Publ. Math. Debrecen
66/3-4 (2005), 283–301

On Hyers–Ulam stability of the generalized Cauchy and Wilson equations

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Abstract. Let G be a topological group, let μ be a complex measure with compact support and let σ be a continuous involution of G. In this paper the Hyers–Ulam stability of the functional inequalities

$$\Big|\int_G f(xty)d\mu(t) - g(x)f(y)\Big| \leq \varepsilon(x),$$

$$\Big|\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)\Big| \leq \varepsilon(y),$$

$x, y \in G$, shall be investigated, where $f, g : G \longrightarrow \mathbb{C}$ and $\varepsilon : G \longrightarrow \mathbb{R}^+$ are continuous functions.

1. Introduction

In many studies concerning functional equations related to the Cauchy equation: $f(xy) = f(x)f(y)$, $x, y \in G$, the main tool is a kind of stability problem inspired by the famous problem proposed in 1940 by S. Ulam (see [17]). More precisely, given a group G and a metric group H with metric d, it is asked if for every function $f : G \longrightarrow H$, such that the function

Mathematics Subject Classification: 39B72.
Key words and phrases: topological group, complex measure, Cauchy equation, Wilson equation, d'Alembert equation, μ-spherical function, Hyers–Ulam stability, superstability.

$(x, y) \longmapsto d(f(xy), f(x)f(y))$ is bounded, there exists a homomorphism $\chi : G \longmapsto H$ such that the function $x \longmapsto d(f(x), \chi(x))$ is bounded.

The first affirmative answer to Ulam's question was given by D. H. Hyers in [10], under the assumption that G and H are Banach spaces.

After Hyers's result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers's result in various directions. The interested reader should refer to [11] for an indepth account on the subject of stability of functional equations.

In the present paper, we shall investigate the Hyers–Ulam stability of the functional equations:

$$\int_G f(xty)d\mu(t) = g(x)f(y) \tag{1.1}$$

and

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y). \tag{1.2}$$

Throughout this paper, G will denote a topological group, μ denote a compactly supported measure on G and σ denote a continuous involution of G. This means that $(\sigma \circ \sigma)(x) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$, for all $x, y \in G$.

We say that μ is σ-invariant if $\langle f \circ \sigma, \mu\rangle = \langle f, \mu\rangle$ for all complex continuous function f on G, where $\langle f, \mu\rangle = \int_G f(t)d\mu(t)$.

A continuous mapping $f, g : G \longrightarrow \mathbb{C}$ will be called a solution of the generalized Cauchy equation if it satisfies

$$\int_G f(xty)d\mu(t) = g(x)f(y), \quad x, y \in G. \tag{1.3}$$

A continuous function $f : G \longrightarrow \mathbb{C}$ is a μ-spherical function if

$$\int_G f(xty)d\mu(t) = f(x)f(y), \quad x, y \in G. \tag{1.4}$$

Classical examples of the integral equation (1.3) is Cauchy equation

$$f(x + y) = f(x)f(y), \quad x, y \in G \tag{1.5}$$

and it is generalization

$$f(x + y) = g(x)f(y), \quad x, y \in G. \tag{1.6}$$

Equation (1.2) is a generalization of d'Alembert's functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x,y\in G, \tag{1.7}$$

Wilson's functional equation

$$f(x+y)+f(x-y)=2f(x)g(y), \quad x,y\in G, \tag{1.8}$$

and was studied by the authors in [7].

In this setting G is an abelian group, $\sigma(x)=-x$ and $\mu=\delta_e$: Dirac measure concentrated at the identity element of G.

Baker, Lawrence and Zorzitto [2] and Baker [1] proved the Hyers–Ulam stability of the functional equation (1.5) i.e., if the Cauchy difference $f(x+y)-f(x)f(y)$ of a complex-valued mapping f defined on a normed space is bounded for all $x,y\in G$, then either f is bounded or $f(x+y)=f(x)f(y)$, for all $x,y\in G$. Such a phenomenon for some functional equation is called superstability.

Székelyhidi [14] and many others considered a generalized version of the previous result [1] and [2], see for example the recent study by R. Badora [3] (stability of K-spherical functions), G. Dolinar [4], R. Ger and P. Šemrl [8] (stability of multiplicative functions with values in semisimple Banach algebra) and by S-M. Jung [12] who considered the case when the Cauchy difference is not bounded.

In [1], Baker also found the superstability of equation (1.7). The result has been extended by L. Székelyhidi [15] and [16].

Our work is organized as follows.

In Section 2, we prove the Hyers–Ulam stability of the functional equation (1.1) (Theorem 2.2).

In Section 3, we study the Hyers–Ulam stability of equation (1.2), where f satisfies the Kannappan type-condition:

$$\mathrm{K}(\mu) \qquad \int_G\int_G f(zsxty)d\mu(t)d\mu(s)=\int_G\int_G f(zsytx)d\mu(t)d\mu(s),$$

for all $x,y,z\in G$ (Theorem 3.2).

