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The quadratic functional equation on groups

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Abstract. In this paper, we study the functional equation $f(xy)+f(xy^{-1}) = 2f(x) + 2f(y)$, where f maps a group into an abelian group. Using the relationship between its Cauchy kernel and solutions of Jensen's functional equation, we deduce many basic reduction formulas and relations, and use them to obtain its general solution on free groups. We solve it on more specific groups including free abelian groups and the general linear groups $GL_n(\mathbb{Z})$.

1. Introduction

Let $(G,\cdot)$ be a group, $(H,+)$ an abelian group, and $f : G \to H$ a mapping. Consider the following functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad \forall\, x, y \in G. \tag{Q}$$

Equation (Q) is known as *the quadratic functional equation* [8]. Any mapping f satisfying equation (Q) is called a *quadratic form.* When G is a normed linear space, the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall\, x, y \in G,$$

which gives a basic algebraic condition that makes a normed linear space an inner product space, is of the form equation (Q); see [1], [2], and [5]. For

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equation (Q), the authors of [4] and [9] study some relations between bimorphisms and quadratic forms. Under the assumptions G is a 2-divisible group, H is a field of characteristic different from 2, and $f : G \to H$ satisfies the subsidiary condition, i.e., $f(xyz) = f(xzy)$ for all $x, y, z \in G$, reference [7] gives some functional equations equivalent to equation (Q), and proves that $f(x) = S(x, x)$, $\forall x \in G$, where $S : G \times G \to H$ is a symmetric bimorphism.

To study equation (Q), we first make a simple observation. Let $e \in G$ and $0 \in H$ be the identity elements. Suppose that f is a solution of equation (Q). Letting $x = y = e$ in equation (Q) we get

$$2f(e) = 0.$$

It is easy to check that

$$\tilde{f}(x) = f(x) - f(e)$$

is also a solution of equation (Q), and it satisfies the normalization condition $\tilde{f}(e) = 0$. Therefore, every solution $f : G \to H$ is of the form

$$f = \tilde{f} + c,$$

where $\tilde{f}$ is a solution which is normalized and $c \in H$ is a constant satisfying $2c = 0$. In our paper, without loss of generality, we are concerned about equation (Q) along with the normalization condition, i.e.,

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad \forall\, x, y \in G, \quad \text{with } f(e) = 0. \tag{1}$$

The aim of this paper is to give the general solution of equation (1) on free groups. Since any group is isomorphic to a factor group of a free group, we can obtain the general solution of equation (1) on a large variety of group by means of pullbacks and factorizations. Using this procedure, we illustrate how it leads to solutions on more specific groups including free abelian groups and the general linear groups $GL_n(\mathbb{Z})$. A result due to Kurepa [9, Remark 2] is also extended by removing the 2 torsion-free condition on H.

2. Preparation

To each map $f : G \to H$, as in [10]–[12], define its Cauchy kernel $A : G \times G \to H$

$$A(x, y) := f(xy) - f(x) - f(y), \quad \forall x, y \in G, \tag{2}$$

and B by

$$B(x_1, \dots, x_\ell) := f(x_1 \dots x_\ell) - \sum_{i=1}^{\ell} f(x_i) - \sum_{1 \le i < j \le \ell} A(x_i, x_j), \tag{3}$$

over all sequences $x_1, \dots, x_\ell$ in G and $\ell \ge 1$.

Theorem 2.1. *Suppose that $f : G \to H$ is a solution of equation (1) and that $A : G \times G \to H$ and B are defined by (2) and (3), respectively. Then for all $x, y, z, x_i, u, v \in G$, $\ell \ge 1$, $m_i, n \in \mathbb{Z}$, $i = 1, \dots, \ell$, the following holds.*

$$f(x^{-1}) = f(x), \tag{4}$$

$$f(xy) = f(yx), \tag{5}$$

$$A(x, y) = A(y, x), \tag{6}$$

$$A(e, u) = 0, \tag{7}$$

$$A(xy, u) + A(xy^{-1}, u) = 2A(x, u), \tag{8}$$

$$A(x^{-1}, y) = -A(x, y), \tag{9}$$

$$A(xy^n z, u) = nA(xyz, u) - (n - 1)A(xz, u), \tag{10}$$

$$A(xyz, u) + A(yxz, u) = 2A(xz, u) + 2A(yz, u) - 2A(z, u), \tag{11}$$

$$A(xyz, u) + A(xzy, u) = 2A(xy, u) + 2A(xz, u) - 2A(x, u), \tag{12}$$

$$f(y^n) = n^2 f(y), \tag{13}$$

$$f(xy^n z) = nf(xyz) - (n - 1)f(xz) + n(n - 1)f(y), \tag{14}$$

$$A(x, x) = 2f(x), \tag{15}$$

$$2B(\cdot,u,v),\ 2B(u,\cdot,v) \text{ and } 2B(u,v,\cdot) : G \to H \text{ are homomorphisms,} \tag{16}$$

$$f(xuvy) = f(xvuy) + 2B(u,v,yx), \tag{17}$$

$$B(x_1) = B(x_1,x_2) = 0, \tag{18}$$

$$B(e,x,y) = B(x,e,y) = B(x,y,e) = 0, \tag{19}$$

$$B(x,y,y) = B(y,x,y) = B(y,y,x) = 0, \tag{20}$$

$$B(x^{-1},y,z) = B(x,y^{-1},z) = B(x,y,z^{-1}) = -B(x,y,z), \tag{21}$$

$$\begin{cases} B(x,y,z) + B(x,z,y) = 0, \\ B(x,y,z) + B(z,y,x) = 0, \\ B(x,y,z) + B(y,x,z) = 0, \end{cases} \tag{22}$$

$$B(x_1,\dots,x_{\ell+1}) = B(x_1,\dots,x_\ell) + A(x_1\dots x_\ell, x_{\ell+1}) - \sum_{i=1}^{\ell} A(x_i, x_{\ell+1}), \tag{23}$$

where π is the natural homomorphism from G onto G/G^2 and the quotient group which is abelian and is of exponent at most 2.

Proof. Let $x = e$ in equation (1). Noting that $f(e) = 0$, we have (4). Switching x and y, we get

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) = f(yx) + f(yx^{-1}).$$

By (4), $f(xy^{-1}) = f(yx^{-1})$ and so we get (5). Relation (5) implies (6):

$$A(x,y) \overset{(2)}{=} f(xy) - f(x) - f(y) \overset{(5)}{=} f(yx) - f(x) - f(y) = A(y,x);$$

i.e., $A(x,y)$ is symmetric.

