatorname{Ann} R\} \subseteq \{x \in R \mid R^2x = 0 = xR^2\} = \operatorname{Ann} R$. Hence $\operatorname{Ann}(R/\operatorname{Ann} R) = 0$ and S-nilpotency of R implies that $R = \operatorname{Ann} R$, so $R = R^2 = 0$. This and Theorem 5 give in particular another proof of the following result obtained in [9] and in [10].

Corollary 6 (cf. [9], Proposition 26, and [10], Proposition 3.8)**.** *There is no nonzero idempotent β-radical filial or left filial ring.*

As it was mentioned above not all β-radical filial rings are nilpotent. However it turns out that they are sums of nilpotent ideals.

Corollary 7. *Every β-radical filial ring R is the sum of its nilpotent ideals.*

Proof. By Theorem 1, $(a)^2 + \mathbb{Z}a = (a)$. This implies that for every positive integer n, $(a)^{n+1} + \mathbb{Z}a^n \subseteq (a)^n = ((a)^2 + \mathbb{Z}a)^n \subseteq (a)^{n+1} + \mathbb{Z}a^n$. Consequently $(a)^{n+1} + \mathbb{Z}a^n = (a)^n$. However $R \in \beta$, so for some n, $a^n = 0$. Then $(a)^n = (a)^{n+1}$ and $(a)^n$ is an idempotent β-radical filial ring. By Corollary 6, $(a)^n = 0$. Hence each principal ideal of R is nilpotent. This obviously gives the result. □

In [4] it was proved that β-semisimple left filial rings (which in fact are reduced rings) are filial. From Theorem 2 (i) and Corollary 7 we get in particular that for β-radical rings the converse implication holds.

Corollary 8. *A β-radical ring is filial if and only if all its subrings are ideals. In particular every β-radical filial ring is left filial.*

Rings in which subrings are ideals were studied in [6]. Corollary 8 shows that the results obtained there apply to β-radical filial rings.

In [4] it was proved that left filial β-radical rings which are algebras over fields are filial. By Proposition 3 also torsion-free β-radical left filial rings are filial. Thus it is rather surprising that there are nilpotent left filial rings which are not filial. Now we give an example of such a ring.

Example. If A is a ring and V is a left A-module, then the set $\begin{pmatrix} A & V \\ 0 & 0 \end{pmatrix}$ of 2×2-matrices of the form $\begin{pmatrix} a & v \\ 0 & 0 \end{pmatrix}$, where $a \in A$ and $v \in V$, is a ring with respect to canonical matrix addition and multiplication. Let p be a prime. Obviously $p\mathbb{Z}_{p^3}$ is a left $p\mathbb{Z}_{p^3}/p^2\mathbb{Z}_{p^3}$-module and $p\mathbb{Z}_{p^3}/p^2\mathbb{Z}_{p^3} \simeq p\mathbb{Z}_{p^2}$. These define the respective structure of $p\mathbb{Z}_{p^2}$-module on $p\mathbb{Z}_{p^3}$ and the ring $R = \begin{pmatrix} p\mathbb{Z}_{p^2} & p\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix}$. Obviously $R^3 = 0$. Note that R is not filial. Indeed, $\begin{pmatrix} p\mathbb{Z}_{p^2} & 0 \\ 0 & 0 \end{pmatrix} \lhd \begin{pmatrix} p\mathbb{Z}_{p^2} & p^2\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix} \lhd \begin{pmatrix} p\mathbb{Z}_{p^2} & p\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix} = R$, but $\begin{pmatrix} p\mathbb{Z}_{p^2} & 0 \\ 0 & 0 \end{pmatrix}$ is not an ideal of R.

Now we will show that every additive subgroup S of R is a left ideal. Let $I = \begin{pmatrix} p\mathbb{Z}_{p^2} & p^2\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix}$. If $S \subseteq I$, then $RS = 0$ and consequently $S <_l R$. If $S \not\subseteq I$, then pS is a nonzero subgroup of the additive group $\begin{pmatrix} 0 & p^2\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix}$, the order of which is equal p. Hence $S \supseteq pS = \begin{pmatrix} 0 & p^2\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix}$. Consequently $RS \subseteq R^2 = \begin{pmatrix} 0 & p^2\mathbb{Z}_{p^3} \\ 0 & 0 \end{pmatrix} \subseteq S$, so $S <_l R$. $\square$

Though all semiprime left filial rings are filial and β-radical left filial rings which are algebras over fields are filial, there are algebras over fields which are left filial but not filial rings.

