Publ. Math. Debrecen
66/3-4 (2005), 313–326

Polynomials and divided differences

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Abstract. Starting with a sequence of recursively defined divided difference operators that differentiate polynomials, we define and then solve two sequences of functional equations whose n-th terms respectively characterize polynomials of degree at most $2n$ and generalized polynomials of degree at most $2n-1$.

1. Introduction

In 1963, J. Aczél [1], see also [2], showed that there is a simple functional equation involving two unknown functions, say f and g, whose general solution (no regularity conditions whatever) is: f is a polynomial of degree at most 2 and g is the derivative of f. In this paper, we extend Aczél's result by showing that there is a sequence $\{E_n\}$ of functional equations, each involving functions f and g, such that for any $n \geq 1$, the general solution of E_n, again without any regularity conditions whatever, is: f is a polynomial of degree at most $2n$ and g is the derivative of f.

We do this by first constructing, in Section 2, a sequence of linear difference operators that differentiate polynomials of successively higher degrees. The expression of the action of these operators leads, in Section 3, to the equations E_n and to related equations E'_n. These equations are

Mathematics Subject Classification: 39B22, 39B52.
Key words and phrases: functional equations, divided differences, polynomials, generalized polynomials.

solved in Section 4 using a result of L. Székelyhidi in [18] and an extension of that result developed by M. Sablik in [13].

The general solution of E'_n, unlike that for E_n, consists of *generalized polynomials* (of degree $\leq 2n-1$) rather than polynomials alone, and so includes highly discontinuous functions. The difference between the two situations already appears for $n=1$: E_1 is Aczél's equation in [2], while E'_1 is Jensen's equation (cf. [8, Chapter 13]). The complete regularity of solutions of E_n, apparently coming "out of nothing", is briefly discussed in Section 3, where it is shown that the regularity is in effect built into the very form of the equation.

A preliminary announcement of the results of this paper appears in [17].

We note that results related to those in this paper, in the sense of connecting polynomial and similar functions with divided differences, appear *inter alia* in papers by Aczél and Kuczma [3], Andersen [4], Davies and Rousseau [5], Deeba and Simeonov [6], Haruki [7], Sablik [12, 13] and Schwaiger [16]. In [10] and [11], Riedel and Sablik characterize polynomial functions by functional equations derived from Flett's mean value theorem. Further examples and references may be found in the book by Sahoo and Riedel [15].

2. The difference operators

We begin by recursively defining a sequence of linear operators on the set of functions from $\mathbb{R}$ to $\mathbb{R}$, as follows: For fixed $c>0$ and any integer n, let:

$$(\delta_c^{(1)} f)(x) = \frac{f(x+c)-f(x-c)}{2c},$$

$$(\delta_c^{(n+1)} f) = \frac{4^n}{4^n-1}(\delta_c^{(n)} f) - \frac{1}{4^n-1}(\delta_{2c}^{(n)} f).$$

An easy induction then yields:

Lemma 1. *For any $n \geq 1$, the operator $\delta_c^{(n)}$ can be expressed in terms of the operators of the form $\delta_c^{(1)}$, as follows:*

$$\delta_c^{(n)} = \sum_{k=1}^{n} a_k^{(n)} \delta_{2^{k-1}c}^{(1)}, \tag{1}$$

where the coefficients $a_k^{(n)}$ are recursively given by the conditions:

$$\begin{aligned} a_0^{(n)} &= 0 = a_{n+1}^{(n)} \quad \text{for all } n \geq 1,\ a_1^{(1)} = 1, \qquad \text{and} \\ a_k^{(n+1)} &= \frac{4^n}{4^n - 1} a_k^{(n)} - \frac{1}{4^n - 1} a_{k-1}^{(n)} \quad \text{for } n \geq 1,\ k = 1, 2, \dots, n+1. \end{aligned} \tag{2}$$

Another induction establishes the following lemma, which gives us an explicit form for the coefficients $a_k^{(n)}$:

Lemma 2. *For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$a_k^{(n)} = \frac{b_{n,k}}{d_n},$$

where d_n and $b_{n,k}$ are the integers recursively given by the conditions:

$$\begin{aligned} & d_1 = 1, \quad \text{and} \quad d_{n+1} = (4^n - 1) d_n && \text{for } n \geq 1, \\ & b_{n,0} = b_{n,n+1} = 0 && \text{for all } n \geq 1,\ b_{1,1} = 1, \qquad \text{and} \\ & b_{n+1,k} = 4^n b_{n,k} - b_{n,k-1} && \text{for } n \geq 1,\ 1 \leq k \leq n-1. \end{aligned}$$

