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# On the Lie derivative of structure Jacobi operator of real hypersurfaces in complex projective space

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**Abstract.** We prove the non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that its structure Jacobi operator has Lie derivative equal to zero.

## 1. Introduction

We will consider connected real hypersurfaces $M$ in complex projective space $\mathbb{C}P^m$, $m \geq 3$, endowed with the metric of constant holomorphic sectional curvature equal to 4.

The problem of classifying such hypersurfaces is still open, although several partial results have been obtained in works due to Lawson, [11], Takagi, [18] and [19], Okumura, [16], Maeda, [12], Kimura, [8], Kon, [9], Cecil and Ryan, [2], Montiel, [13], Montiel and Romero, [14], and Berndt, [1], among others. In [15] there is a survey of the most important results in this line.

The fact of a Riemannian manifold being a real hypersurface in $\mathbb{C}P^m$ yields hard restrictions to its intrinsic geometry. For example, it cannot be Einstein, thus its sectional curvature is not constant. It neither can be a locally symmetric space.

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Therefore some weaker intrinsic conditions have been studied (Ricci-parallelness, [9], harmonic curvature, [10], cyclic-Ryan, [6], and so on).

These results are very important steps in the problem of classification of such submanifolds but there are still many open problems, for instance, if the structure vector field $\xi$ is not principal. That is, following Berndt's notation, if the real hypersurface is not Hopf.

The Jacobi operator $R_X$ with respect to a unit vector field $X$ is defined as $R_X = R(.\,,X)X$, where $R$ is the curvature tensor field on $M$. Then we see that $R_X$ is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y)+R(Y,\dot\gamma)\dot\gamma = 0$ along a geodesic $\gamma$ in $M$. It is well-known that the notion of Jacobi vector fields involves many important geometric properties. Cho and Ki, [3], show that the Jacobi operator $R_\xi$ with respect to the structure vector field $\xi$ of a geodesic hypersphere is represented by $R_\xi = k(I - \eta \otimes \xi)$ where $I$ denotes the identity transformation and $k$ is a constant and they give a local structure theorem of a real hypersurface of $\mathbb{C}P^m$ satisfying $R_\xi = k(I - \eta \otimes \xi)$ where $k$ is a function.

These authors study in [3] and [4] real hypersurfaces of $\mathbb{C}P^m$ in terms of the commutativity of $R_\xi$ and $\phi$ when $A\xi$ is a principal curvature vector field on $M$. In [3] they also study Hopf real hypersurfaces of $\mathbb{C}P^m$ such that the Jacobi operator $R_\xi$ is diagonalizable by a parallel orthonormal frame field along each trajectory of $\xi$ and at the same time their eigenvalues are constant along each trajectory of $\xi$.

In this line of characterizing real hypersurfaces of $\mathbb{C}P^m$ in terms of $R_\xi$ it is natural to consider the problem about the parallelism and the invariance, or Lie parallelism. In [17] we prove the non-existence of real hypersurfaces in nonflat complex space forms with parallel structure Jacobi operator. By the expression of $R_\xi$ in a geodesic hypersphere of $\mathbb{C}P^m$ it can be proved that $(\mathcal{L}_X R_\xi)\phi X = -2\cot^3 r\xi$, where $r$ is the radius of the hypersphere, $0 < r < \pi/2$, $k = \cot^2 r$ and $X$ is a tangent vector field on $M$ orthonormal to $\xi$. Then, $R_\xi$ is not Lie parallel for geodesic hyperspheres of $\mathbb{C}P^m$.

The purpose of this paper is to classify real hypersurfaces in $\mathbb{C}P^m$ whose structure Jacobi operator is Lie parallel, that is its Lie derivative with respect to any tangent vector field vanishes. Concretely we will prove the following

**Theorem.** *There exist no real hypersurfaces in* $\mathbb{C}P^m$, $m \geq 3$, *such that* $\mathcal{L}_X R_\xi = 0$, *for every* $X$ *tangent to* $M$.

The theorem assures that there exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that every tangent vector field is a collineation for the structure Jacobi operator.