Let Δ be a division ring. In [4] it was proved that the matrix ring $M = \begin{pmatrix} \Delta & 0 \\ \Delta & 0 \end{pmatrix}$ is left filial and it is filial if and only if $\Delta \simeq \mathbb{Z}_p$ for a prime p. In [10] it was shown that defining on the group direct sum $R = \Delta \oplus \Delta$ the multiplication $(a,r)(b,s) = (a(b+s), r(b+s))$, $a,b,r,s \in \Delta$ one obtains a ring which also is left filial and is filial if and only if $\Delta \simeq \mathbb{Z}_p$. It turns out that in fact $M \simeq R$. One can easily check that the map $(a,r) \longrightarrow \begin{pmatrix} a+r & 0 \\ r & 0 \end{pmatrix}$ is an isomorphism of these rings.

In the following result we obtain for some fields F a complete classification of F-algebras which are simultaneously filial and left filial rings.

Recall that a ring A is called *strongly regular* ([5]) if for every $a \in A$ there is $x \in A$ such that $a = xa^2$. It is well known that the class of all strongly regular rings is radical.

Theorem 9. *Let F be a field, which is not isomorphic to $\mathbb{Z}_p$ for every prime p, and R be an F-algebra. Then R is a filial and left filial ring if and only if $R = S \oplus T$, where S is a strongly regular ideal of R and T is an ideal of R such that $T^2 = 0$.*

Proof. To get the "if" part it suffices to note that the assumptions on S and T force that if $K <_l L <_l R$, then $K = A \oplus B$, where $A <_l S$ and B is a subring of T.

Now we will prove the "only if" part. Let $T = \beta(R)$ and S be the strongly regular radical of R. Obviously $T \cap S = 0$. By [4], Theorem 18, $RT = 0$ and $\bar{R} = R/T$ is strongly regular. Note now that for every $t \in T$, $\mathbb{Z}t \lhd tR^\star \lhd R$, so $\mathbb{Z}t \lhd R$. Consequently $tR \subseteq \mathbb{Z}t$. Since tR is an F-subspace of R, the assumption on F implies that $tR = 0$. Thus also $TR = 0$. Let $r \in R$ and $\bar{r}$ be the image of r in $\bar{R}$. Since $\bar{R}$ is strongly regular, there is an idempotent a in $\bar{R}$ such that $\bar{r} \in a\bar{R}$. Lift a to an idempotent e in R. If for some $x \in R$, $ex \in T$, then $ex = e(ex) \in RT = 0$. Hence $eR \cap T = 0$. Since $\bar{R}$ is strongly regular, eR is strongly regular and $(eR + T)/T \lhd \bar{R}$.

Thus $R(eR) \subseteq eR + T$ and $R(eR) = R(eR)^2 \subseteq (eR + T)eR \subseteq eR + TR = eR$. These show that $eR \lhd R$ and $eR \subseteq S$. Consequently $r \in eR + T \subseteq S + T$. Hence $R = S \oplus T$. The result follows. □

Note that the conditions on S and T in the above theorem are left-right symmetric. This shows that the ring R is also right filial (with obvious meaning of this notion). One may ask whether, generally, a ring which is filial and left filial must be right filial. Theorem 9 can be also considered as a result concerning the problem when filiality of a ring R implies its left filiality. Obviously the necessary condition is that $\bar{R} = R/\beta(R)$ is left filial. One may ask whether this assumption (plus the assumption that R is filial) is already sufficient. From results in [4] it follows that $\bar{R}$ is left filial if and only $\bar{R}/S(\bar{R})$ is a commutative filial domain, where $S(\bar{R})$ is the strongly regular radical of $\bar{R}$. Now we will show that the answer to the both above mentioned questions is positive when $\bar{R} = S(\bar{R})$.