Clearly,

$$A(e,u) = f(eu) - f(u) - f(e) = 0$$

and we get (7). We get (8) by the following direct computation

$$\begin{aligned} &A(xy,u) + A(xy^{-1},u) \\ &\quad = (f(xyu) - f(xy) - f(u)) + (f(xy^{-1}u) - f(xy^{-1}) - f(u)) \end{aligned}$$

$$
\begin{aligned}
&= (f(xyu) + f(xy^{-1}u)) - (f(xy) + f(xy^{-1})) - 2f(u) \\
&= (f(uxy) + f(uxy^{-1}) - 2f(x) - 2f(y) - 2f(u) \text{ (by (5))} \\
&= 2f(ux) + 2f(y) - 2f(x) - 2f(y) - 2f(u) \\
&= 2f(xu) - 2f(x) - 2f(u) \text{ (by (5))} \\
&= 2A(x,u).
\end{aligned}
$$

In the following, for convenience, denote by $S(G,H)$ the set of all solutions to Jensen's functional equation

$$
F(xy) + F(xy^{-1}) = 2F(x), \quad \forall x, y \in G, \tag{24}
$$

with the normalization condition $F(e) = 0$. Clearly, $S(G,H)$ is an abelian group. Relations (7) and (8) show that $A(\cdot, u)$ is in $S(G,H)$ for any given $u \in G$. So we can use the results in [10] and [11] and get (9)–(12).

Clearly, relation (13) is trivial for $n = 0, 1$. Suppose that it is true for $n \le k$ $(k \ge 1)$. Then

$$
\begin{aligned}
f(y^{k+1}) &= -f(y^{k-1}) + 2f(y^k) + 2f(y) \\
&= (-(k-1)^2 + 2k^2 + 2)f(y) = (k+1)^2 f(y).
\end{aligned}
$$

By induction, this proves (13) is true for any non-negative integer n. By (4), it is true for all $n \in \mathbb{Z}$. Since

$$
\begin{aligned}
f(xy^n z) &= f(zxy^n) \text{ (by (5))} \\
&= f(zx) + f(y^n) + A(zx, y^n) \\
&= f(zx) + n^2 f(y) + nA(zx, y) \quad \text{(by (13) \& (10))} \\
&= nf(xyz) - (n-1)f(xz) + n(n-1)f(y) \quad \text{(by (5))},
\end{aligned}
$$

we have proved (14).

We use (13) to get

$$
A(x,x) = f(x^2) - 2f(x) = 2^2 f(x) - 2f(x) = 2f(x),
$$

which proves (15). To derive (16), first observe that

$$
\begin{aligned}
B(x,y,z) &= f(xyz) - f(x) - f(y) - f(z) - A(x,y) - A(x,z) - A(y,z) \\
&= f(xyz) - f(x) - (A(y,z) + f(y) + f(z)) - A(x,y) - A(x,z)
\end{aligned}
$$

$$= f(xyz) - f(x) - f(yz) - A(x,y) - A(x,z)$$
$$= A(x,yz) - A(x,y) - A(x,z). \tag{25}$$

Similarly, we can check

$$B(x,y,z) = A(xy,z) - A(x,z) - A(y,z). \tag{26}$$

It follows from (26) (resp. (25)) that $B(\cdot,\cdot,u)$ (resp. $B(u,\cdot,\cdot)) : G\times G \to H$ is the Cauchy kernel of $A(\cdot,u)$ (resp. $A(u,\cdot)) : G \to H$ for any given $u \in G$. Thus, from relations (6)–(8) and (2.10) in [10], we get (16). We have (17) since

$$\begin{aligned} f(xuvy) - f(xvuy) &= f(uvyx) - f(vuyx) \text{ (by (5))} \\ &= (A(uv,yx) + f(uv) + f(yx)) \\ &\quad - (A(vu,yx) + f(vu) + f(yx)) \\ &= A(uv,yx) - A(vu,yx) \text{ (by (5))} \\ &= 2B(u,v,yx), \end{aligned}$$

where the last equality comes from (26) and (2.11) in [10]. The usual convention $\sum_\emptyset = 0$ implies (18) by (2) and (3). Thus, by (25), (7), (10), (9), (6), and (11) (or (12), letting $z = e$), we get

$$B(e,x,y) = A(e,xy) - A(e,x) - A(e,y) = 0, \tag{27}$$

$$B(x,y,y) = A(x,y^2) - 2A(x,y) = 0, \tag{28}$$

$$B(x^{-1},y,z) = A(x^{-1},yz) - A(x^{-1},y) - A(x^{-1},z) = -B(x,y,z), \tag{29}$$

$$\begin{aligned} B(x,y,z) + B(x,z,y) = A(x,yz) &+ A(x,zy) \\ &- 2(A(x,y) + A(x,z)) = 0. \end{aligned} \tag{30}$$

This shows the first equality hold in (19)–(22) respectively. By (26)–(30), it is easy to check the rest in (19)–(22) are true.

To show (23), it suffices to note that

$$B(x_1,\ldots,x_{\ell+1}) = f(x_1\ldots x_{\ell+1}) - \sum_{i=1}^{\ell} f(x_i) - f(x_{\ell+1})$$

$$
\begin{aligned}
&- \sum_{1\le i<j\le \ell} A(x_i,x_j) - \sum_{i=1}^{\ell} A(x_i,x_{\ell+1}) \\
&= f(x_1\dots x_\ell) + f(x_{\ell+1}) + A(x_1\dots x_\ell, x_{\ell+1}) - \sum_{i=1}^{\ell} f(x_i) \\
&\quad - f(x_{\ell+1}) - \sum_{1\le i<j\le \ell} A(x_i,x_j) - \sum_{i=1}^{\ell} A(x_i,x_{\ell+1}) \\
&= B(x_1,\dots,x_\ell) + A(x_1\dots x_\ell, x_{\ell+1}) - \sum_{i=1}^{\ell} A(x_i,x_{\ell+1}).
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. *Suppose that $f : G \to H$ is a solution of equation* (1)*. Then*

$$
\begin{aligned}
f(x^{m_1}y^{n_1}\dots x^{m_\ell}y^{n_\ell}) &= \left(\sum_{i=1}^{\ell} m_i\right)^2 f(x) + \left(\sum_{i=1}^{\ell} n_i\right)^2 f(y) \\
&\quad + \left(\sum_{i=1}^{\ell} m_i\right)\left(\sum_{i=1}^{\ell} n_i\right)(f(xy)-f(x)-f(y))
\end{aligned}
\tag{31}
$$

holds for all $x,y\in G$, $\ell\ge 1$, $m_i,n_i\in\mathbb{Z}$, $i=1,\dots,\ell$.