## 2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class $C^\infty$ unless otherwise stated. Let $M$ be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let $N$ be a locally defined unit normal vector field of $M$. Let $\nabla$ be the Levi–Civita conection on $M$ and $(J, g)$ the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field $X$ tangent to $M$ we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$. That is, we have

$$\begin{aligned} \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

for vector fields $X$, $Y$ tangent to $M$. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of $J$ we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any vector fields $X$, $Y$ tangent to $M$, where $A$ denotes the Weingarten endomorphism of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & - 2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \tag{2.6}$$

for any vector fields $X$, $Y$ and $Z$ tangent to $M$, where $R$ is the curvature tensor of $M$.

In the sequel we shall call $\mathbb{D}$ the distribution on $M$ given by all vectors orthogonal to $\xi$ at any point of $M$. We will need the following results

**Theorem 2.1** ([7]). *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, such that $A\phi + \phi A = 0$.*

**Theorem 2.2** ([17]). *There exist no real hypersurfaces $M$ in $\mathbb{C}P^m$, $m \geq 3$, such that the Weingarten endomorphism is given by $A\xi = \xi + \beta U$, where $U$ is a vector field orthonormal to $\xi$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$, $AX = -X$ for every tangent vector $X$ orthogonal to $\mathrm{Span}\{\xi, U, \phi U\}$, where $\beta$ is a nonvanishing smooth function defined on $M$.*

## 3. Some lemmas

Our hypothesis gives $0 = (\mathcal{L}_X R_\xi)Y = \nabla_X(R_\xi(Y)) - \nabla_{R_\xi(Y)}X - R_\xi(\nabla_X Y) + R_\xi(\nabla_Y X)$ for any vector fields $X$, $Y$ tangent to $M$. From the equations in Section 2 this equation yields

$$\begin{aligned} & - g(\phi AX,Y)\xi - g(\xi,Y)\phi AX + g(\nabla_X(A\xi),\xi)AY + g(A\xi,\phi AX)AY \\ & + g(A\xi,\xi)\nabla_X(AY) - g(\nabla_X(AY),\xi)A\xi - g(AY,\phi AX)A\xi \\ & - g(AY,\xi)\nabla_X(A\xi) + g(\xi,Y)\nabla_\xi X - g(A\xi,\xi)\nabla_{AY}X \\ & + g(AY,\xi)\nabla_{A\xi}X - g(A\xi,\xi)A\nabla_X Y + g(\nabla_X Y,A\xi)A\xi \\ & - g(\nabla_Y X,\xi)\xi + g(A\xi,\xi)A\nabla_Y X - g(\nabla_Y X,A\xi)A\xi = 0. \end{aligned} \tag{3.1}$$

Taking the scalar product of this equation and $\xi$ we obtain

$$\begin{aligned}&-g(\phi AX,Y)+g(\phi AX,A\xi)g(AY,\xi)-g(AY,\phi AX)g(A\xi,\xi)\\&+g(\xi,Y)g(\nabla_\xi X,\xi)-g(A\xi,\xi)g(\nabla_{AY}X,\xi)\\&+g(AY,\xi)g(\nabla_{A\xi}X,\xi)-g(\nabla_Y X,\xi)=0\end{aligned}\tag{3.2}$$

for any vector fields $X,Y$ tangent to $M$.

**Theorem 3.1.** *Let $E$ be a subspace of $\mathbb{D}$ that is both holomorphic and $A$-invariant. Let $G=\{X\in E\mid(\phi A+A\phi)X=0\}$ and let $F$ be its orthogonal complement in $E$. Then $AX=\sigma X$ for all $X\in F$ where $\sigma$ is the number satisfying $1+\alpha\sigma=0$. Furthemore, there is a principal basis for $G$ of the form $\{X_i,\phi X_i\}$ with corresponding principal curvatures $\lambda_i$ and $-\lambda_i$. In particular, $F$ and $G$ are $A$-invariant.*