Proposition 10. *If R is a filial ring such that $R/\beta(R)$ is strongly regular, then R is left and right filial.*

Proof. Since the assumptions are left-right symmetric, it suffices to prove that R is left filial. Suppose that $K <_l L <_l R$. We have to show that $K <_l R$. Since R is filial, by Theorem 2 (ii) and Corollary 7, $K \cap \beta(R) \lhd R$. Passing to the factor ring $R/(K \cap \beta(R))$ we can assume that $K \cap \beta(R) = 0$. Now $(K + \beta(R))/\beta(R) <_l (L + \beta(R))/\beta(R) <_l R/\beta(R)$, so since $R/\beta(R)$ is strongly regular, $(K + \beta(R))/\beta(R) <_l R/\beta(R)$. Hence $RK \subseteq K + \beta(R)$ and $(K + \beta(R))/\beta(R) \simeq K/(K \cap \beta(R)) \simeq K$. Since left ideals of strongly regular rings are idempotent, we get that $K^2 = K$. Consequently $RK = RK^2 \subseteq K^2 + \beta(R)K = K + \beta(R)K$. Now $\beta(R)K = \beta(R)K^2 \subseteq \beta(R) \cap LK \subseteq \beta(R) \cap K = 0$. Hence $RK \subseteq K$, so $K <_l R$. □

3. Filiality and some ring constructions.

In this section we describe direct sums of copies of a ring, matrix rings and polynomial rings which are filial or left filial rings.

From Theorem 2 it follows that if R is a left filial ring, then every nil subring of R is a left ideal of R and that if R is filial, then every subring of a nilpotent ideal of R is an ideal of R (Theorem 2 (i) and Corollary 7 give that if R is filial, then every subring of $\beta(R)$ is an ideal of R. However the stated weaker result is sufficient for us here).

Proposition 11. *Let R be a ring. Then*

(i) *for every integer $n \geq 2$, the ring $M_n(R)$ of $n \times n$-matrices over R is left filial if and only if $R^2 = 0$;*

(ii) *the polynomial ring $R[x]$ is filial (left filial) if and only if $R^2 = 0$.*

Proof. (i) Clearly Re_{21}, where e_{21} is the respective matrix unit, is a nilpotent subring of $M_n(R)$. Hence if $M_n(R)$ is left filial, then $Re_{21} <_l M_n(R)$. Clearly this holds if and only if $R^2 = 0$.

(ii) If $R[x]$ is filial or left filial, then so is $R[\bar{x}] = R[x]/x^4R[x]$. Note that $S = \{r\bar{x}^2 + r\bar{x}^3 \mid r \in R\}$ is a subring of the nilpotent ideal $\bar{x}R[\bar{x}]$ of $R[\bar{x}]$. Consequently $S <_l R[\bar{x}]$. However $R\bar{x} \cdot S = R^2\bar{x}^3$. Clearly $R^2\bar{x}^3 \subseteq S$ if and only if $R^2 = 0$.

The other implications are obvious. $\square$

Now we will describe left filial rings which are sums of copies of a ring.

Theorem 12. *For a given ring R the following conditions are equivalent*

(i) *$R \oplus R$ is a left filial ring;*

(ii) *$R\beta(R) = 0$ and $R/\beta(R)$ is a strongly regular ring;*

(iii) *for every $r \in R$, $Rr = Rr^2$.*

Proof. (i) $\Rightarrow$ (ii). Clearly $S = \{(x, x) \mid x \in \beta(R)\}$ is a nil subring of $R \oplus R$. Hence $S <_l R \oplus R$. This obviously implies that $R\beta(R) = 0$. Now we can factor out $\beta(R)$ and assume that $\beta(R) = 0$. Obviously R is left filial, so by Theorem 2 (i), R is a reduced ring (i.e. R contains no nonzero nilpotent elements). By Theorem 1 (ii), for every $r \in R$, $R^\star r = \mathbb{Z}r + R^\star r^2$ and $R^\star r^2 = \mathbb{Z}r^2 + R^\star r^4$. Consequently $I = (R^\star r)^4 \subseteq R^\star r^4$. Obviously $(R^\star r/I) \oplus (R^\star r/I)$ is also a left filial ring. Hence $(R^\star r/I) \cdot \beta(R^\star r/I) = 0$. However $\beta(R^\star r/I) = R^\star r/I$, so $R^\star r^2 \subseteq I \subseteq R^\star r^4$. In particular $r^2 = xr^3$ for some $x \in R$. Thus $(r - xr^2)r = 0$ and $(r - xr^2)^2 = 0$. Since R is reduced, $r - xr^2 = 0$. Hence R is strongly regular.

(ii) $\Rightarrow$ (iii). Since $R/\beta(R)$ is strongly regular, for every $r \in R$ there exists $x \in R$ such that $r - xr^2 \in \beta(R)$. Now $R\beta(R) = 0$, so $R(r - xr^2) = 0$. This implies that $Rr = Rr^2$.