Proof. It follows from (10) and (13) that

$$
\begin{aligned}
f(x^{m_1}y^{n_1}) &= f(x^{m_1}) + f(y^{n_1}) + A(x^{m_1},y^{n_1}) \\
&= m_1^2 f(x) + n_1^2 f(y) + m_1n_1(f(xy)-f(x)-f(y)).
\end{aligned}
$$

This shows (31) holds for $\ell=1$. Suppose it holds for $\ell=k$, $k\ge 1$. Then from (25), (26) and (2.6) in [10], we obtain

$$
\begin{aligned}
f(x^{m_1}y^{n_1}\dots x^{m_k}y^{n_k}x^{m_{k+1}}y^{n_{k+1}}) &= f(x^{m_1}y^{n_1}\dots x^{m_k+m_{k+1}}y^{n_k+n_{k+1}}) \\
&\quad + 2B(y^{n_k},x^{m_{k+1}},y^{n_{k+1}}x^{m_1}y^{n_1}\dots y^{n_{k-1}}x^{m_k}) \quad \text{(by (18))} \\
&= f(x^{m_1}y^{n_1}\dots x^{m_k+m_{k+1}}y^{n_k+n_{k+1}}) + 2\sum_{i=1}^{k} B(y^{n_k},x^{m_{k+1}},x^{m_i})
\end{aligned}
$$

$$+ 2\sum_{i=1}^{k-1} B(y^{n_k}, x^{m_{k+1}}, y^{n_i}) + 2B(y^{n_k}, x^{m_{k+1}}, y^{n_{k+1}}) \quad \text{(by (16))}$$
$$= f(x^{m_1}y^{n_1}\dots x^{m_k+m_{k+1}}y^{n_k+n_{k+1}}) + 2\sum_{i=1}^{k} n_k m_{k+1} m_i B(y,x,x)$$
$$+ 2\sum_{i=1}^{k-1} n_k m_{k+1} n_i B(y,x,y) + 2n_k m_{k+1} n_{k+1} B(y,x,y)$$
$$= f(x^{m_1}y^{n_1}\dots x^{m_k+m_{k+1}}y^{n_k+n_{k+1}}) \quad \text{(by (20))},$$

which concludes the proof of (31) by induction. □

The following extends a result due to Kurepa [9, Remark 2] by removing the 2 torsion-free condition on H.

Corollary 2.3. *Let $f : G \to H$ be a solution of equation* (1)*. Then f vanishes on the commutator subgroup $[G,G]$ of G, i.e., $f([G,G]) = \{0\}$. In particular, if $G = [G,G]$ then equation* (1) *has only the trivial solution $f \equiv 0$.*

Proof. Put $m_1 = n_1 = 1$ and $m_2 = n_2 = -1$ in (31) to get $f(xyx^{-1}y^{-1}) = 0$ for all $x, y \in G$. Let $x_i := u_i^{-\epsilon_i} v_i^{-\epsilon_i} u_i^{\epsilon_i} v_i^{\epsilon_i}$ with $u_i, v_i \in G$ and $\epsilon_i := \pm 1$, $i = 1, 2, \dots, k$. From (16), (17), and (20), we obtain

$$\begin{aligned} f(x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_k^{\epsilon_k}) &= f(x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_{k-1}^{\epsilon_{k-1}} u_k^{-\epsilon_k} v_k^{-\epsilon_k} u_k^{\epsilon_k} v_k^{\epsilon_k}) \\ &= f(x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_{k-1}^{\epsilon_{k-1}}) + 2B(u_k^{-\epsilon_k}, v_k^{-\epsilon_k}, x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_{k-1}^{\epsilon_{k-1}} u_k^{-\epsilon_k} v_k^{-\epsilon_k}) \\ &= f(x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_{k-1}^{\epsilon_{k-1}}), \end{aligned}$$

which ends our proof by simple induction. □

Note that Higman's group (see [3])

$$G = \langle a_1, a_2, a_3, a_4; \quad a_1^{-1}a_2a_1 = a_2^2,\ a_2^{-1}a_3a_2 = a_3^2,\ a_3^{-1}a_4a_3 = a_4^2;\ a_4^{-1}a_1a_4 = a_1^2\rangle$$

is such an example satisfying $G = [G,G]$.

Remark 2.4. 1) Relation (6) shows that $A : G \times G \to H$ is symmetric. We need only give the properties of $A(\cdot, u) : G \to H$, where $u \in G$ is arbitrarily given, because $A(u,\cdot) : G \to H$ has similar properties by the symmetry of A.

2) Using a similar argument, we can see that $\Psi_k : G \to H$ is also in $S(G,H)$ where

$$\Psi_k(x) := B(u_1,\dots,u_{k-1},x,u_{k+1},\dots,u_\ell) - B(u_1,\dots,u_{k-1},e,u_{k+1},\dots,u_\ell)$$

for any given $k = 1,\dots,\ell$, $u_i \in G$. Thus we can apply all the results in [10] and [11] to Ψ_k and get the corresponding relations for $\ell \geq 3$.

3. The general solution on free groups

Let $\langle \mathcal{A} \rangle$ denote the free group generated by $\mathcal{A}$. When $|\mathcal{A}|$ =1, $\langle \mathcal{A} \rangle$ is cyclic. By (13), $f(a^n) = n^2 f(a)$, $\forall n \in \mathbb{Z}$ and $a \in \mathcal{A}$. Then it is easy to see that the general solution of equation (1) is $f(x) = n^2 h$, where $x = a^n$ and $h \in H$ is an arbitrary constant. Therefore, in the following, we shall focus our attention on $|\mathcal{A}| \geq 2$.

Theorem 3.1. *Let $G = \langle \mathcal{A} \rangle$ and $f : G \to H$ be a solution of equation* (1)*. Then f is even and has the representation*

$$\begin{aligned} f(x) = \sum_{i=1}^{n} f(a_i) &+ \sum_{1\leq i<j\leq n} \epsilon_i\epsilon_j A(a_i,a_j) \\ &+ \sum_{1\leq i<j<k\leq n} \epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k) + g(\pi(x)), \end{aligned} \tag{32}$$

where $x = a_1^{\epsilon_1}\dots a_n^{\epsilon_n}$ with $a_i \in \mathcal{A}$ and $\epsilon_i = \pm 1$ is a writing of $x \in G$. Here, π is the natural homomorphism from G onto G/G^2, the factor group of G when all squares in G are equated to e; the mapping $g : G/G^2 \to H$ satisfies

$$2g = 0,\ g(\pi(a_1\dots a_\ell)) = 0 \quad \text{if } \ \ell \leq 3; \tag{33}$$

and A, B are defined by (2) *and* (3)*.*

Conversely, if $f : \mathcal{A} \to H$, $A : \mathcal{A}\times\mathcal{A} \to H$ and $B : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to H$ satisfy

$$A(a,a) = 2f(a), \quad A(a,b) = A(b,a) \tag{34}$$

and

$$B(a,a,b)=0, \tag{35}$$

$$\begin{cases} B(a,b,c)+B(a,c,b)=0, \\ B(a,b,c)+B(c,b,a)=0, \end{cases} \tag{36}$$

respectively, for all $a,b,c\in\mathcal{A}$*; and* $g: G/G^2\to H$ *satisfies* (33)*, then* (32) *defines a mapping* $f: G\to H$ *which is a solution of equation* (1).

Furthermore, the representation (32) *is unique and can be compressed into*

$$\begin{aligned} f(a_1^{m_1}\dots a_\ell^{m_\ell}) &= \sum_{i=1}^{\ell} m_i^2 f(a_i) + \sum_{1\le i<j\le \ell} m_i m_j A(a_i,a_j) \\ &+ \sum_{1\le i<j<k\le \ell} m_i m_j m_k B(a_i,a_j,a_k) + g(\pi(a_1^{m_1}\dots a_\ell^{m_\ell})) \end{aligned} \tag{37}$$

for all $a_i\in\mathcal{A}$, $m_i\in\mathbb{Z}$.