Proof. Take any orthonormal principal basis $S$ for $E$. Let $V$ be the vector space spanned by the elements of $S$ that are actually in $G$. The orthogonal complement, $W$ of $V$ (in $E$) is spanned by the remaining vectors, namely those $Y\in S$ satisfying $(A\phi+\phi A)Y\neq 0$. Consideration of (3.2) for $X$ and $Y$ in $E$ yields $g((I+\alpha A)X,(A\phi+\phi A)Y)=0$. In particular, if $X$ is one of the basis vectors spanning $W$, the corresponding eigenvalue must be $\sigma$. (Otherwise we would have $g(X,(A\phi+\phi A)Y)=0$ for all $Y\in E$ and setting $Y=(A\phi+\phi A)X$ would produce a contradiction.) Since $V$ is a subspace of $G$, we must have that $F$ is a subspace of $W$. Thus we have shown that $A=\sigma I$ on $F$. Consequently $F$ and hence $G$ are $A$-invariant. The statement about the basis of $G$ now follows from the condition $A\phi=-\phi A$ which holds on $G$. $\square$

Let us suppose that $\xi$ is principal ($A\xi=\alpha\xi$).

Let us call $G=\{X\in\mathbb{D}/(\phi A+A\phi)X=0\}$. We can decompose $\mathbb{D}_p=F_p\oplus G_p$, for any $p\in M$, where at any $p\in M$, $F_p$ is the orthogonal complement of $G_p$ in $\mathbb{D}_p$. Thus from the above Theorem 3.1 $G$ is a holomorphic subspace of $\mathbb{D}$ at any point.

Then if $F=\{0\}$, in a neighborhood where $M$ is a Hopf real hypersurface we have $G=\mathbb{D}$ and as $(A\phi+\phi A)\xi=0$, $A\phi+\phi A=0$ which is impossible by Theorem 2.1.

If $F\neq\{0\}$ in $p$ (this is true in a neighborhood of $p$), then as $\mathbb{D}$ is holomorphic and $A$-invariant, by the above Theorem 3.1 we conclude that

$F$ is holomorphic and $AX = \sigma X$ for all $X \in F_p$ where $\sigma$ is the nonzero number satisfying $1 + \alpha\sigma = 0$. But it is well-known that for a Hopf real hypersurface such that $X$ and $\phi X$ are principal vectors in $\mathbb{D}$ with the same principal curvature $\sigma$ we have $\sigma^2 = \alpha\sigma + 1$ which contradicts $1 + \alpha\sigma = 0$. Thus the case $F \neq 0$ cannot occur and there are no Hopf real hypersurfaces satisfying the hypothesis of the Main Theorem.

Therefore we suppose that there exist a unit $U \in \mathbb{D}$ such that $A\xi = \alpha\xi + \beta U$, for a certain nonvanishing function $\beta$ at least on a neighbourhood of any point, that is, take a point where $A\xi - g(A\xi,\xi)\xi \neq 0$ and work in a neighborhood where this holds. From now on, every computation is made locally.

**Lemma 3.1.** *In the above conditions, $A\phi U = -(1/\alpha)\phi U$ and $AU = \beta\xi + \gamma U$, where $\gamma$ satisfies $(\alpha\gamma - 1)(\alpha\gamma - (\beta^2 - 1)) = 0$.*

Proof. We take $Y = U$, $X = \xi$ in (3.1) and obtain $\alpha\beta g(AU, \phi U) = 0$. Thus either $\alpha = 0$ or $g(AU, \phi U) = 0$. In (3.1) we take $Y = \phi U$ and get

$$\begin{aligned} &- g(AX,U)\xi + g(\nabla_X(A\xi),\xi)A\phi U + \beta g(U,\phi AX)A\phi U + \alpha\nabla_X(A\phi U)\\ &- \alpha\nabla_{A\phi U}X - \alpha A\nabla_X\phi U - \alpha g(U,AX)A\xi + \beta g(\nabla_X\phi U,U)A\xi\\ &- g(\nabla_{\phi U}X,\xi)\xi + \alpha A\nabla_{\phi U}X - \alpha g(\nabla_{\phi U}X,\xi)A\xi\\ &- \beta g(\nabla_{\phi U}X,U)A\xi = 0 \end{aligned} \tag{3.3}$$

for any vector field $X$ tangent to $M$.