We therefore have:

$$d_n = \prod_{m=1}^{n-1} (4^m - 1) \quad \text{for all } n \geq 2, \qquad \text{and}$$

$$b_{n,k} = (-1)^{k-1} 2^{(n-k)(n-k+1)} \prod_{m=1}^{k-1} \frac{4^{n-m} - 1}{4^m - 1} \quad \text{for all } n \geq 1,\ 1 \leq k \leq n.$$

The following table, brief as it is, is enough to illustrate the very rapid growth of the d_n and $b_{n,k}$.

		$b_{n,k}$					
d_n	n	$k=1$	2	3	4	5	6
1	1	1					
3	2	4	-1				
45	3	64	-20	1			
2835	4	4096	-1344	84	-1		
722925	5	1048576	-348160	22848	-340	1	
739552275	6	1073741824	-357564416	23744512	-371008	1364	-1

It follows that the numbers $a_k^{(n)}$ can be expressed in the form

$$a_k^{(n)} = (-1)^{k-1} \prod_{m=1}^{n-k} \left(\frac{4^m}{4^m - 1} \right) \prod_{m=1}^{k-1} \frac{1}{4^m - 1},$$

whence, since $\prod_{m=1}^{\infty} \left(\frac{4^m}{4^m-1} \right)$ converges, the $a_k^{(n)}$ remain bounded.

The latter recursion (2) leads to two properties for sums of $a_k^{(n)}$ that we will need:

Lemma 3. *For $n \geq 1$, $0 \leq i \leq n-1$, let $S_{n,i} = \sum_{k=1}^{n} 4^{(k-1)i} a_k^{(n)}$. Then for all n, we have $S_{n,0} = 1$, while $S_{n,i} = 0$ for any $i > 0$.*

Proof. We consider two cases and proceed by induction:
Case 1: $i = 0$.

Then for $n = 1$, we have:

$$S_{1,0} = 4^0 a_1^{(1)} = 1,$$

and for $n \geq 1$, we have:

$$S_{n+1,0} = \sum_{k=1}^{n+1} 4^{0(k-1)} a_k^{(n+1)} = \sum_{k=1}^{n+1} a_k^{(n+1)}$$

$$= \sum_{k=1}^{n+1} \left(\frac{4^n}{4^n - 1} a_k^{(n)} - \frac{1}{4^n - 1} a_{k-1}^{(n)} \right)$$

$$= \sum_{k=1}^{n+1} \frac{4^n}{4^n - 1} a_k^{(n)} - \sum_{k=1}^{n+1} \frac{1}{4^n - 1} a_{k-1}^{(n)}$$

$$= \frac{4^n}{4^n - 1} \sum_{k=1}^{n} a_k^{(n)} - \frac{1}{4^n - 1} \sum_{j=1}^{n+1} a_{j-1}^{(n)}$$

$$= \frac{4^n}{4^n - 1} \sum_{k=1}^{n} a_k^{(n)} - \frac{1}{4^n - 1} \sum_{k=1}^{n} a_k^{(n)}$$

$$= \frac{4^n - 1}{4^n - 1} \sum_{k=1}^{n} a_k^{(n)} = \frac{4^n - 1}{4^n - 1} = 1.$$

Case 2: Let $i \geq 1$,then we have $n \geq 2$ and for $i = n-1$, we obtain:

$$\begin{aligned}
S_{n,n-1} &= \sum_{k=1}^{n} 4^{(n-1)(k-1)} a_k^{(n)} \\
&= \sum_{k=1}^{n} 4^{(n-1)(k-1)} \left(\frac{4^{n-1}}{4^{n-1} - 1} a_k^{(n-1)} - \frac{1}{4^{n-1} - 1} a_{k-1}^{(n-1)} \right) \\
&= \frac{4^{n-1}}{4^{n-1} - 1} \sum_{k=1}^{n} 4^{(n-1)(k-1)} a_k^{(n-1)} - \frac{1}{4^{n-1} - 1} \sum_{k=1}^{n} 4^{(n-1)(k-1)} a_{k-1}^{(n-1)} \\
&= \frac{4^{n-1}}{4^{n-1} - 1} \sum_{k=1}^{n} 4^{(n-1)(k-1)} a_k^{(n-1)} - \frac{1}{4^{n-1} - 1} \sum_{k=0}^{n-1} 4^{(n-1)k} a_k^{(n-1)} \\
&= \frac{4^{n-1}}{4^{n-1} - 1} \sum_{k=1}^{n} 4^{(n-1)(k-1)} a_k^{(n-1)} - \frac{4^{n-1}}{4^{n-1} - 1} \sum_{k=0}^{n-1} 4^{(n-1)(k-1)} a_k^{(n-1)} = 0.
\end{aligned}$$