Proof. Let $f: G\to H$ be a solution of equation (1). Then f, A and B in (2) and (3) have the properties asserted in Theorem 2.1. In particular f is even.

In view of (3), (4), (6), and (9), we have

$$f(a_1^{\epsilon_1}\dots a_n^{\epsilon_n}) = B(a_1^{\epsilon_1},\dots,a_n^{\epsilon_n}) + \sum_{i=1}^{n} f(a_i) + \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A(a_i,a_j).$$

From (7) and (8) and applying Theorem 3 in [11] to the mapping $A(\cdot,u)$: $G\to H$ for any fixed $u\in G$, we arrive at

$$\begin{aligned} A(x_1^{m_1}\dots x_\ell^{m_\ell},u) &= \sum_{i=1}^{\ell} m_i A(x_i,u) + \sum_{1\le i<j\le \ell} m_i m_j (A(x_i x_j,u) \\ &\quad - A(x_i,u) - A(x_j,u)) + g_u(\pi(x_1^{m_1}\dots x_\ell^{m_\ell})), \end{aligned} \tag{38}$$

where $g_u : G/G^2\to H$ satisfies

$$2g_u=0, \quad \text{and} \quad g_u(\pi(x_1\dots x_\ell))=0 \quad \text{when } \ell\le 2. \tag{39}$$

Repeatedly using (23), we obtain

$$B(a_1^{\epsilon_1},\ldots,a_n^{\epsilon_n}) = \sum_{k=2}^{n-1} A(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}, a_{k+1}^{\epsilon_{k+1}}) - \sum_{k=3}^{n}\sum_{j=1}^{k-1} A(a_j^{\epsilon_j}, a_k^{\epsilon_k})$$

$$= \sum_{k=3}^{n} \epsilon_k A(a_1^{\epsilon_1}\ldots a_{k-1}^{\epsilon_{k-1}}, a_k) - \sum_{k=3}^{n}\sum_{j=1}^{k-1} \epsilon_j\epsilon_k A(a_j,a_k) \quad \text{(by (9))}$$

$$= \sum_{k=3}^{n} \epsilon_k \Bigg\{ \sum_{j=1}^{k-1} \epsilon_j A(a_j,a_k) + \sum_{i<j<k} \epsilon_i\epsilon_j \big(A(a_ia_j,a_k) - A(a_i,a_k) - A(a_j,a_k)\big)$$

$$+ g_{a_k}\big(\pi(a_1^{\epsilon_1}\ldots a_{k-1}^{\epsilon_{k-1}})\big)\Bigg\} - \sum_{k=3}^{n}\sum_{j=1}^{k-1} \epsilon_j\epsilon_k A(a_j,a_k) \quad \text{(by (38))}$$

$$= \sum_{k=3}^{n} \epsilon_k \Bigg\{ \sum_{i<j<k} \epsilon_i\epsilon_j B(a_i,a_j,a_k) + g_{a_k}\big(\pi(a_1^{\epsilon_1}\ldots a_{k-1}^{\epsilon_{k-1}})\big)\Bigg\} \quad \text{(by (26))},$$

where π is the natural homomorphism from G onto G/G^2, and $g_{a_k} : G/G^2 \to H$ satisfies (39) for $k = 3,\ldots,n$. Since $g_{a_3}(\pi(a_1^{\epsilon_1}a_2^{\epsilon_2})) = 0$,

$$\begin{aligned} f(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}) &= \sum_{1\le i<j<k\le n} \epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k) + \sum_{i=1}^{n} f(a_i) \\ &+ \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A(a_i,a_j) + \sum_{k=3}^{n-1} \epsilon_{k+1} g_{a_{k+1}}(\pi(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k})). \end{aligned} \tag{40}$$

Now we want to define $F_i : G \to H$ $(i = 1,2)$ by

$$F_1(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}) := \sum_{i=1}^{n} f(a_i) + \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A(a_i,a_j) \tag{41}$$

and

$$F_2(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}) := \sum_{1\le i<j<k\le n} \epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k), \tag{42}$$

respectively. Using (15), (20), (21) and (22), we can check that F_i $(i = 1,2)$ are well-defined. In fact, for any $b \in \mathcal{A}$, we have

$$F_1(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}\cdot bb^{-1}\cdot a_{k+1}^{\epsilon_{k+1}}\ldots a_n^{\epsilon_n}) = \sum_{i=1}^{n} f(a_i) + \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A(a_i,a_j) + f(b)$$

$$+ f(b) + \sum_{i=1}^{k} \epsilon_i(A(a_i,b) - A(a_i,b)) + \sum_{i=k+1}^{n} \epsilon_i(A(b,a_i) - A(b,a_i)) - A(b,b)$$

$$= F_1(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}),$$

and

$$\begin{aligned}
F_2(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}\cdot bb^{-1}\cdot a_{k+1}^{\epsilon_{k+1}}\ldots a_n^{\epsilon_n}) &= \sum_{1\le i<j<k\le n} \epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k)\\
&+ \sum_{1\le i<j\le k} \epsilon_i\epsilon_j(B(a_i,a_j,b) - B(a_i,a_j,b))\\
&+ \sum_{k+1\le i<j\le n} \epsilon_i\epsilon_j(B(b,a_i,a_j) - B(b,a_i,a_j))\\
&+ \sum_{i=1}^{k}\sum_{j=k+1}^{n} \epsilon_i\epsilon_j(B(a_i,b,a_j) - B(a_i,b,a_j))\\
&- \sum_{i=1}^{k} B(a_i,b,b) - \sum_{j=k+1}^{n} B(b,b,a_j) = F_2(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}).
\end{aligned}$$

Similarly, we can verify

$$F_1(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}\cdot b^{-1}b\cdot a_{k+1}^{\epsilon_{k+1}}\ldots a_n^{\epsilon_n}) = F_1(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}),$$

$$F_2(a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}\cdot b^{-1}b\cdot a_{k+1}^{\epsilon_{k+1}}\ldots a_n^{\epsilon_n}) = F_2(a_1^{\epsilon_1}\ldots a_n^{\epsilon_n}).$$

Thus if x and y have the same reduced form then $F_i(x) = F_i(y)$, $i = 1, 2$. Therefore we have proved F_i $(i = 1, 2)$ are well-defined. For convenience, let

$$x = a_1^{\epsilon_1}\ldots a_n^{\epsilon_n},\ y = b_1^{\eta_1}\ldots b_p^{\eta_p},\ a_i, b_j \in \mathcal{A},$$

$$\epsilon_i, \eta_j = \pm 1,\ i = 1,\ldots,n,\ j = 1,\ldots,p.$$

Since

$$\begin{aligned}
F_1(xy) + F_1(xy^{-1}) &= \sum_{i=1}^{n} f(a_i) + \sum_{j=1}^{p} f(b_i) + \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A(a_i,a_j)\\
&+ \sum_{1\le i<j\le p} \eta_i\eta_j A(b_i,b_j) + \sum_{i=1}^{n}\sum_{j=1}^{p} \epsilon_i\eta_j A(a_i,b_j)
\end{aligned}$$

$$+\sum_{i=1}^{n} f(a_i)+\sum_{j=1}^{p} f(b_i)+\sum_{1\le i<j\le n}\epsilon_i\epsilon_j A(a_i,a_j)$$
$$+\sum_{1\le i<j\le p}(-\eta_j)(-\eta_i)A(b_j,b_i)+\sum_{i=1}^{n}\sum_{j=1}^{p}\epsilon_i(-\eta_j)A(a_i,b_j)$$
$$=2F_1(x)+2F_1(y),$$

$F_1 : G \to H$ satisfies equation (1).