Let K be a compact subgroup of the group $\mathrm{Aut}(G)$ of all mappings of G onto G that are automorphisms. Let dk be the normalized Haar measure on K, and consider

$$\int_K f(xk\cdot y)dk=f(x)g(y), \quad x,y\in G, \tag{1.9}$$

where $k \cdot x$ denote the action of $k \in K$ on $x \in G$.

The Hyers–Ulam stability of this equation was considered by Badora in [3]. Consider the group $\tilde{G} = G \times_s K$: the semidirect product of G and K, where the topology is the product topology and the group operation is given by

$$(g_1, k_1)(g_2, k_2) = (g_1 k_1 \cdot g_2, k_1 k_2), \tag{1.10}$$

$K_s = \{e\} \times K$ is a closed compact subgroup of $\tilde{G}$. So the above functional equation is closely related to the following functional equation

$$\int_{K_s} f(xky)dk = f(x)g(y) \tag{1.11}$$

on $\tilde{G}$, and consequently Badora's Theorems (Theorem 1 and Theorem 3) of [3] are special cases of the authors's results.

2. Hyers–Ulam stability of the equation $\int_G f(xty)d\mu(t) = g(x)f(y)$

Theorem 2.1. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be a continuous function. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions such that*

$$\Big| \int_G f(xty)d\mu(t) - g(x)f(y) \Big| \leq \varepsilon(x), \tag{2.1}$$

for all $x, y \in G$.

If f is unbounded, then g is a μ-spherical function.

Proof. Assume that the pair f, g satisfies the inequality (2.1), then by using Fubini's theorem ([9] (14.25) Theorem), we get

$$\Big| \int_G \int_G f(xtysz)d\mu(t)d\mu(s) - \int_G g(xty)d\mu(t)f(z) \Big| \leq \int_G \varepsilon(xty)d|\mu|(t)$$

and

$$\Big| \int_G \int_G f(xtysz)d\mu(t)d\mu(s) - g(x) \int_G f(ytz)d\mu(t) \Big| \leq \varepsilon(x)\|\mu\|.$$

Hence we conclude that

$$\Big| f(z) \int_G g(xty)d\mu(t) - g(x) \int_G f(ysz)d\mu(s) \Big| \leq \varepsilon(x)\|\mu\| + \int_G \varepsilon(xty)d|\mu|(t).$$

Consequently we obtain

$$
\begin{aligned}
|f(z)|&\Big|\int_G g(xty)d\mu(t)-g(x)g(y)\Big| \\
&\le \Big|f(z)\int_G g(xty)d\mu(t)-g(x)\int_G f(ysz)d\mu(s)\Big| \\
&\quad + |g(x)|\Big|\int_G f(ysz)d\mu(s)-g(y)f(z)\Big| \\
&\le |g(x)|\varepsilon(y)+\int_G \varepsilon(xty)d|\mu|(t)+\varepsilon(x)\|\mu\|.
\end{aligned}
$$

Since f is unbounded, then we get

$$\int_G g(xty)d\mu(t)=g(x)g(y), \quad x,y\in G.$$

This completes the proof. □

In particular we have the following corollary which generalizes the result obtained by Baker in [1]. The statement (ii) is proved in [5].

Corollary 2.1. *Let $\alpha\in\mathbb{C}^*$. Let $\varepsilon: G\longrightarrow\mathbb{R}^+$ be a continuous function and let $f: G\longrightarrow\mathbb{C}$ be a continuous function such that*

$$\Big|\int_G f(xty)d\mu(t)-\alpha f(x)f(y)\Big|\le\varepsilon(x), \tag{2.2}$$

for all $x,y\in G$. Then,

(i) *f is either bounded function, or αf is a μ-spherical function.*

(ii) *If $\varepsilon(x)=\delta$, $\delta\in\mathbb{R}^+$, then either $|f(x)|\le\frac{\|\mu\|+\sqrt{\|\mu\|^2+4\delta|\alpha|}}{2|\alpha|}$, $x\in G$, or αf is a μ-spherical function.*

Lemma 2.1. *Let $\alpha\in\mathbb{C}$ such that $\|\mu\|<|\alpha|$. Let $a\in G$, $\epsilon\in\mathbb{R}^+$ and let $f: G\longrightarrow\mathbb{C}$ be a continuous function such that*

$$\Big|\int_G f(atx)d\mu(t)-\alpha f(x)\Big|\le\epsilon, \tag{2.3}$$

for all $x\in G$, then there exits exactly one solution $\mathcal{F}\in C(G)$ of

$$\int_G \mathcal{F}(atx)d\mu(t)=\alpha\mathcal{F}(x), \tag{2.4}$$

for all $x \in G$ such that $\mathcal{F} - f$ is bounded. $\mathcal{F}$ satisfies

$$\big|\mathcal{F}(x) - f(x)\big| \leq \frac{\epsilon}{|\alpha| - \|\mu\|}, \tag{2.5}$$

for all $x \in G$.