Take $X = \xi$ in (3.3) and the scalar product by $\xi$. This yields

$$-\beta - \alpha\beta g(A\phi U, \phi U) = 0. \tag{3.4}$$

Thus $\alpha$ does not vanish and $g(A\phi U, \phi U) = -(1/\alpha)$. Similarly we get $g(A\phi U, U) = 0$.

Take $Y \in \mathbb{D}_U = \mathbb{D} \cap \operatorname{Span}\{U, \phi U\}^\perp$ in (3.1) and the scalar product by $\xi$. We have

$$-g(\phi AX, Y) - \alpha g(AY, \phi AX) - \alpha g(\nabla_{AY}X, \xi) - g(\nabla_Y X, \xi) = 0 \tag{3.5}$$

for any vector field $X$ tangent to $M$. If in (3.5) we take $X = \xi$ we obtain $\alpha\beta g(AY, \phi U) = 0$, for any $Y \in \mathbb{D}_U$. This and (3.4) give the result about $A\phi U$. From (3.5), taking $X = U$ we have

$$g(AU, \phi Y) + \alpha g(AU, \phi AY) = 0 \tag{3.6}$$

for any $Y \in \mathbb{D}_U$. If $X = \phi U$ in (3.5) we get $-\alpha g(AY, \phi A\phi U)+$ $\alpha g(\phi U, \phi A^2 Y) + g(\phi U, \phi AY) = 0$, and taking $A\phi U = -(1/\alpha)\phi U$ we get

$$g(AU, AY) = 0 \tag{3.7}$$

for any $Y \in \mathbb{D}_U$. Take $Y = U$ in (3.1) and the scalar product by $\xi$. Thus for any vector field $X$ tangent to $M$ we obtain

$$\begin{aligned} &-g(\phi AX, U) + \beta^2 g(\phi AX, U) - \alpha g(AU, \phi AX) \\ &-\alpha g(\nabla_{AU} X, \xi) + \beta g(\nabla_{A\xi} X, \xi) - g(\nabla_U X, \xi) = 0. \end{aligned} \tag{3.8}$$

If in (3.8) we take $X \in \mathbb{D}_U$ we get $-\alpha g(AU, \phi AX) + \alpha g(X, \phi A^2 U) -$ $\beta g(X, \phi A^2 \xi) + g(X, \phi AU) = 0$, for any $X \in \mathbb{D}_U$. From (3.6) and (3.7) this yields $\beta^2 g(\phi X, AU) = 0$, for any $X \in \mathbb{D}_U$. Thus $AU \in \text{Span}\{\xi, U\}$, that is, for a certain smooth function $\gamma$, $AU = \beta\xi + \gamma U$.

Now take $X = \phi U$, $Y = U$ in (3.1) and the scalar product by $\xi$. This gives $((\beta^2 - 1)/\alpha) + \alpha g(AU, AU) - \alpha\beta^2 - \beta^2\gamma = 0$. That is, $(\alpha\gamma - 1)(\alpha\gamma -$ $(\beta^2 - 1)) = 0$, and this finishes the proof. □

Anyway, we obtain that $\mathbb{D}_U$ is a holomorphic and $A$-invariant distribution.

**Lemma 3.2.** *With the above conditions, for any arbitrary tangent vector $X$ we get $X(\alpha\gamma - \beta^2) = 0$, that is, $\alpha\gamma - \beta^2$ is constant.*

Proof. If we take in (3.1) $Y = U$ and the scalar product by $U$, we obtain the result. □

**Lemma 3.3.** *If $X \in \mathbb{D}_U$ is such that $AX = \lambda X$, then:*

1. $X(\beta) = \beta g(\nabla_U U, X)$
2. $X(\gamma) = (\gamma - \lambda) g(\nabla_U U, X)$
3. $(\phi U)(\beta) + \beta g(\nabla_U \phi U, U) = 1 + \gamma\alpha + 2(\gamma/\alpha)$
4. $(\phi U)(\gamma) + (\gamma + (1/\alpha)) g(\nabla_U \phi U, U) = \beta(\gamma - (2/\alpha))$.