(iii) $\Rightarrow$ (i). For arbitrary $r_1, r_2 \in R$, $Rr_1^2 = Rr_1$ and $Rr_2^2 = Rr_2$. Hence $(R \oplus R)(r_1, r_2)^2 = (R \oplus R)(r_1, r_2)$ and Theorem 1 (ii) implies that $R \oplus R$ is left filial. $\square$

Remark. Note that the class of rings satisfying the conditions of Theorem 12 is the largest subclass of the class of left filial rings which is closed under direct sums (direct products).

We conclude with a description of filial rings which are sums of copies of a ring and filial matrix rings.

Let $\mathcal{C} = \{R \mid \text{ for every } I \lhd R,\ RI = I^3 = IR\}$. It is easy to observe that if $R \in \mathcal{C}$, then for every $I \lhd R$ and every integer $n \geq 2$, $I^2 = I^n$. In particular for every $a \in R$, $(a)^2 = R(a) = (a)R$. Hence, since $(a) = \mathbb{Z}a + R(a) + (a)R$, $(a) = \mathbb{Z}a + (a)^2$. Consequently Theorem 1 (i) implies that all rings from $\mathcal{C}$ are filial.

Proposition 13. *For arbitrary $A, B \in \mathcal{C}$, $A \oplus B \in \mathcal{C}$ and for every integer $n \geq 2$, $M_n(A) \in \mathcal{C}$.*

Proof. Let $I \lhd A \oplus B$ (respectively, $I \lhd M_n(A)$) and let $\bar{I}$ be the ideal of $A^\star \oplus B^\star$ (respectively, $M_n(A^\star)$) generated by I. Clearly $\bar{I} = J_1 \oplus J_2$ for some $J_1 \lhd A$, $J_2 \lhd B$ (respectively, $\bar{I} = M_n(J)$ for some $J \lhd A$). Since $A, B \in \mathcal{C}$, $AJ_1 = {J_1}^9 = J_1A$, $BJ_2 = {J_2}^9 = J_2B$ and $AJ = J^9 = JA$. Hence $(A \oplus B)\bar{I} = \bar{I}^9 = \bar{I}(A \oplus B)$ (respectively, $M_n(A)\bar{I} = \bar{I}^9 = \bar{I}M_n(A)$). Since $\bar{I}^3 \subseteq I$, $(A \oplus B)I \subseteq (A \oplus B)\bar{I} = \bar{I}^9 \subseteq I^3$. Hence $(A \oplus B)I = I^3$. Similarly $I(A \oplus B) = I^3$ and $M_n(A)I = I^3 = IM_n(A)$. $\square$

Theorem 14. *Let R be a ring and n an integer ≥ 2. The following conditions are equivalent*

(i) *$R \in \mathcal{C}$;*

(ii) *$R \oplus R$ is a filial ring;*

(iii) *$M_n(R)$ is a filial ring.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow directly from Proposition 13 and the fact that $\mathcal{C}$ consists of filial rings.

To get (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) take $I \lhd R$. Factoring out I^3 we can assume that $I^3 = 0$ and then we have to show that $RI = IR = 0$. Now $S = \{(i,i) \mid i \in I\}$ is a subring of the nilpotent ideal $I \oplus I$ of $R \oplus R$. Hence (ii) gives that $S \lhd R \oplus R$. This obviously implies that $RI = IR = 0$.

Also $T = Ie_{11}$, where e_{11} is the respective matrix unit, is a subring of the nilpotent ideal $M_n(I)$ of $M_n(R)$. Hence (iii) gives that $T \lhd M_n(R)$. This implies that $RI = IR = 0$. $\square$

Remarks. In [2], Proposition 6, it was shown that $R \oplus R$ is a filial ring if and only if for every $a \in R$, $[a] = [a]^2$, where $[a] = Ra + aR + RaR$. Let us observe that the condition $[a] = [a]^2$ is equivalent to the condition that for every integer $n \geq 2$, $[a] = [a]^n$ and this is equivalent to $Ra + aR \subseteq (RaR)^n$ for every $n \geq 2$. Further, it is easy to see that this is equivalent to the fact that $R \in \mathcal{C}$. Thus one can apply Theorem 14 to get another proof of Proposition 6 from [2] or, conversely, apply that proposition to get another proof of a part of Theorem 14.

One can check that $R \in \mathcal{C}$ if and only if for every $I \lhd R$, $RI = RI^2 = IR = I^2R$. This condition may be considered as an analog of condition (iii) in Theorem 12.

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