Proof. It follows from the inequality (2.3) that

$$\Big|\int_G \frac{f}{\|\mu\|}(atx)d\frac{\mu}{\|\mu\|} - \frac{\alpha}{\|\mu\|}\frac{f(x)}{\|\mu\|}\Big| \leq \frac{\epsilon}{\|\mu\|^2}, \tag{2.6}$$

for all $x \in G$. Thus we find the new equation that

$$\Big|\int_G g(atx)d\nu(t) - \beta g(x)\Big| \leq \epsilon', \tag{2.7}$$

for all $x \in G$, where $g = \frac{f}{\|\mu\|}$, $\nu = \frac{\mu}{\|\mu\|}$, $\beta = \frac{\alpha}{\|\mu\|}$ and $\epsilon' = \frac{\epsilon}{\|\mu\|^2}$.

Notice that $|\beta| > 1$ and $\|\nu\| \leq 1$.

Now for any $x \in G$ and every $n \in \mathbb{N}^*$, we define the function:

$$G_n(x) = \beta^{-n}\int_G \cdots \int_G \int_G g(at_1a \dots t_{n-1}at_nx)d\nu(t_1)\dots d\nu(t_{n-1})d\nu(t_n)$$

and we prove by induction the following inequality

$$\big|\beta^n G_n(x) - \beta^n g(x)\big| \leq \frac{\epsilon'(1-|\beta|^n)}{1-|\beta|}. \tag{2.8}$$

For $n = 1$, it is the inequality (2.7). For all positive integer n, it holds

$$\begin{aligned}
&\big|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}g(x)\big| \\
&\quad\leq \big|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}G_n(x)\big| + \big|\beta^{n+1}G_n(x) - \beta^{n+1}g(x)\big| \\
&\quad\leq |\beta|\frac{\epsilon'(1-|\beta|^n)}{1-|\beta|} + \big|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}G_n(x)\big|.
\end{aligned}$$

Since

$$\begin{aligned}
&|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}G_n(x)| \\
&\quad= \Big|\int_G\int_G \cdots \int_G g(at_1at_2\dots at_{n+1}x)d\nu(t_1)d\nu(t_2)\dots d\nu(t_{n+1})
\end{aligned}$$

$$- \beta \int_G \int_G \cdots \int_G g(at_1at_2 \ldots at_nx)d\nu(t_1)d\nu(t_2)\ldots d\nu(t_n)\Big|$$
$$\leq \int_G \cdots \int_G \Big| \int_G g(at_1at_2 \ldots at_{n+1}x)d\nu(t_1) - \beta g(at_2a \ldots at_{n+1}x)\Big|$$
$$d|\nu|(t_2)\ldots d|\nu|(t_{n+1}).$$

In view of (2.7) and $\|\nu\| \leq 1$, we get

$$|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}G_n(x)| \leq \epsilon'.$$

Which implies that

$$|\beta^{n+1}G_{n+1}(x) - \beta^{n+1}g(x)| \leq \frac{\epsilon'(1-|\beta|^{n+1})}{1-|\beta|}.$$

Consequently we get

$$|G_n(x) - g(x)| \leq \frac{\epsilon'}{|\beta|-1}, \quad x \in G, \ n \in \mathbb{N}. \tag{2.9}$$

From the inequality (2.7), we obtain

$$|G_{n+1}(x) - G_n(x)| \leq |\beta|^{-(n+1)}\epsilon'. \tag{2.10}$$

Since $|\beta| > 1$, hence $(G_n(x))$ is a Cauchy sequence for each $x \in G$ and it follows that there exists a limit function

$$G_\mu(x) = \lim_{n\to+\infty} G_n(x). \tag{2.11}$$

By (2.9) we obtain

$$|G_\mu(x) - g(x)| \leq \frac{\epsilon'}{|\beta|-1}, \tag{2.12}$$

for all $x \in G$. In what follows we prove that

$$\int_G G_\mu(atx)d\nu(t) = \beta G_\mu(x), \quad x \in G. \tag{2.13}$$

The convergence in (2.11) is uniform, then for all $x \in G$

$$\int_G G_\mu(atx)d\nu(t) = \lim_{n\to+\infty} \int_G G_n(atx)d\nu(t)$$
$$= \lim_{n\to+\infty} \beta^{-n} \int_G \int_G \cdots \int_G \int_G g(at_1at_2\ldots at_natx)d\nu(t_1)d\nu(t_2)\ldots d\nu(t_n)d\nu(t)$$

$$= \beta \lim_{n\to+\infty} \beta^{-(n+1)} \int_G \dots \int_G \int_G g(at_1 \dots at_n at_{n+1} x) d\nu(t_1) \dots d\nu(t_n) d\nu(t_{n+1})$$

$$= \beta \lim_{n\to+\infty} G_{n+1}(x) = \beta G_\mu(x).$$

It follows that

$$G_\mu(x) = \beta^{-n} \int_G \dots \int_G G_\mu(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n), \tag{2.14}$$

for all $x \in G$.