For $i \leq n-1$, we obtain:

$$S_{n+1,i} = \sum_{k=1}^{n+1} 4^{i(k-1)} a_k^{(n+1)} = \sum_{k=1}^{n+1} 4^{i(k-1)} \left(\frac{4^n}{4^n - 1} a_k^{(n)} - \frac{1}{4^n - 1} a_{k-1}^{(n)} \right)$$

$$=\frac{4^n}{4^n-1}\sum_{k=1}^{n}4^{i(k-1)}a_k^{(n)}-\frac{1}{4^n-1}\sum_{k=1}^{n+1}a_{k-1}^{(n)}$$

$$=\frac{4^n}{4^n-1}S_{n,i}-\frac{1}{4^n-1}\sum_{j=1}^{n}4^{ij}a_j^{(n)}$$

$$=-\frac{4^i}{4^n-1}\sum_{j=1}^{n}4^{i(j-1)}a_j^{(n)}=-\frac{4^i}{4^n-1}S_{n,i}=0. \qquad \square$$

A simple calculation shows that $\delta_c^{(1)}$ applied to the affine function $f(x)=ax+b$ yields a, and applied to a polynomial of degree less than or equal to 2 yields its derivative. Similarly $\delta_c^{(2)}$ is easily seen to differentiate polynomials of degree less than or equal to 4. Further investigation leads to:

Theorem 4. *For each $n\geq 1$ and any $c>0$, the operator $\delta_c^{(n)}$ differentiates all polynomials of degree less than or equal to $2n$.*

Proof. First we note that since $\delta_c^{(n)}$ is linear, it suffices to study its behavior on monomials. In view of equation (1), it is enough to consider $\delta_{2^{k-1}y}^{(1)}$, and for simplicity of notation it helps to separate the cases of odd and even powers. We consider the case of even powers, $f(x)=x^{2j}$.

$$\left(\delta_{2^{k-1}y}^{(1)}x^{2j}\right)=\frac{(x+2^{k-1}y)^{2j}-(x-2^{k-1}y)^{2j}}{2\cdot 2^{k-1}y}. \tag{3}$$

Using the binomial theorem we obtain after some simplification,

$$\left(\delta_{2^{k-1}y}^{(1)}(x^{2j})\right)=\sum_{l=1}^{j}4^{(k-1)(l-1)}\binom{2j}{2l-1}x^{2j-(2l-1)}y^{2l-2}. \tag{4}$$

Now, using (4) in (1) we obtain

$$\left(\delta_y^{(n)}(x^{2j})\right)=\sum_{l=1}^{j}\binom{2j}{2l-1}x^{2j-(2l-1)}y^{2l-2}\sum_{k=1}^{n}4^{(k-1)(l-1)}a_k^{(n)}. \tag{5}$$

Finally, by Lemma 3, the right-hand side of (4) vanishes for $l = 2, \dots, j$ as long as $j \le n$. This implies that the terms in (4) which involve y all disappear. The only term left is $2^j x^{2j-1}$, which occurs when $l = 1$. A similar argument works in the case of odd powers of x. It is also easily seen that if we apply $\delta_c^{(n)}$ to a polynomial of degree higher than $2n$, terms containing c remain and thus $\delta_c^{(n)}$ does not differentiate polynomials of degree greater than $2n$. □

3. The functional equations

Note that Theorem 4 can be restated as follows: If f is a polynomial of degree less than or equal to $2n$, then

$$\delta_c^{(n)}(f)(x) = f'(x). \tag{6}$$

Thus the converse of Theorem 4 becomes the question: If f satisfies the functional differential equation (6), is f necessarily a polynomial of degree less than or equal to $2n$?