To verify that F_2 also satisfies equation (1), first observe that F_2 is even by (22). Indeed,

$$\begin{aligned}F_2(x^{-1}) &= \sum_{1\le i<j<k\le n}(-\epsilon_k)(-\epsilon_j)(-\epsilon_i)B(a_k,a_j,a_i)\\ &= -\sum_{1\le i<j<k\le n}\epsilon_k\epsilon_j\epsilon_i B(a_k,a_j,a_i)\\ &= \sum_{1\le i<j<k\le n}\epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k)=F_2(x).\end{aligned}\tag{43}$$

Using the definition of F_2, we obtain

$$\begin{aligned}F_2(xy) =& \sum_{1\le i<j<k\le n}\epsilon_i\epsilon_j\epsilon_k B(a_i,a_j,a_k)+\sum_{1\le i<j<k\le p}\eta_i\eta_j\eta_k B(b_i,b_j,b_k)\\ &+\sum_{1\le i<j\le n}\sum_{k=1}^{p}\epsilon_i\epsilon_j\eta_k B(a_i,a_j,b_k)+\sum_{i=1}^{n}\sum_{1\le j<k\le p}\epsilon_i\eta_j\eta_k B(a_i,b_j,b_k)\\ =& F_2(x)+F_2(y)+E_{x,y},\end{aligned}\tag{44}$$

where

$$E_{x,y}=\sum_{1\le i<j\le n}\sum_{k=1}^{p}\epsilon_i\epsilon_j\eta_k B(a_i,a_j,b_k)+\sum_{i=1}^{n}\sum_{1\le j<k\le p}\epsilon_i\eta_j\eta_k B(a_i,b_j,b_k).$$

Similarly

$$F_2(xy^{-1})=F_2(x)+F_2(y^{-1})+E_{x,y^{-1}},\tag{45}$$

where

$$E_{x,y^{-1}}=\sum_{1\le i<j\le n}\sum_{k=1}^{p}\epsilon_i\epsilon_j(-\eta_k)B(a_i,a_j,b_k)$$

$$+\sum_{i=1}^{n}\sum_{1\le j<k\le p}\epsilon_i(-\eta_k)(-\eta_j)B(a_i,b_k,b_j).$$

So we have from (22)

$$E_{x,y}+E_{x,y^{-1}}=0. \tag{46}$$

It follows that from (43)–(46)

$$F_2(xy)+F_2(xy^{-1})=2F_2(x)+2F_2(y).$$

This proves that F_2 is a solution of equation (1). Therefore $\tilde{g}:=f-F_1-F_2$ is also a solution of equation (1). By (40) we have

$$\tilde{g}(x)=\sum_{k=3}^{n-1}\epsilon_{k+1}g_{a_{k+1}}(\pi(a_1^{\epsilon_1}\dots a_k^{\epsilon_k})). \tag{47}$$

Because $g_{a_{k+1}}$ $(k=3,\dots,n-1)$ satisfy (39), we have

$$2\tilde{g}=0. \tag{48}$$

Thus

$$\tilde{g}(xy)+\tilde{g}(xy^{-1})=2\tilde{g}(x)+2\tilde{g}(y)=0=2\tilde{g}(x).$$

This shows that $\tilde{g}$ is also a solution of (24). By (2.2) in [10], $\tilde{g}$ is actually a mapping from G/G^2 to H, i.e., $\tilde{g}=g\circ\pi$, where $g:G/G^2\to H$. Clearly g has property (33). Combining (40) and (47), we obtain (32), completing the proof of the first part of this theorem. It follows from (6), (15) and (20) that we can compress (32) to the form (37) by induction.

To prove the converse of this theorem, suppose that $f:G\to H$ is defined by (32). Then such an f is well-defined since we have shown F_i $(i=1,2)$ in (41) and (42) are well-defined by (34)–(36). Observe that since g satisfies (33), f defined by (32) is indeed an extension of f on $\mathcal{A}$. (Here, for simplicity, we still use the same symbol f to denote the extension of f on $\mathcal{A}$.) Also

$$\begin{aligned} f(e)&=0,\\ f(a_1a_2)&=f(a_1)+f(a_2)+A(a_1,a_2), && (49)\\ f(a_1a_2a_3)&=f(a_1)+f(a_2)+f(a_3)+A(a_1,a_2)\\ &\quad+A(a_2,a_3)+A(a_1,a_3)+B(a_1,a_2,a_3). && (50)\end{aligned}$$

Since

$$g(\pi(xy)) + g(\pi(xy^{-1})) = 2g(\pi(xy)) = 0 = 2g(\pi(x)) + 2g(\pi(y)),$$

where we have used $\pi(y) = \pi(y^{-1})$ and $2g = 0$, the mapping $g \circ \pi : G \to H$ does satisfy equation (1). By the proof of the first part, $F_i (i = 1, 2)$ are solutions of equation (1). Therefore, f, being the sum of F_1, F_2, and $g \circ \pi$,

$$f(x) = F_1(x) + F_2(x) + g \circ \pi(x), \quad \forall x \in G,$$

is a solution of equation (1). This completes the proof of the converse. Finally, it follows from (49) and (50) that both A and B are unique. Therefore g in (32) is also unique, which implies the uniqueness of the representation (32). □

In particular, suppose $G = \langle a, b\rangle$ is the free group on two generators a and b. By Corollary 2.2 and Theorem 3.1, we get the following.

Corollary 3.2. *Suppose that $G = \langle a, b\rangle$ is the free group on two generators a, b and that $f : G \to H$ is a solution of equation* (1). *Then f has the representation*

$$\begin{aligned} f(a^{m_1}b^{n_1} \dots a^{m_\ell}b^{n_\ell}) = {} & \left(\sum_{i=1}^{\ell} m_i\right)^2 f(a) + \left(\sum_{i=1}^{\ell} n_i\right)^2 f(b) \\ & + \left(\sum_{i=1}^{\ell} m_i\right)\left(\sum_{i=1}^{\ell} n_i\right)(f(ab) - f(a) - f(b)) \end{aligned} \tag{51}$$

for all $\ell \geq 1$, $m_i, n_i \in \mathbb{Z}$. Conversely, each mapping initialized at the three words a, b and ab can be extended to a solution f of equation (1) *by taking* (51) *as its defining formula.*

In particular, the above f can be factored through the abelianization $\langle a, b\rangle^{ab}$ of $\langle a, b\rangle$.