Now we will show that G_μ is the unique continuous solution of (2.13) which satisfies (2.12). Let $H_\mu : G \longrightarrow \mathbb{C}$ be another continuous mapping which satisfies (2.13) as well as (2.12). In view of (2.9) and (2.14), we have

$$\begin{aligned}
|G_\mu(x) - H_\mu(x)| &= \Big|\beta^{-n} \int_G \dots \int_G G_\mu(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n) \\
&\quad - \beta^{-n} \int_G \dots\dots \int_G H_\mu(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n)\Big| \\
&\leq \Big|\beta^{-n} \int_G \dots \int_G G_\mu(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n) \\
&\quad - \beta^{-n} \int_G \dots\dots \int_G g(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n)\Big| \\
&\quad + \Big|\beta^{-n} \int_G \dots \int_G H_\mu(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n) \\
&\quad - \beta^{-n} \int_G \dots\dots \int_G g(at_1 \dots at_n x) d\nu(t_1) \dots d\nu(t_n)\Big| \\
&\leq |\beta|^{-n} \left(\frac{\epsilon'}{|\beta| - 1} + \frac{\epsilon'}{|\beta| - 1} \right),
\end{aligned}$$

for all $x \in G$ and all $n \in \mathbb{N}^*$. Thus $G_\mu = H_\mu$.

Consequently $\mathcal{F} = \|\mu\| G_\mu$ is the unique mapping which satisfies

$$\int_G \mathcal{F}(atx) d\mu(t) = \alpha \mathcal{F}(x)$$

and

$$|\mathcal{F}(x) - f(x)| \leq \frac{\epsilon}{|\alpha| - \|\mu\|},$$

for all $x \in G$. This completes the proof. □

Remark 2.1. By adapting the proof used in Lemma 2.1, we obtain the similarly results as in Lemma 2.1 for the functional equation.

$$\int_G f(xta)d\mu(t) = \alpha f(x), \quad x \in G. \tag{2.15}$$

In the following theorem we shall investigate the stability of equation (1.1) by using an idea from the paper [13] in which J. Schwaiger has proved the stability of the functional equation of homogeneity:

$$F(\gamma \cdot x) = M(\gamma)F(x), \quad x \in X,\ \gamma \in G, \tag{2.16}$$

where G is a semigroup with unit, $X \neq \emptyset$ is a set and $\cdot\ G \times X \longrightarrow X$ is a semigroup action of G on X.

Theorem 2.2. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be an arbitrary continuous function. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions such that*

$$\Big| \int_G f(xty)d\mu(t) - g(x)f(y)\Big| \leq \varepsilon(x), \tag{2.17}$$

for all $x, y \in G$.

Suppose that there is $a \in G$ such that

$$\int_G \int_G f(xtasy)d\mu(t)d\mu(s) = \int_G \int_G f(atxsy)d\mu(t)d\mu(s), \quad x, y \in G$$

and $|g(a)| > \|\mu\|$. Then there exists one solution $\mathcal{F} \in C(G)$ of

$$\int_G \mathcal{F}(xty)d\mu(t) = g(x)\mathcal{F}(y) \tag{2.18}$$

such that $\mathcal{F} - f$ is bounded. $\mathcal{F}$ satisfies

$$|\mathcal{F}(x) - f(x)| \leq \frac{\varepsilon(a)}{|g(a)| - \|\mu\|}, \tag{2.19}$$

for all $x, y \in G$.

Proof. By putting $x = a$ in (2.17), we get

$$\Big| \int_G f(atx)d\mu(t) - g(a)f(x)\Big| \leq \varepsilon(a), \tag{2.20}$$

for all $x \in G$. In view of Lemma 2.1, there exists a unique continuous mapping $\mathcal{F} : G \longrightarrow \mathbb{C}$ such that

$$\int_G \mathcal{F}(atx)d\mu(t) = g(a)\mathcal{F}(x) \tag{2.21}$$

and

$$|\mathcal{F}(x) - f(x)| \leq \frac{\varepsilon(a)}{|g(a)| - \|\mu\|}, \tag{2.22}$$

for all $x \in G$.

Now we show that $\mathcal{F}$ is a solution of equation (2.18). From the proof of Theorem 2.2, we have $\mathcal{F}(x) = \|\mu\| G_\mu(x)$, where

$$\begin{aligned} G_\mu(x) &= \lim_{n\to+\infty} \left(\frac{g(a)}{\|\mu\|}\right)^{-n} \int_G \cdots \int_G \frac{f}{\|\mu\|}(at_1 \ldots at_n x)d\nu(t_1)\ldots d\nu(t_n) \\ &= \lim_{n\to+\infty} G_n(x) \end{aligned}$$

and $\nu = \frac{\mu}{\|\mu\|}$.