Proof. Codazzi equation gives $(\nabla_X A)U - (\nabla_U A)X = 0$. If we develop this equality and take the scalar product by $\xi$ we obtain the first equality and the scalar product by $U$ gives the second one. Also Codazzi equation yields $(\nabla_{\phi U} A)U - (\nabla_U A)\phi U) = 2\xi$. If we take the scalar product by $\xi$, the third equality follows and the scalar product by $U$ gives the last one. □

**Lemma 3.4.** *If $X \in \mathbb{D}_U$ is such that $AX = \lambda X$, then:*

1. $X(\alpha) + \beta g(\nabla_\xi X, U) = 0$
2. $X(\beta) + (\gamma - \lambda) g(\nabla_\xi X, U) = 0.$

Proof. Codazzi equation gives $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. If we develop this equality and take the scalar product by $\xi$ we get the first equation. Taking the scalar product by $U$ we get the second one. □

**Proposition 3.1.** *If $Y \in \mathbb{D}_U$ is such that $AY = \lambda Y$, for every $X$ tangent to $M$ we get $(1 + \alpha\lambda) g(X, (\phi A + A\phi)Y) = 0$.*

Proof. In (3.1) we take $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$, and $X$ tangent to $M$. The scalar product of the result and $\xi$ gives the proposition. □

## 4. Proof of Main Theorem

We will continue with the notations appearing in Section 3.

Let us call now $G_U = \{X \in \mathbb{D}_U \mid (\phi A + A\phi)X = 0\}$. Let us write $\mathbb{D}_U = F_U \oplus G_U$, where $F_U$ is the orthogonal complement of $G_U$ in $\mathbb{D}_U$. Thus at any point, from Theorem 3.1 $G_U$ is a holomorphic subspace of $\mathbb{D}_U$. As $\mathbb{D}_U$ is holomorphic, the same is true for $F_U$.

From Proposition 3.1 we have three possibilities:

Case A: $\dim G_U = 0$, that is, from Theorem 3.1, $A = -(1/\alpha)\mathrm{Id}.$ on $\mathbb{D}_U$.

Case B: $\dim G_U = 2m - 4$. That is, $G_U = \mathbb{D}_U$.

Case C: $0 < \dim G_U < 2m - 4$. Notice that this case only occurs if $m \geq 4$.

For Cases A and C we have:

**Lemma 4.1.** *With the conditions of either Case A or Case C, we have $\alpha^2\beta^2 = 1 + \alpha\gamma$.*

Proof. Codazzi equation implies $(\nabla_X A)Y - (\nabla_Y A)X = -2g(\phi X, Y)\xi$ for any $X, Y \in F_U$. If we develop this equation and take the scalar product by $\xi$ we obtain

$$g([Y, X], U) = (2/\alpha^2\beta) g(\phi X, Y). \tag{4.1}$$

If we take the above Codazzi equation and its scalar product by $U$ we get

$$g([Y,X],U) = (2\beta/(1+\alpha\gamma))g(\phi X,Y). \tag{4.2}$$

The result follows from (4.1) and (4.2). $\square$

From Lemma 4.1 and the equation $(\alpha\gamma-1)(\alpha\gamma-(\beta^2-1)) = 0$ of Lemma 3.1, we have in this Case

$$(\alpha^2\beta^2-2)(\alpha^2-1) = 0. \tag{4.3}$$

**Lemma 4.2.** *If we are either in Case A or in Case C we have for any* $X \in F_U$, $(1+\alpha\gamma-\beta^2)g(\nabla_\xi X,U) = 0$.