Remark 3.3. By Theorem 2.1, it is easy to check that $2A$ is a bimorphism if, and only if, $f(xyz) = f(xzy)$ for all $x, y, z \in G$. This result appeared in [1], [6], [7], and [9]. By Theorem 3.1, not every $2A$ needs to be a bimorphism. However, In particular, if G is a group generated by 2 elements, $2A$ must be a bimorphism by Corollary 2.2. Here notice that Corollary 2.2 holds for any (not necessarily free) group G generated by 2 elements. This generalizes Theorem 1 in [9].

4. The general solutions on some special groups

Consider the commuting diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & H_1 \\ \phi \big\downarrow & & \big\uparrow \psi \\ G_2 & \xrightarrow{f_2} & H_2 \end{array}$$

where ϕ, ψ are homomorphisms. Then it is not hard to check that if f_2 is a solution of equation (1) on G_2, so is f_1 on G_1, and that if f_1 is a solution of equation (1) on G_1, ϕ is an epimorphism and ψ is a monomorphism, then f_2 is a solution of equation (1) on G_2.

Therefore, suppose that G_1 is a group, that S is any subset of G_1 and $G_2 = G_1/\langle S\rangle^{G_1}$, where $\langle S\rangle$ denotes the subgroup generated by S and $\langle S\rangle^{G_1}$ the normal closure of $\langle S\rangle$ in G_1, and that $\pi : G_1 \to G_2$ is the canonical surjection. Then $f_2 : G_2 \to H$ is a solution of equation (1) if, and only if, so is $f_1 := f_2 \circ \pi$ on G_1. Thus, to obtain all solutions $f_2 : G_2 \to H$ of equation (1), it suffices to filter all solutions $f : G_1 \to H$ by the factorization criterion $f(xwy) = f(xy)$, $\forall x, y \in G_1$, $\forall w \in S$, equivalently

$$f(xw) = f(x), \quad \forall x \in G_1,\ \forall w \in S, \tag{52}$$

by (5). It is easy to see (52) is also equivalent to

$$f(w) = 0 \quad \text{and} \quad A(\cdot, w) = 0, \quad \forall w \in S. \tag{53}$$

Indeed, for $f(xw) = f(x) + f(w) + A(x, w)$, in view of (52), we get $f(w) + A(x, w) = 0$, $\forall x \in G_1$. From (6) and (7), we get (53). On the other hand, from (53), $f(xw) = A(x, w) + f(x) + f(w) = f(x)$. Therefore, through this process, we can solve equation (1) on groups presented by generators and defining relations.

In the following, by $\langle \mathcal{A}\rangle^{ab}$ we denote the abelianization of the free group $\langle \mathcal{A}\rangle$.

Theorem 4.1. $f : \langle \mathcal{A}\rangle^{ab} \to H$ *is a solution of equation* (1) *if and only if it has the representation*

$$f(x) = \sum_{i=1}^{\ell} m_i^2 f(a_i) + \sum_{1\le i<j\le \ell} m_i m_j A(a_i, a_j) + g(\pi(x)) \tag{54}$$

for all $x = a_1^{m_1} \dots a_\ell^{m_\ell} \in \langle \mathcal{A} \rangle^{ab}$*. Here,* $a_1, \dots, a_\ell \in \mathcal{A}$*,* $f : \mathcal{A} \to H$ *and* $A : \mathcal{A} \times \mathcal{A} \to H$ *satisfy* (34)*),* π *is the natural epimorphism from* $\langle \mathcal{A} \rangle^{ab}$ *to* $\langle \mathcal{A} \rangle^{ab} / (\langle \mathcal{A} \rangle^{ab})^2$*, and* $g : \langle \mathcal{A} \rangle^{ab} / (\langle \mathcal{A} \rangle^{ab})^2 \to H$ *is an arbitrary mapping satisfying*

$$2g = 0, \ g(\pi(a_1 \dots a_\ell)) = 0, \quad \text{if } \ \ell \leq 2. \tag{55}$$

Proof. By the preceding discussion, the general solutions are given by (37), where $f : \langle \mathcal{A} \rangle \to H$, satisfies the factorization criterion (52) with $S = \{aba^{-1}b^{-1} \mid a, b \in \langle \mathcal{A} \rangle\}$. By Proposition 5.1 in Appendix, condition (52) is equivalent to the following

$$f(xaba^{-1}b^{-1}) = f(x), \quad \forall x \in \langle \mathcal{A} \rangle, \ \forall a, b \in \mathcal{A}. \tag{56}$$

By (37) and Theorem 2.1, if we let $x = c_1^{m_1} \dots c_\ell^{m_\ell}$ with $c_i \in \mathcal{A}$ and $m_i \in \mathbb{Z}$, then

$$\begin{aligned} f(xaba^{-1}b^{-1}) &= f(xaa^{-1}bb^{-1}) + 2B(b, a^{-1}, b^{-1}xa) \quad \text{(by (17))} \\ &= f(x) + 2B(b, a^{-1}, b^{-1}) + 2B(b, a^{-1}, a) + 2\sum_{i=1}^{\ell} m_i B(b, a^{-1}, c_i) \\ &= f(x) + 2\sum_{i=1}^{\ell} m_i B(a, b, c_i) \end{aligned}$$

by (16), (20), and (22). Therefore, condition (56) is equivalent to

$$2B(a, b, c) = 0, \quad \forall a, b, c \in \mathcal{A}. \tag{57}$$

Let

$$g_0(x) := \sum_{1 \leq i < j < k \leq \ell} m_i m_j m_k B(a_i, a_j, a_k), \quad a_i, a_j, a_k \in \mathcal{A},$$

where $x = a_1^{m_1} \dots a_\ell^{m_\ell} \in \langle \mathcal{A} \rangle$ with $a_i \in \mathcal{A}$, $m_i \in \mathbb{Z}$. Completely similar to F_2 in (42), we can show that g_0 is well-defined. It also follows from (57) that $2g_0 = 0$, which implies that g_0 is a function on $\langle \mathcal{A} \rangle^{ab} / (\langle \mathcal{A} \rangle^{ab})^2$ as in the proof of Theorem 3.1. Clearly g_0 has property (55). Thus g_0 can be absorbed into the last term g in (37). Therefore (37) can be reduced to (54), which completes the proof of this theorem. □