For all $x, y \in G$, we have

$$\begin{aligned} &\Big|\int_G G_n(xty)d\nu(t) - \frac{g(x)}{\|\mu\|}G_n(y)\Big| = \left|\frac{g(a)}{\|\mu\|}\right|^{-n} \\ &\times \Big|\int_G\int_G \cdots \int_G\int_G \frac{f}{\|\mu\|}(at_1at_2\ldots at_n xty)d\nu(t_1)d\nu(t_2)\ldots d\nu(t_n)d\nu(t) \\ &\quad - \frac{g(x)}{\|\mu\|}\int_G\int_G \cdots \int_G \frac{f}{\|\mu\|}(at_1at_2\ldots at_n y)\Big| \\ &= \left|\frac{g(a)}{\|\mu\|}\right|^{-n} \Big|\int_G\int_G \cdots \int_G\int_G \frac{f}{\|\mu\|}(xtat_1\ldots at_n y)d\nu(t)d\nu(t_1)\ldots d\nu(t_n) \\ &\quad - \frac{g(x)}{\|\mu\|}\int_G\int_G \cdots \int_G \frac{f}{\|\mu\|}(at_1at_2\ldots at_n y)d\nu(t_1)d\nu(t_2)\ldots d\nu(t_n)\Big| \\ &\leq \left|\frac{g(a)}{\|\mu\|}\right|^{-n} \int_G \cdots \int_G \Big|\int_G \frac{f}{\|\mu\|}(xtat_1\ldots at_n y)d\nu(t) \\ &\quad - \frac{g(x)}{\|\mu\|}\frac{f}{\|\mu\|}(at_1at_2\ldots at_n y)\Big|d|\nu|(t_1)\ldots d|\nu|(t_n) \leq \left|\frac{g(a)}{\|\mu\|}\right|^{-n}\frac{\varepsilon(x)}{\|\mu\|^2}. \end{aligned}$$

This shows that

$$\Big|\int_G G_n(xty)d\nu(t) - \frac{g(x)}{\|\mu\|}G_n(y)\Big| \longrightarrow 0,$$

when n goes to $+\infty$ and we obtain

$$\int_G G_\mu(xty)d\nu(t) = \frac{g(x)}{\|\mu\|}G_\mu(y), \tag{2.23}$$

for all $x \in G$. Which proves that $\mathcal{F}$ is a solution of equation (2.18). This establishes the theorem. □

Remark 2.2. Similarly to Theorem 2.2, we can easily proved the Hyers–Ulam stability of the functional inequality

$$\Big|\int_G f(xty)d\mu(t) - f(x)g(y)\Big| \leq \varepsilon(y), \quad x, y \in G,$$

under the condition that there exists $a \in G$ such that

$$\int_G\int_G f(xtasy)d\mu(t)d\mu(s) = \int_G\int_G f(xtysa)d\mu(t)d\mu(s), \quad x, y \in G$$

and $|g(a)| > \|\mu\|$.

3. Hyers–Ulam stability of the equation: $\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y),\ x, y \in G$

Throughout this section $f : G \longrightarrow \mathbb{C}$ is assumed to be a continuous function which satisfies the condition $\mathrm{K}(\mu)$ and μ is a σ-invariant measure on G.

Theorem 3.1. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be a continuous function. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions such that*

$$\Big|\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)\Big| \leq \varepsilon(y), \tag{3.1}$$

for all $x, y \in G$.

If f is unbounded, then g is a solution of the generalized d'Alembert functional equation

$$\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y), \tag{3.2}$$

for all $x, y \in G$.

Proof. Let f, g satisfies (3.1), then for all $x, y, z \in G$,

$$\begin{aligned}
&|2f(z)|\Big|\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) - 2g(x)g(y)\Big| \\
&\quad\le \Big|\int_G\int_G f(zsxty)d\mu(s)d\mu(t) + \int_G\int_G f(zs\sigma(y)t\sigma(x))d\mu(t)d\mu(s) \\
&\quad - 2f(z)\int_G g(xty)d\mu(t)\Big| + \Big|\int_G\int_G f(zsxt\sigma(y))d\mu(s)d\mu(t) \\
&\quad + \int_G\int_G f(zsyt\sigma(x))d\mu(t)d\mu(s) - 2f(z)\int_G g(xt\sigma(y))d\mu(t)\Big| \\
&\quad + \Big|\int_G\int_G f(zsxty)d\mu(s)d\mu(t) + \int_G\int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) \\
&\quad - 2g(y)\int_G f(ztx)d\mu(t)\Big| + \Big|\int_G\int_G f(zsyt\sigma(x))d\mu(s)d\mu(t) \\
&\quad + \int_G\int_G f(zs\sigma(y)t\sigma(x))d\mu(t)d\mu(s) - 2g(y)\int_G f(zt\sigma(x))d\mu(t)\Big| \\
&\quad + 2|g(y)|\Big|\int_G f(zsx)d\mu(s) + \int_G f(zs\sigma(x))d\mu(s) - 2f(z)g(x)\Big|.
\end{aligned}$$