Proof. In (3.1) we take $X \in F_U$, $Y = \xi$ and the scalar product of the result by $U$ gives the result. $\square$

*Remarks.* a) From Lemma 4.1 and (4.3) we get that either $\alpha^2\beta^2 = 2$ and $\gamma = (1/\alpha)$ or $\alpha^2 = 1$ and $\gamma = (\beta^2-1)/\alpha$.

b) From Lemma 4.2 we have $(2-\beta^2)g(\nabla_\xi X,U) = 0$ if $\gamma\alpha = 1$.

c) From Lemmas 4.1 and 4.2 we obtain that $\beta^2(\alpha^2-1)g(\nabla_\xi X,U) = 0$.

Now we begin with Case A. From (4.3) we have two Subcases for Case A:

Subcase A$_1$: $\alpha^2 = 1$. In this case, maybe changing $\xi$ by $-\xi$, if necessary, we can suppose that $\alpha = 1$. Thus $\gamma = \beta^2 - 1$. This kind of real hypersurfaces cannot occur by Theorem 2.2.

Subcase A$_2$: $\alpha^2 \neq 1$ at some point. Thus we work in a neighborhood where this holds. Then $\alpha^2\beta^2 - 2$ is identically zero so that $\alpha\gamma = 1$. From Lemma 3.2, $\beta$ is constant and from third equation in Lemma 3.3, $\beta g(\nabla_U\phi U,U) = 2(1+(1/\alpha)^2)$. From last equation in Lemma 3.3, as $\gamma$ is constant, we get $(2/\alpha)g(\nabla_U\phi U,U) = -(\beta/\alpha)$, thus $4\alpha^2+4 = -\alpha^2\beta^2$, which is impossible. Then we obtain

**Proposition 4.1.** *Case A cannot appear.*

From now on we consider Cases B and C. All computations are going to be made in a neighborhood of any point. In these cases from Theorem 3.1 we have the following situation: $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \gamma U$ (with $(\alpha\gamma-1)(\alpha\gamma-(\beta^2-1)) = 0$), $A\phi U = -(1/\alpha)\phi U$, and for every $X \in G_U$

such that there exists a function $\lambda(X)$ such that $AX = \lambda(X)X$, $A\phi X = -\lambda(X)\phi X$. Thus we obtain the following results where $\lambda$ denotes $\lambda(X)$:

**Lemma 4.3.** *In Cases B and C, for any $X \in G_U$ such that $AX = \lambda X$ we get:*

1. $\alpha\lambda g(\nabla_X \phi X, X) = 0$
2. $\alpha\lambda g(\nabla_{\phi X} X, \phi X) = 0$
3. $\lambda X(\alpha) + \alpha X(\lambda) = 0$
4. $(\alpha\lambda + \alpha\gamma - \beta^2)(g(\nabla_{\phi X} X, U) - g(\nabla_X \phi X, U)) = 0.$

Proof. If in (3.1) we take $Y = \phi X$ and the scalar product by $X$ we get first item. Taking the scalar product by $\phi X$ we have the third equality.

If in (3.1) we take $X = \phi X$, $Y = X$ and the scalar product by $\phi X$ we obtain the second equation and fourth one follows taking the scalar product by $U$. $\square$

We begin to study Case B. Now from Lemma 4.3 we obtain the following subcases:

Subcase $B_1$: $\lambda = 0$ on a neighborhood of every point.

Subcase $B_2$: $g(\nabla_X \phi X, X) = g(\nabla_{\phi X} X, \phi X) = 0$.

Let us begin with Subcase $B_1$. From the fourth item in Lemma 4.3 we get $(\alpha\gamma - \beta^2)(g(\nabla_{\phi X} X, U) - g(\nabla_X \phi X, U)) = 0$.

**Lemma 4.4.** *If $\lambda = 0$, we have:*

1. $\beta(g(\nabla_X \phi X, U) - g(\nabla_{\phi X} X, U)) = 2$
2. $\gamma g(\nabla_{\phi X} X, U) = \gamma g(\nabla_X \phi X, U)$

*for any $X \in G_U$.*

Proof. Codazzi equation gives $(\nabla_{X+\phi X} A)X - (\nabla_X A)(X + \phi X) = 2\xi$. If we develop this equality and take its scalar product by $\xi$, as $X(\lambda) = (\phi X)(\lambda) = 0$, we obtain first item. The second one is obtained taking the scalar product by $U$. $\square$

From Lemma 4.3 and Lemma 4.4 the case $\gamma\alpha = \beta^2 - 1$ does not occur because the fourth equation in Lemma 4.3 yields $g(\nabla_{\phi X} X, U) = g(\nabla_X \phi X, U)$ which contradicts first equality in Lemma 4.4.