As in [10], [11], and [13], consider the general linear group $GL_n(\mathbb{Z})$ ($n \geq 2$) of $n \times n$ invertible matrices over the integers. Here we use the same notations in [10]–[12]. Let $a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $GL_2(\mathbb{Z})$ is generated by a, b with some defining relations; see [10]. In $GL_n(\mathbb{Z})$ ($n \geq 3$), for $i \neq j$, let $e_{ij}(p)$ be the matrix whose diagonal entries are 1, the (i,j)-th entry is p, and elsewhere 0. For diagonal matrices, let $[\alpha]_i$ denote the matrix whose (i,i)-th entry is α, and elsewhere 1; $[\alpha,\beta]_{ij} = [\alpha]_i[\beta]_j$, $d_{ij}(\alpha) = [\alpha, \alpha^{-1}]_{ij}$, and $[\alpha_1, \alpha_2, \ldots, \alpha_n] = [\alpha_1]_1[\alpha_2]_2 \ldots [\alpha_n]_n$. Here $p \in \mathbb{Z}$ and α, α_i, β are units of $\mathbb{Z}$. Then $GL_n(\mathbb{Z})$ is generated by the above matrices with the following relations (see [11] and [12]):

$$e_{ij}(p)e_{ij}(q) = e_{ij}(p+q), \tag{58}$$

$$e_{ij}(p)e_{k\ell}(q) = e_{k\ell}(q)e_{ij}(p) \quad (i \neq \ell, j \neq k), \tag{59}$$

$$e_{ij}(p)e_{jk}(q) = e_{jk}(q)e_{ij}(p)e_{ik}(pq) \quad (i \neq k), \tag{60}$$

$$e_{ij}(\alpha - 1)e_{ji}(1) = d_{ij}(\alpha)e_{ji}(\alpha)e_{ij}(1 - \alpha^{-1}), \tag{61}$$

$$e_{ij}(x)[\alpha_1, \alpha_2, \ldots, \alpha_n] = [\alpha_1, \alpha_2, \ldots, \alpha_n]e_{ij}(\alpha_i^{-1}x\alpha_j), \tag{62}$$

$$[\alpha_1, \alpha_2, \ldots, \alpha_n][\beta_1, \beta_2, \ldots, \beta_n] = [\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n]. \tag{63}$$

Using these facts, we can get the general solution of equation (1) on $GL_n(\mathbb{Z})$ for $n \geq 2$.

Theorem 4.2. *A mapping $f : GL_2(\mathbb{Z}) \to H$ is a solution of equation* (1) *if and only if it is given by*

$$\begin{aligned} f(a^{m_1}b^{n_1} \ldots a^{m_\ell}b^{n_\ell}) &= \left(\sum_{i=1}^{\ell} m_i\right)^2 f(a) + \left(\sum_{i=1}^{\ell} n_i\right)^2 f(b) \\ &\quad + \left(\sum_{i=1}^{\ell} m_i\right)\left(\sum_{i=1}^{\ell} n_i\right) A(a,b) \end{aligned} \tag{64}$$

with $4f(a) = 4f(b) = 2A(a,b) = 0$.

For $n \geq 3$, $f : GL_n(\mathbb{Z}) \to H$ is a solution of equation (1) *if and only if $f = f_1 \circ \det$, where $f_1 : \{\pm 1\} \to H$ s a solution of equation* (1) *such that $f_1(1) = 0$, $f_1(-1) = \alpha$. Here α is any element in H satisfying $4\alpha = 0$.*

Proof. Suppose, first of all, that $n = 2$. Let $f : \langle a, b\rangle \to H$ satisfy equation (1). By Corollary 3.2, f has the representation (51) and is actually a mapping on $\langle a, b\rangle^{ab}$. Working at this level, as in [10], we get $b^2 = e$. Clearly $a^2 = e$. The factorization criterion says $f(a^2) = f(b^2) = A(\cdot, a^2) = A(\cdot, b^2) = 0$, i.e., $4f(a) = 4f(b) = 2A(\cdot, a) = 2A(\cdot, b) = 0$ by (13) and (10). By Corollary 3.2 and Remark 3.3, $2A(\cdot, a)$ is a homomorphism. Thus condition $2A(\cdot, a) = 0$ is equivalent to $2A(a, a) = 0$ and $2A(b, a) = 0$. But $2A(a, a) = 0$ is just $4f(a) = 0$ by (15). By the symmetry of A, the condition $2A(\cdot, a) = 0$ is reduced to $2A(a, b) = 0$. Similarly, condition $2A(\cdot, b) = 0$ also can be reduced to $2A(a, b) = 0$. This completes the proof of the first part of this theorem.

For $n \geq 3$, let $f : GL_n(\mathbb{Z}) \to H$ be a solution of equation (1). Then, for any $u \in G$, the mapping $A(\cdot, u) : GL_n(\mathbb{Z}) \to H$ must be a homomorphism by Theorem 4 in [11] since it satisfies (24) on $GL_n(\mathbb{Z})$; that is,

$$A(xy, u) = A(x, u) + A(y, u), \quad \forall x, y \in G. \tag{65}$$

Now

$$\begin{aligned} f(xe_{ik}(pq)y) &= f(xe_{ij}^{-1}(p)e_{jk}^{-1}(q)e_{ij}(p)e_{jk}(q)y) \quad \text{(by (60))} \\ &= f(xe_{ij}^{-1}(p)e_{ij}(p)e_{jk}^{-1}(q)e_{jk}(q)y) \\ &\quad + 2B(e_{jk}^{-1}(q), e_{ij}(p), e_{jk}(q)yxe_{ij}^{-1}(p)) \quad \text{(by (17))} \\ &= f(xy) \text{ (by (26) \& (65))}. \end{aligned}$$

Hence

$$f(xe_{ik}(p)y) = f(xy). \tag{66}$$

By (61), $d_{ij}(\alpha)$ is a product of matrices of type e_{ij}, and so from (66) we get

$$f(xd_{ij}(\alpha)y) = f(xy). \tag{67}$$

Therefore,

$$\begin{aligned} &f(x[\alpha_1, \alpha_2, \ldots, \alpha_n]y) \\ &\quad = f(x[\alpha_1, \alpha_2, \ldots, \alpha_n]d_{12}(\alpha_2)d_{13}(\alpha_3)\ldots d_{1n}(\alpha_n)y) \quad \text{(by (67))} \\ &\quad = f(x[\alpha_1]_1[\alpha_2]_2\ldots[\alpha_n]_n[\alpha_2]_1[\alpha_2^{-1}]_2[\alpha_3]_1[\alpha_3^{-1}]_3\ldots[\alpha_n]_1[\alpha_n^{-1}]_n y) \\ &\qquad\quad \text{(by the definitions of } [\alpha_1, \ldots, \alpha_n] \text{ and } d_{ij}(\alpha)) \end{aligned}$$

$$= f(x[\alpha_1\alpha_2\dots\alpha_n]_1 y) \quad \text{(by (63))}.$$

Since $\det[\alpha_1, \alpha_2, \dots, \alpha_n] = \alpha_1\alpha_2\dots\alpha_n$, we have proved

$$f(x[\alpha_1, \alpha_2, \dots, \alpha_n]y) = f(x[\det[\alpha_1, \alpha_2, \dots, \alpha_n]]_1 y). \tag{68}$$