In view of the condition $\mathrm{K}(\mu)$, we get

$$\begin{aligned}
&|2f(z)|\Big|\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) - 2g(x)g(y)\Big| \\
&\le 2\varepsilon(y)\|\mu\| + 2|g(y)|\varepsilon(x) + \int_G \varepsilon(xty)d|\mu|(t) + \int_G \varepsilon(xt\sigma(y))d|\mu|(t),
\end{aligned} \tag{3.3}$$

for all $x, y \in G$. Since f is unbounded then we obtain that g is a solution of equation (3.2). This ends the proof. □

The statement (ii) of the following corollary is proved in [6].

Corollary 3.1. *Let $\alpha \in \mathbb{C}^*$ and assume that a continuous function $f : G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$\Big| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2\alpha f(x)f(y) \Big| \le \varepsilon(y), \tag{3.4}$$

for all $x, y \in G$. Then

(i) *f is either bounded, or αf is a solution of equation (3.2).*

(ii) *If $\varepsilon(y) = \delta$, then either $|f(x)| \le \frac{\|\mu\|+\sqrt{\|\mu\|^2+2\delta|\alpha|}}{2|\alpha|}$, $x \in G$, or αf is a solution of (3.2).*

Theorem 3.2. *Let $\varepsilon : G \longrightarrow \mathbb{R}^+$ be a continuous function. Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions which satisfies the inequality*

$$\Big| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \Big| \le \varepsilon(y), \tag{3.5}$$

for all $x, y \in G$. Suppose furthermore that there exists $a \in G$ such that $|g(a)| > \|\mu\|$. Then there exists exactly one solution $\mathcal{F} \in C(G)$ of

$$\int_G \mathcal{F}(xty)d\mu(t) + \int_G \mathcal{F}(xt\sigma(y))d\mu(t) = 2\mathcal{F}(x)g(y) \tag{3.6}$$

such that $\mathcal{F} - f$ is bounded. $\mathcal{F}$ satisfies

$$|\mathcal{F}(x) - f(x)| \le \frac{\varepsilon(a)}{2(|g(a)| - \|\mu\|)}, \tag{3.7}$$

for all $x, y \in G$.

Proof. With $F(x) = \frac{f(x)}{\|\mu\|}$, $\nu = \frac{\mu}{\|\mu\|}$ and by putting a in place of y, inequality (3.5) yields

$$\Big| \int_G F(xta)d\nu(t) + \int_G F(xt\sigma(a))d\nu(t) - \beta F(x) \Big| \le \frac{\varepsilon(a)}{\|\mu\|^2}, \tag{3.8}$$

where $\beta = \frac{2g(a)}{\|\mu\|}$.

Now for any $x \in G$ and every $n \in \mathbb{N}$, we define by induction the following sequence

$$G_1(x) = \int_G F(xta)d\nu(t) + \int_G F(xt\sigma(a))d\nu(t),$$

and

$$G_{n+1}(x) = \int_G G_n(xta)d\nu(t) + \int_G G_n(xt\sigma(a))d\nu(t), \quad \text{for } n \geq 1. \tag{3.9}$$

In order to prove the convergence of the function sequence $\beta^{-n}G_n(x)$, we need to show by induction the following inequalities:

$$|G_{n+1}(x) - \beta G_n(x)| \leq \frac{2^n \varepsilon(a)}{\|\mu\|^2}, \tag{3.10}$$

$$|G_n(x) - \beta^n F(x)| \leq \frac{\varepsilon(a)}{\|\mu\|^2}(2^{n-1} + 2^{n-2}|\beta| + \cdots + |\beta|^{n-1}), \ n \geq 1 \tag{3.11}$$

and

$$|\beta^{-(n+1)}G_{n+1}(x) - \beta^{-n}G_n(x)| \leq \frac{\varepsilon(a)}{\|\mu\|^2|\beta|}(\frac{2}{|\beta|})^n. \tag{3.12}$$