If $\gamma\alpha = 1$, the second item in Lemma 4.4 yields $g(\nabla_{\phi X} X, U) = g(\nabla_X \phi X, U)$ and we have a new contradiction with Lemma 4.4. Thus we obtain

**Proposition 4.2.** *Subcase* $\mathrm{B}_1$ *does not occur.*

Let us study Subcase $\mathrm{B}_2$: for any $X \in \mathbb{D}_U$ such that $AX = \lambda X$, $\lambda \neq 0$, $g(\nabla_{\phi X} X, \phi X) = g(\nabla_X \phi X, X) = 0$. We have:

**Lemma 4.5.** *In Subcase* $\mathrm{B}_2$ *we obtain:*

1. $X(\lambda) = (\phi X)(\lambda) = 0$
2. $\lambda(\phi X)(\alpha) + \alpha(\phi X)(\lambda) = 0$
3. $-(\phi U)(\lambda) + ((1/\alpha) - \lambda) g(\nabla_{\phi X} \phi U, \phi X) = 0$

*for any* $X \in \mathbb{D}_U$ *such that* $AX = \lambda X$, $\lambda \neq 0$.

Proof. To obtain the second equality, take $X = \phi X$, $Y = X$ in (3.1) and the scalar product by $X$.

By Codazzi equation $(\nabla_X A)\phi X - (\nabla_{\phi X} A) X = -2\xi$. Developing this equation and taking the scalar product by $\phi X$ we get $X(\lambda) = 0$. Taking its scalar product by $X$ we have $(\phi X)(\lambda) = 0$, finishing the first item.

Codazzi equation also yields $(\nabla_{X+\phi U} A)\phi X - (\nabla_{\phi X} A)(X + \phi U) = -2\xi$. If we develop this equation and take the scalar product by $\phi X$, bearing in mind that $X(\lambda) = 0$ and $g(\nabla_{\phi X} X, \phi X) = 0$, we finish the proof. $\square$

If we now take $X = \phi X$, $Y = \phi U$ in (3.1) and the scalar product by $\phi X$, from the third equation of Lemma 4.5 we get $(\phi U)(\lambda) = 0$. From the fourth equality in Lemma 4.3 we have $(\alpha\lambda + \alpha\gamma - \beta^2)(g(\nabla_{\phi X} X, U) - g(\nabla_X \phi X, U)) = 0$. Thus we have two new subcases:

Subcase $\mathrm{B}_{21}$: $g(\nabla_{\phi X} X, U) = g(\nabla_X \phi X, U)$ for any $X \in \mathbb{D}_U$ such that $AX = \lambda X$, $\lambda \neq 0$.

Subcase $\mathrm{B}_{22}$: $\alpha\lambda + \alpha\gamma - \beta^2 = 0$.

Let us consider Subcase $\mathrm{B}_{21}$: Codazzi equation yields $(\nabla_X A)\xi - (\nabla_\xi A) X = -\phi X$. Taking its scalar product by $\phi X$ we get

$$\beta g(\nabla_X U, \phi X) - 2\lambda g(\nabla_\xi X, \phi X) = -1 - \lambda^2 - \alpha\lambda. \tag{4.4}$$

If we rewrite (4.4) taking $\phi X$, $-X$ and $-\lambda$ instead $X$, $\phi X$ and $\lambda$, respectively, we obtain

$$\beta g(\nabla_{\phi X} U, X) + 2\lambda g(\nabla_\xi \phi X, X) = 1 + \lambda^2 - \alpha\lambda. \tag{4.5}$$

In this Subcase, (4.4) and (4.5) imply $2+2\lambda^2=0$ which is impossible. Thus we have obtained

**Proposition 4.3.** *Subcase* $B_{21}$ *does not occur.*

Let us deal with Subcase $B_{22}$. As $\alpha\lambda+\alpha\gamma-\beta^2=0$ for every $X$ such that $AX=\lambda X$, as now $A\phi X=-\lambda\phi X$, we also get $-\alpha\lambda+\alpha\gamma-\beta^2=0$, thus $\lambda=0$ which gives a contradiction. Thus we have

**Proposition 4.4.** *Case B does not occur.*

We continue with Case C.