More generally, suppose f, g are two functions satisfying

$$\delta_c^{(n)}(f)(x) = g(x). \tag{7}$$

for all x and all $c \neq 0$. Multiplying (7) by $2^n c$, and using equation (1), (7) becomes:

$$2^n c g(x) = \sum_{k=1}^{n} a_k^{(n)} 2^{n-k} (f(x + 2^{k-1}c) - f(x - 2^{k-1}c)). \qquad E_n$$

So if f, g satisfy E_n for all x, c, does f have to be a polynomial of degree less than or equal to $2n$ and g its derivative?

Note that for $n = 1$ E_n reduces to

$$2cg(x) = f(x + c) + f(x - c).$$

In [2], J. Aczél showed that this equation characterizes quadratic polynomials and their derivatives, without assuming any regularity conditions.

Actually, as the second author noted in [14], the effect of getting high regularity of solutions out of nothing becomes a little bit less mysterious when we look more carefully at the left-hand side of the above equation. For a fixed x the number $2cg(x)$ may be considered as a value of the real linear mapping $c \to 2g(x)c$. This observation is consistent with the origin of the equation, since g replaces the derivative of f, and multiplication actually is the action of the corresponding differential on the increment. Now, suppose that $2g(x)$ is an endomorphism of $\mathbb{R}$ for every x, and replace multiplication on the left-hand side by $2g(x)(c)$. Then (cf. [12]) the solution will be a pair of generalized polynomial functions of order 2 and 1, not necessarily continuous. From this point of view, high regularity of solutions is due to the choice of linear (and hence highly regular) functions among all possible homomorphisms.

We arrive at another sequence of functional equations by multiplying equation (6) by $2c$ and then differentiating with respect to c to get:

$$2f'(x) = \sum_{k=1}^{n} a_k^{(n)} \left(f'(x + 2^{k-1}c) + f'(x - 2^{k-1}c) \right).$$

Taking $g = f'$ with a polynomial f of degree less than or equal to $2n$ and using Theorem 4 shows that any polynomial g of degree $\leq 2n - 1$ satisfies

$$2g(x) = \sum_{k=1}^{n} a_k^{(n)} \left(g(x + 2^{k-1}c) + g(x - 2^{k-1}c) \right), \qquad E'_n$$

and the question is whether such polynomials are the only solutions of the functional equations E'_n.

For $n = 1$ equation E'_n reduces to

$$2g(x) = g(x + c) + g(x - c)$$

which is Jensen's equation and thus has non-polynomial as well as (affine) polynomial solutions. As will be seen, this conclusion extends to the solutions of E'_n for all $n \geq 1$.

4. The solutions

It turns out that equation E_n' is easier to solve, so we will treat it first. For its solution we need a lemma due to L. Székelyhidi (cf. [18], Theorem 9.5).

Lemma 5. *Let G, S be commutative groups, n a nonnegative integer and let S be uniquely divisible by $n!$. Further, let φ_i, ψ_i be additive functions from G into G with the property that* $\mathrm{Ran}(\varphi_i) \subseteq \mathrm{Ran}(\psi_i)$ $(i = 1, \dots, n+1)$, *where* $\mathrm{Ran}(\varphi_i)$ *stands for the range of φ_i. Then if $h, h_i : G \to S$ $(i = 1, \dots, n+1)$ satisfy*

$$h(x) + \sum_{i=1}^{n+1} h_i(\varphi_i(x) + \psi_i(t)) = 0$$

then h is a generalized polynomial of degree at most n.

Using the above lemma and the definition (from Kuczma [5; Chapter 13.4]), that a function from $\mathbb{R}^k$ into $\mathbb{R}$ is k-additive if it is additive in each variable, we now prove:

Theorem 6. *A function g satisfies the functional equation E_n' if and only if g is a generalized polynomial of degree at most $2n-1$, that is*

$$g(x) = \sum_{k=0}^{2n-1} A_k^d(x), \tag{8}$$

where $A_k^d(x)$ is the diagonal of a symmetric k-additive function A_k.