In view of (3.8) and (3.9), we have

$$\begin{aligned}
&|G_2(x) - \beta G_1(x)| \\
&= \Big| \int_G \int_G F(xtasa)d\nu(t)d\nu(s) + \int_G \int_G F(xtas\sigma(a))d\nu(t)d\nu(s) \\
&+ \int_G \int_G F(xt\sigma(a)sa)d\nu(t)d\nu(s) + \int_G \int_G F(xt\sigma(a)s\sigma(a))d\nu(t)d\nu(s) \\
&- \beta \int_G F(xta)d\nu(t) - \beta \int_G F(xt\sigma(a))d\nu(t)\Big| \leq \Big| \int_G \int_G F(xtasa)d\nu(t)d\nu(s) \\
&+ \int_G \int_G F(xtas\sigma(a))d\nu(t)d\nu(s) - \beta \int_G F(xta)d\nu(t)\Big| \\
&+ \Big| \int_G \int_G F(xt\sigma(a)sa)d\nu(t)d\nu(s) + \int_G \int_G F(xt\sigma(a)s\sigma(a))d\nu(t)d\nu(s) \\
&- \beta \int_G F(xt\sigma(a))d\nu(t)\Big| \leq 2\frac{\varepsilon(a)}{\|\mu\|^2}, \quad \text{since } \|\nu\| \leq 1.
\end{aligned}$$

Now assume that (3.10) is true for same $n \geq 1$. For the case $n+1$,

$$|G_{n+2}(x) - \beta G_{n+1}(x)| = \Big| \int_G G_{n+1}(xta)d\nu(t) + \int_G G_{n+1}(xt\sigma(a))d\nu(t)$$

$$-\beta\int_G G_n(xta)d\nu(t)-\beta\int_G G_n(xt\sigma(a))d\nu(t)\Big|\le 2\frac{2^n\varepsilon(a)}{\|\mu\|^2}=\frac{2^{n+1}\varepsilon(a)}{\|\mu\|^2}.$$

This proves the inequality (3.10).

In view of (3.8) the inequality (3.11) is trivial for $n=1$. Now assume that (3.11) holds for some $n\ge 1$. For the case $n+1$,

$$\begin{aligned}|G_{n+1}(x)-\beta^{n+1}F(x)|&\le|G_{n+1}(x)-\beta G_n(x)|+|\beta||G_n(x)-\beta^nF(x)|\\&\le 2^n\frac{\varepsilon(a)}{\|\mu\|^2}+|\beta|\frac{\varepsilon(a)}{\|\mu\|^2}(2^{n-1}+2^{n-2}|\beta|+\cdots+|\beta|^{n-1})\\&=\frac{\varepsilon(a)}{\|\mu\|^2}(2^n+2^{n-1}|\beta|+2^{n-2}|\beta|^2+\cdots+|\beta|^n).\end{aligned}$$

Which proves (3.11).

The left hand side of the inequality (3.12) can be written

$$|\beta^{-(n+1)}G_{n+1}(x)-\beta^{-n}G_n(x)|=|\beta|^{-(n+1)}|G_{n+1}(x)-\beta G_n(x)|,$$

consequently from (3.10), we obtain (3.12).

Now by using (3.12), we deduce the convergence of the sequence $\beta^{-n}G_n(x)$ and we can define a new continuous mapping

$$G_\mu(x)=\lim_{n\to+\infty}\beta^{-n}G_n(x),\quad x\in G. \tag{3.13}$$

By definition

$$\beta^{-(n+1)}G_{n+1}(x)=\beta^{-1}\int_G\beta^{-n}G_n(xta)d\nu(t)+\beta^{-1}\int_G\beta^{-n}G_n(xt\sigma(a))d\nu(t),$$

which implies that

$$\beta G_\mu(x)=\int_G G_\mu(xta)d\nu(t)+\int_G G_\mu(xt\sigma(a))d\nu(t),\quad x\in G. \tag{3.14}$$

In view of (3.11), we get

$$|G_\mu(x)-F(x)|\le\frac{\varepsilon(a)}{2\|\mu\|(|g(a)|-\|\mu\|)},\quad x\in G. \tag{3.15}$$

Now we are going to show that G_μ is the unique continuous solution of equation (3.14) and inequality (3.15). Let H_μ be another continuous mapping which satisfies (3.14) as well as (3.15). The proof follows from the inequality

$$|G_\mu(x) - H_\mu(x)| \leq \left(\frac{\|\mu\|}{|g(a)|}\right)^n \frac{\varepsilon(a)}{\|\mu\|} \frac{1}{(|g(a)| - \|\mu\|)}, \tag{3.16}$$

which can be easily proved by induction on $n = 0, 1, \ldots$

To conclude this proof, we show that $G_\mu(x)$ satisfies the equation

$$\int_G G_\mu(xty)d\nu(t) + \int_G G_\mu(xt\sigma(y))d\nu(t) = 2G_\mu(x)\frac{g(y)}{\|\mu\|}, \quad x, y \in G. \tag{3.17}$$

Thus we need to show by induction the inequality

$$\Big|\int_G \beta^{-n} G_n(xty)d\nu(t) + \int_G \beta^{-n} G_n(xt\sigma(y))d\nu(t) - 2\beta^{-n} G_n(x)\frac{g(y)}{\|\mu\|}\Big| \leq \frac{\varepsilon(y)}{\|\mu\|^2}\left(\frac{\|\mu\|}{|g(a)|}\right)^n, \quad x, y \in G,\ n \in \mathbb{N}^*. \tag{3.18}$$