Lemma 4.1 implies $\alpha^2\beta^2=1+\alpha\gamma$.

From Lemma 4.2, $(1+\alpha\gamma-\beta^2)g(\nabla_\xi X,U)=0$ for any $X\in F_U$. Following the same reasoning as in Case A we have no real hypersurfaces satisfying our condition unless $\alpha=1$.

Let us now suppose $\alpha=1$. As $G_U\neq\{0\}$, there exists $X\in G_U$ such that $AX=\lambda X$ and $A\phi X=-\lambda\phi X$. Now From Lemma 4.3 we get: i) $\lambda g(\nabla_X\phi X,X)=0$, ii) $\lambda g(\nabla_{\phi X}X,\phi X)=0$, iii) $X(\lambda)=0$, iv) $(\lambda+\gamma-\beta^2)(g(\nabla_{\phi X}X,U)-g(\nabla_X\phi X,U))=0$.

Thus either $\lambda=0$ or $g(\nabla_{\phi X}X,\phi X)=g(\nabla_X\phi X,X)=0$. If $\lambda=0$ the above equation iv) gives $(\gamma-\beta^2)(g(\nabla_{\phi X}X,U)-g(\nabla_X\phi X,U))=0$. Therefore as in Lemma 4.4 we obtain for any $X\in G_U$ such that $AX=0$: v) $\beta(g(\nabla_X\phi X,U)-g(\nabla_{\phi X}X,U))=2$, vi) $\gamma g(\nabla_{\phi X}X,U)=\gamma g(\nabla_X\phi X,U)$. From these last equations and equation iv) the case $\gamma=\beta^2-1$ does not occur. Similarly, if $\gamma=1$ we obtain a contradiction. Thus there exists no $X\in G_U$ such that $AX=0$.

Take now $X\in G_U$ such that $AX=\lambda X$, $\lambda\neq 0$, from i) above $g(\nabla_X\phi X,X)=g(\nabla_{\phi X}X,\phi X)=0$. For such an $X$ as in Lemma 4.5 we get: vii) $X(\lambda)=(\phi X)(\lambda)=0$, viii) $-(\phi U)(\lambda)+(1-\lambda)g(\nabla_{\phi X}\phi U,\phi X)=0$. From equation iv), $(\lambda+\gamma-\beta^2)(g(\nabla_{\phi X}X,U)-g(\nabla_X\phi X,U))=0$ for such an $X$.

The possibility of being $g(\nabla_{\phi X}X,U)=g(\nabla_X\phi X,U)=0$ gives a contradiction as in Case B.

Now if $\lambda+\gamma-\beta^2=0$, suppose that $\gamma=\beta^2-1$. Then $\lambda=1$, and $A\phi X=-\phi X$. Thus $-1+\gamma-\beta^2=0$ which gives a contradiction.

Thus suppose $\lambda+\gamma-\beta^2=0$ and $\gamma=1$. Then $\lambda=\beta^2-1$. If we take in (3.1) $X=\phi U$, $Y=U$ and the scalar product by $U$, as

$(\phi U)(\alpha) = (\phi U)(\gamma) = 0$, we get $(\phi U)(\beta) = 0$. Thus $(\phi U)(\lambda) = 0$ and from third and fourth equations in Lemma 3.3 we get $\beta g(\nabla_U \phi U, U) = 4$ and $2g(\nabla_U \phi U, U) = -\beta$. Thus we have $8 = -\beta^2$, which is impossible. Summing up we have

**Proposition 4.5.** *Case C does not occur.*

This finishes the proof of the Theorem. $\square$

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