Proof. The 'only if' part follows immediately from Lemma 5. If, on the other hand, g is given by (8), we can again use the linearity of E_n' to consider each term separately. So for $g(x) = A_j^d(x)$, we get

$$2A_j^d(x) = \sum_{k=1}^{n} a_k^{(n)} \sum_{l=0}^{j} \binom{k}{l} \Big[A_j(x^{j-l}, c^l) 2^{(k-1)l} - (-1)^l A_j(x^{j-l}, c^l) 2^{(k-1)l} \Big],$$

where $A_k(x^{k-l}, c^l) = A_k(x, \dots, x, c, \dots, c)$ with $k-l$ occurrences of x and l occurrences of c. Now if j is even, then we have

$$A_{2m}^d(x) = \sum_{k=1}^{n} a_k^{(n)} \sum_{l=0}^{m} \binom{k}{l} \Big[A_{2m}(x^{2m-2l}, c^{2l}) 2^{(k-1)2l} \Big],$$

which can be rearranged to

$$A_{2m}^{d}(x)=\sum_{l=0}^{m}\left(\begin{array}{c}k\\ l\end{array}\right)\left[A_{2m}(x^{2m-2l},c^{2l})\sum_{k=1}^{n}4^{(k-1)l}a_{k}^{(n)}\right],$$

and the desired result follows from Lemma 3. A similar argument works for the case of odd j. □

In order to solve the functional equations E_n, we need a generalization of Lemma 5. Such a generalization was proved by the second author in [13] and appears below as Lemma 7.

To state the lemma we adopt the following notation: G and H are commutative groups, and $SA^i(G;H)$ stands for the group of all i-additive, symmetric mappings from G^i into H, $I\geq 2$, while $SA^0(G;H)$ denotes the family of constant functions from G into H and $SA^1(G;H)=\mathrm{Hom}(G;H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G;G)\times\mathrm{Hom}(G;G)$ containing all the pairs (α,β) for which $\mathrm{Ran}(\alpha)\subset\mathrm{Ran}(\beta)$. The symbol $\#S$ stands for cardinality of a set S. We also adopt the convention that a sum over an empty set of indices equals 0.

Lemma 7. *Fix $N\in\mathbb{N}\cup\{0\}$ and let $I_0,\dots,I_N$ be finite subsets of $\mathcal{I}$. Suppose further that H is uniquely divisible by $N!$ and let functions $\varphi_i : G\to SA^i(G;H)$, $i\in\{0,\dots,N\}$ and $\psi_{i,(\alpha,\beta)} : G\to SA^i(G;H)$, $(\alpha,\beta)\in I_i$, $i\in\{0,\dots,N\}$ satisfy*

$$\varphi_N(x)(y^N)+\sum_{i=0}^{N-1}\varphi_i(x)(y^i)=\sum_{i=0}^{N}\sum_{(\alpha,\beta)\in I_i}\psi_{i,(\alpha,\beta)}(\alpha(x)+\beta(y))(y^i)$$

for every $x,y\in G$. Then φ_N is a generalized polynomial of degree at most equal to

$$\sum_{i=0}^{N}\#\left(\bigcup_{s=i}^{N}I_s\right)\leq\sum_{i=0}^{N}(i+1)\#I_i.$$

Remark 8. According to I. Pawlikowska (cf. [9], Lemat 2.2), the upper bound for the degree of φ_N can be lowered by 1 with no change in the original proof.

We will also need the following result:

Lemma 9. *For every $k \in \mathbb{N}$, if $B \in SA^k(\mathbb{R};\mathbb{R})$ satisfies*

$$B(x^{k-1}, y) = yB(x^{k-1}, 1) \tag{9}$$

for every $x, y \in \mathbb{R}$, then B is k-linear, i.e.

$$B(v_1, \ldots, v_k) = v_1 \cdot \cdots \cdot v_k B(1^k)$$

for every $v_1, \ldots, v_k \in \mathbb{R}$, where 1^k is the k-tuple $(1, 1, \ldots, 1)$.

Proof. B_y defined by $B_y(v_1, \ldots, v_{k-1}) := B(v_1, \ldots, v_{k-1}, y)$, obviously is $k-1$-additive and symmetric. Moreover $B_y^d(x) := B(x^{k-1}, y) = yB_1^d(x^{k-1})$ by assumption. Thus by the polarization formula (see, e.g. [18, Lemma 1.4]),

$$\begin{aligned} B_y(v_1, \ldots, v_{k-1}) &= \frac{1}{(k-1)!} \sum_{S \subseteq \{1, \ldots, k-1\}} (-1)^{k-|S|} B_y^d \left(\sum_{l \in S} v_l \right) \\ &= yB_1^d(v_1, \ldots, v_{k-1}). \end{aligned}$$

This means $B(v_1, \ldots, v_{k-1}, y) = yB(v_1, \ldots, v_{k-1}, 1)$ for all $v_1, \ldots, v_{k-1}, y$. By the symmetry of B we get the desired result. □

We can now prove our principal result.