For $n = 1$,

$$\begin{aligned}
&\frac{1}{|\beta|}\Big|\int_G G_1(xty)d\nu(t) + \int_G G_1(xt\sigma(y))d\nu(t) - 2G_1(x)\frac{g(y)}{\|\mu\|}\Big| \\
&= \frac{1}{|\beta|}\Big|\int_G\int_G F(xtysa)d\nu(t)d\nu(s) + \int_G\int_G F(xtys\sigma(a)))d\nu(t)d\nu(s) \\
&+ \int_G\int_G F(xt\sigma(y)sa)d\nu(t)d\nu(s) + \int_G\int_G F(xt\sigma(y)s\sigma(a))d\nu(t)d\nu(s) \\
&- 2\int_G F(xsa)d\nu(s)\frac{g(y)}{\|\mu\|} - 2\int_G F(xs\sigma(a))d\nu(s)\frac{g(y)}{\|\mu\|}\Big| \\
&\leq \frac{1}{|\beta|}\Big|\int_G\int_G F(xsaty)d\nu(t)d\nu(s) + \int_G\int_G F(xsat\sigma(y))d\nu(t)d\nu(s) \\
&- 2\int_G F(xsa)d\nu(s)\frac{g(y)}{\|\mu\|}\Big| \\
&+ \Big|\int_G\int_G F(xs\sigma(a)ty)d\nu(t)d\nu(s) + \int_G\int_G F(xs\sigma(a)t\sigma(y))d\nu(t)d\nu(s) \\
&- 2\int_G F(xs\sigma(a))d\nu(s)\frac{g(y)}{\|\mu\|}\Big|,
\end{aligned}$$

since F satisfies $\mathrm{K}(\nu)$.

In view of (3.5) and $\|\nu\| \leq 1$, we obtain

$$\Big|\int_G \beta^{-1}G_1(xty)d\nu(t) + \int_G \beta^{-1}G_1(xt\sigma(y))d\nu(t) - 2\beta^{-1}G_1(x)\frac{g(y)}{\|\mu\|}\Big|$$

$$\leq \frac{1}{|\beta|}2\frac{\varepsilon(y)}{\|\mu\|^2} = \frac{\|\mu\|}{|g(a)|}\frac{\varepsilon(y)}{\|\mu\|^2}.$$

Which proves (3.18) for $n = 1$.

Now assume that (3.18) is true for some $n \geq 1$. For the case $n + 1$,

$$\Big|\int_G \beta^{-(n+1)}G_{n+1}(xty)d\nu(t) + \int_G \beta^{-(n+1)}G_{n+1}(xt\sigma(y))d\nu(t)$$

$$- 2\beta^{-(n+1)}G_{n+1}(x)\frac{g(y)}{\|\mu\|}\Big| = \frac{1}{|\beta|}\Big|\int_G\int_G \beta^{-n}G_n(xtysa)d\nu(t)d\nu(s)$$

$$+\int_G\int_G \beta^{-n}G_n(xtys\sigma(a))d\nu(t)d\nu(s) + \int_G\int_G \beta^{-n}G_n(xt\sigma(y)sa)d\nu(t)d\nu(s)$$

$$+ \int_G\int_G \beta^{-n}G_n(xt\sigma(y)s\sigma(a))d\nu(t)d\nu(s)$$

$$- 2\int_G \beta^{-n}G_n(xta)d\nu(t)\frac{g(y)}{\|\mu\|} - 2\int_G \beta^{-n}G_n(xt\sigma(y))d\nu(t)\frac{g(y)}{\|\mu\|}\Big|$$

$$\leq \frac{1}{|\beta|}2\left(\frac{\|\mu\|}{|g(a)|}\right)^n\frac{\varepsilon(y)}{\|\mu\|^2} = \left(\frac{\|\mu\|}{|g(a)|}\right)^{n+1}\frac{\varepsilon(y)}{\|\mu\|^2}.$$

Which completes the proof of (3.18).

Consequently $\mathcal{F}(x) = \|\mu\|G_\mu(x)$, $x \in G$ is the unique continuous function, which satisfies (3.6) and (3.7). This ends the proof. □

Remark 3.1. In Theorem 3.2. we can replace the condition that f satisfies the condition $\mathrm{K}(\mu)$ by the weaker condition that there exists $a \in G$ such that

$$|g(a)| > \|\mu\|$$

and

$$\int_G\int_G f(xsaty)d\mu(t)d\mu(s) = \int_G\int_G f(xsyta)d\mu(t)d\mu(s),$$

$$\int_G \int_G f(xs\sigma(a)ty)d\mu(t)d\mu(s) = \int_G \int_G f(xsyt\sigma(a))d\mu(t)d\mu(s),$$

for all $x, y \in G$.

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(Received March 4, 2003; revised June 10, 2004)