Let $x = g_1g_2\dots g_\ell$ be a writing of x as a product of generating matrices of types e_{ij}, d_{ij} or $[\alpha_1, \alpha_2, \dots, \alpha_n]$; and let $h_1, h_2, \dots, h_m$ be the subsequence of diagonal matrices obtained by removing those of types e_{ij} and d_{ij}. Then we obtain

$$\begin{aligned} f(x) &= f(g_1g_2\dots g_\ell) = f(h_1h_2\dots h_m) \quad \text{(by (66) \& (67))} \\ &= f([\det(h_1h_2\dots h_m)]_1) \quad \text{(by (63) \& (68))} \\ &= f([\det x]_1). \end{aligned}$$

Thus

$$f(x) = f([\det x]_1) =: f_1(\det x), \quad \forall x \in GL_n(\mathbb{Z}). \tag{69}$$

Since the mapping $\det : GL_n(\mathbb{Z}) \to \{\pm 1\}$ is an epimorphism, as mentioned first this section, $f_1 : \{\pm 1\} \to H$ is a solution of equation (1), which has been solved at the beginning of Section 3. The rest of this theorem is very easy to get by Theorem 2.1. □

Remark 4.3. From [10]–[12], we see that there is much difference between equations (24) and $F(xy) + F(y^{-1}x) = 2F(x)$, $\forall x, y \in G$, with $F(e) = 0$. However, equation (Q) and the following functional equation

$$f(xy) + f(y^{-1}x) = 2f(x) + 2f(y), \quad \forall\, x, y \in G, \tag{70}$$

are equivalent in the sense that they have the same general solutions. Clearly, it suffices to see that equation (1) is equivalent to equation (70) with $f(e) = 0$. Indeed, if $f : G \to H$ is a solution of equation (1), then from (5), f is also a solution of equation (70) with $f(e) = 0$. On the other hand, similar to equation (1), if $f : G \to H$ is a solution of equation (70) with $f(e) = 0$, then f has property (5). Thus f also satisfies equation (1).

Remark 4.4. In [13], Stetkær effectively uses the observation that functions satisfying the Jensen equation (24) are indeed functions on the quotient group $G/[G, [G, G]]$ to perform computations on groups like $GL_n(\mathbb{R})$. The same observation can be made about equation (1) using Theorem 2.1.

5. Appendix

Proposition 5.1. *Let G and H be groups. If G is generated by $\mathcal{A}$ and $f : G \to H$ is arbitrary, then*

$$f(xuvu^{-1}v^{-1}y) = f(xy), \quad \forall x, y, u, v \in G, \tag{71}$$

is equivalent to the simpler

$$f(xaba^{-1}b^{-1}y) = f(xy), \quad \forall x, y \in G, \ \forall a, b \in \mathcal{A}. \tag{72}$$

Proof. Clearly (71) implies (72). Now suppose that (72) is true. Then it is easy to derive

$$f(xa^{\epsilon}b^{\eta}a^{-\epsilon}b^{-\eta}y) = f(xy), \quad \forall x, y \in G, \ \forall a, b \in \mathcal{A}, \ \epsilon, \eta = \pm 1. \tag{73}$$

Let $u = a_1^{\epsilon_1} \dots a_m^{\epsilon_m}$ and $v = b_1^{\eta_1} \dots b_n^{\eta_n}$ with $a_i, b_j \in \mathcal{A}$, $\epsilon_i, \eta_j = \pm 1$, $i = 1, \dots, m, j = 1, \dots, n$. So (73) shows that (71) holds for $m = n = 1$. Suppose it holds for $m \leq \ell$, $n \leq k$, $\ell, k \geq 1$. For convenience, let $u_1 = a_1^{\epsilon_1} \dots a_\ell^{\epsilon_\ell}$, $v_1 = b_1^{\eta_1} \dots b_k^{\eta_k}$. Then by the inductive assumption we have

$$\begin{aligned}
&f(x \cdot a_1^{\epsilon_1} \dots a_\ell^{\epsilon_\ell} b_1^{\eta_1} \dots b_{k+1}^{\eta_{k+1}} a_\ell^{-\epsilon_\ell} \dots a_1^{-\epsilon_1} b_{k+1}^{-\eta_{k+1}} \dots b_1^{-\eta_1} \cdot y) \\
&\quad = f(xu_1v_1u_1^{-1}b_{k+1}^{\eta_{k+1}} \cdot b_{k+1}^{-\eta_{k+1}} u_1 b_{k+1}^{\eta_{k+1}} u_1^{-1} \cdot b_{k+1}^{-\eta_{k+1}} v_1^{-1} y) \\
&\quad = f(xu_1v_1u_1^{-1} \cdot b_{k+1}^{\eta_{k+1}} b_{k+1}^{-\eta_{k+1}} \cdot v_1^{-1} y) \\
&\quad = f(x \cdot u_1v_1u_1^{-1}v_1^{-1} \cdot y) = f(xy).
\end{aligned}$$

Similarly we can check that

$$f(x \cdot a_1^{\epsilon_1} \dots a_\ell^{\epsilon_\ell} a_{\ell+1}^{\epsilon_{\ell+1}} b_1^{\eta_1} \dots b_k^{\eta_k} a_\ell^{-\epsilon_\ell} \dots a_1^{-\epsilon_1} b_k^{-\eta_k} \dots b_1^{-\eta_1} \cdot y) = f(xy).$$

This proves (71) is true by induction, which completes the proof of Proposition 5.1. □

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References

[1] J. Aczél and J. Dhombres, Functional Equations in Several Variables, *New York – Sydney*, 1989.

[2] M. Fréchet, Sur la definition axiomatique d'une class d'espaces vectoriels distancies applicables vectorielement sur l'espaces de Hilbert, *Ann. Maths.* **36** (1935), 705–718.

[3] G. Higman, A finitely generated infinite simple group, *J. London Math. Soc.* **26** (1951), 61–64.

[4] M. Hosszú, A remark on the square norm, *Aequaitones Math.* **2** (1963), 190–193.

[5] P. Jordan and J. Von Neumann, On inner products in linear metric soaces, *Ann. Maths.* **36** (1935), 719–723.

[6] Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.* **27** (1995), 368–372.

[7] Pl. Kannappan, On quadratic functional equation, *Internal. J. Math.* & *Statist. Sci.* **9** (2000), 10–35.

[8] S. Kurepa, Quadratic and sesquilinear functionals, *Glasnik Mat.-Fiz. Astr.* **20** (1965), 79–93.

[9] S. Kurepa, On bimorphisms and quadratic forms on groups, *Aequationes Math.* **9** (1973), 30–45.

[10] C. T. Ng, Jensen's functional equation on groups, *Aequationes Math.* **39** (1990), 85–99.

[11] C. T. Ng, Jensen's functional equation on groups, II, *Aequationes Math.* **58** (1999), 311–320.

[12] C. T. Ng, Jensen's functional equation on groups, III, *Aequationes Math.* **62** (2001), 143–159.

[13] H. Stetkær, On Jensen's functional equation on groups, *Aequationes Math.* **66** (2003), 100–118.

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