Theorem 10. *The functions f and g satisfy the functional equation E_n if and only if f is a polynomial of degree at most $2n$ and $g = f'$.*

Proof. We first apply Lemma 7 and Remark 8 to equation E_n (with $N = 1$, $\varphi_1 = 2^n g$, $\varphi_0 = 0$, $I_1 = \emptyset$, $I_0 = \{(\mathrm{id}, \pm 2^{k-1}\mathrm{id}) : k \in \{1, \ldots, n\}\}$ and $\psi_{0,(\mathrm{id}, \pm 2^{k-1}\mathrm{id})} = \pm a_k^{(n)} 2^{n-k} f$), and find that g is a generalized polynomial of degree at most $2n-1$. Next we fix x in E_n and apply the difference operator Δ_y^2 to both sides. The left-hand side vanishes and we have, with $\phi_k := \Delta_{2^{k-1}y}^2 f$,

$$\sum_{k=1}^{n} 2^{n-k} a_k^{(n)} \Big(\phi_k(x + 2^{k-1}c) - \phi_k(x - 2^{k-1}c) \Big) = 0.$$

Replacing x by $x+c$ and applying Lemma 5, we find that ϕ_1 is a generalized polynomial of degree at most $2n-2$. Thus f is a generalized polynomial of degree at most $2n$. Thus we have $g(x) = \sum_{l=0}^{2n-1} A_l^d(x)$ and $f(x) = \sum_{l=0}^{2n} B_l^d(x)$, where A_l^d and B_l^d are the diagonals of symmetric l-additive functions. Substituting this back into equation E_n, we get

$$2^n c \sum_{l=0}^{2n-1} A_l^d(x) = \sum_{k=1}^{n} a_k^{(n)} 2^{n-k} \sum_{l=0}^{2n} \left(B_l^d(x + 2^{k-1}c) - B_l^d(x - 2^{k-1}c) \right).$$

Using the abbreviation

$$B_{l,i} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2B_l(x^{l-i}, c^i) & \text{if } i \text{ is odd,} \end{cases}$$

and the addition formula for B_k^d, we obtain

$$2^n c \sum_{l=0}^{2n-1} A_l^d(x) = \sum_{k=1}^{n} a_k^{(n)} 2^{n-k} \sum_{l=0}^{2n} \sum_{i=0}^{l} \binom{l}{i} 2^{i(k-1)} B_{l,i}.$$

Rearranging the sums on the right hand side above yields

$$2^n c \sum_{l=0}^{2n-1} A_l^d(x) = \sum_{l=0}^{2n} \sum_{i=0}^{l} \binom{l}{i} B_{l,i} \sum_{k=1}^{n} a_k^{(n)} 2^{n-k} 2^{i(k-1)}. \tag{10}$$

We have $B_{l,i} = 0$ for even i, and if $i = 2j+1$, then after some simplification we obtain

$$\sum_{k=1}^{n} a_k^{(n)} 2^{n-k} 2^{i(k-1)} = 2^{n-1} S_{n,j}.$$

Using Lemma 3, we infer that the sum is non-zero for $j = 0$, or $i = 1$. This means that (10) becomes

$$c \sum_{l=0}^{2n-1} A_l^d(x) = \sum_{l=1}^{2n} l B_l(x^{l-1}, c).$$

Equating terms of equal degree in x on both sides yields that

$$cA_l^d(x) = (l+1) B_{l+1}(x^l, c), \text{ for all } x, c \in \mathbb{R},$$

implying $B_{l+1}(x^l, c) = cB_{l+1}(x^l, 1)$. Thus in view of Lemma 9, B_l we have $B_l^d(x) = b_l x^l$, for $l = 1 \ldots 2n$. This in turn implies that $A_l(x^l) = (l+1) b_{l+1} x^l$, and thus f is a polynomial of degree at most $2n$ and $g(x) = f'(x)$, and our proof is complete. □

Remark 11. It should be noted that, while the motivation for our equations come from real valued functions, in view of Lemma 5 and Lemma 7, the results will also hold on fields of characteristic zero.

Acknowledgments. The authors would like to thank the referees for their valuable comments and suggestions.

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(Received April 12, 2003; revised February 20, 2004)