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Uniformly noncreasy Orlicz spaces

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Abstract. The geometric properties which are considered are closely related to the fixed point property (see [14] and [20]). Criteria in order that Orlicz spaces are uniformly noncreasy are given. It is shown that uniform noncreaseness and orthogonal uniform convexity are not comparable one to another in general. It is also proved that uniformly monotone Köthe space with a uniformly monotone dual is orthogonally uniformly convex and the converse is not true. It is noticed that orthogonal uniform convexity is not self-dual property.

1. Introduction

Uniform rotundity and uniform smoothness of Banach spaces play an essential role in the theory of Banach spaces and their numerous applications. In particular each of these properties guarantees normal structure and reflexivity which in turn imply the fixed point property for nonexpansive mappings (see [7]). On the other hand, there are some methods which allow us to establish the fixed point property for Banach spaces without normal structure. One of them is uniform noncreaseness. It was introduced by Prus in [20]. He has proved that it implies the fixed point property both for a Banach space and its dual and it does not imply normal structure (see [20]).

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Throughout this paper X is a real Banach space. As usual, $S(X)$ and $B(X)$ stand for the unit sphere and the unit ball of X, respectively. Given a functional $x^* \in S(X^*)$ and a scalar $\delta \in [0,1]$ we set

$$S(x^*, \delta) = \{x \in B(X) : x^*(x) \geq 1 - \delta\}.$$

For any two functionals $x^*, y^* \in S(X^*)$ and a scalar $\delta \in [0,1]$ we put

$$S(x^*, y^*, \delta) = S(x^*, \delta) \cap S(y^*, \delta).$$

We say that a Banach space $(X, \|\cdot\|_X)$ has a *crease*, if there are two distinct functionals $x^*, y^* \in S(X^*)$ with $\operatorname{diam} S(x^*, y^*, 0) > 0$. This means that the sphere $S(X)$ contains a segment of positive length which lies on two different hyperplanes supporting the ball $B(X)$. $(X, \|\cdot\|_X)$ is called *noncreasy* ($X \in (NC)$) whenever $S(X)$ has not creases, i.e. for every $x^*, y^* \in S(X^*)$ with $x^* \neq y^*$ we have $\operatorname{diam} S(x^*, y^*, 0) = 0$. Obviously all spaces with $\dim X \leq 2$ are noncreasy.

It is clear that $X \in (NC)$ whenever it is rotund or smooth (we refer to [5] for definitions of rotundity and smoothness). Moreover, that implication can be reversed in Orlicz function spaces L_Φ over nonatomic finite measure space (see [4]).

Definition 1. A Banach space $(X, \|\cdot\|_X)$ is uniformly noncreasy ($X \in (UNC)$) if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\operatorname{diam} S(x^*, y^*, \delta) \leq \varepsilon$, whenever $x^*, y^* \in S(X^*)$ and $\|x^* - y^*\|_{X^*} \geq \varepsilon$. We put $\operatorname{diam} \emptyset = -\infty$.

Recall that X is *uniformly convex* ($X \in (UC)$), if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in S(X)$ the inequality $\|x - y\|_X > \varepsilon$ implies $\|x + y\|_X < 2(1 - \delta)$ (see [5]). X is *uniformly smooth* ($X \in (US)$), if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $x \in S(X)$ and y with $\|y\|_X \leq \delta$ we have $\|x + y\|_X + \|x - y\|_X < 2 + \varepsilon \|y\|_X$ (see [5]).

It is easy to see that each of these two properties ((UC), (US)) implies uniform noncreaseness (see [20]).

2. Orlicz spaces

In this section we present criteria for uniform noncreaseness of Orlicz function spaces equipped with the Luxemburg and with the Orlicz norm. We consider both the case of finite and infinite nonatomic measure space. We start with some notations and definitions.

Let (T, Σ, μ) be a measure space with a σ-finite, complete nonatomic measure μ. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T. We say that $\Phi : \mathbb{R} \longrightarrow [0, \infty]$ is an *Orlicz function,* if $\Phi(0) = 0, \Phi$ is a convex, even, left continuous on $[0, \infty)$ and Φ is a function, which is not identically equal to zero and infinity.

Let Φ be an Orlicz function, p be its right derivative and q be the right-inverse function of p. Then we call $\Psi(v) = \int_0^{|v|} q(s)\, ds$ *the complementary function* of Φ. It is known that $\Psi(v) = \sup_{u>0} \{u|v| - \Phi(u)\}$ for every $v \in \mathbb{R}$. Moreover

$$uv = \Phi(u) + \Psi(v) \tag{1}$$

whenever $v = p(u)$ or $u = q(v)$.

The *Orlicz function space* L_Φ is defined to be the set

$$L_\Phi = \left\{ x \in L^0 : I_\Phi(cx) = \int_T \Phi(cx(t)) d\mu < \infty \text{ for some } c > 0 \right\}.$$

This space is usually considered with the *Luxemburg* norm

$$\|x\|_\Phi = \inf \{\varepsilon > 0 : I_\Phi(x/\varepsilon) \le 1\}$$

or with the equivalent *Orlicz* norm

$$\|x\|_\Phi^o = \sup \left\{ \left| \int_T x(t) y(t) d\mu \right| : y \in L_\Psi,\ I_\Psi(y) \le 1 \right\}.$$

It is more convenient for our consideration to make use of the *Amemiya* norm

$$\|x\|_\Phi^A = \inf \left\{ \frac{1}{k} (1 + I_\Phi(kx)) : k > 0 \right\}.$$

We have $\|x\|_\Phi^A = \|x\|_\Phi^o$ for an arbitrary Orlicz function Φ (see [10]). Denote $L_\Phi = (L_\Phi, \|\cdot\|_\Phi)$ and $L_\Phi^o = (L_\Phi, \|\cdot\|_\Phi^o)$. If Ψ is finitely valued (or equivalently if $\lim_{u\to\infty}\Phi(u)/u=\infty$), then for every $x\in L_\Phi\setminus\{0\}$ we have $k^{**}(x)<\infty$ and

$$\|x\|^o_\Phi=\frac{1}{k}\big(1+I_\Phi(kx)\big)\quad\text{for any } k\in K(x)=[k^*_x,k^{**}_x]\,,\ \text{where}\tag{2}$$

$k^*_x=\inf\left\{k>0\,:\,I_\Psi\left(p(k|x|)\right)\geq 1\right\}$ and $k^{**}_x=\sup\left\{k>0\,:\,I_\Psi\left(p(k|x|)\right)\leq 1\right\}$ (see [3]). We say that an Orlicz function Φ satisfies *the* Δ_2*-condition for all* u (*for large* u) if there is a constant $k>2$ (there are $u_0>0$ with $\Phi(u_0)<\infty$ and $k>2$) such that

$$\Phi(2u)\leq k\Phi(u)$$

for every $u\in\mathbb{R}$ (for every $|u|\geq u_0$), respectively. We will use abbreviations $\Phi\in\Delta_2^a$, $\Phi\in\Delta_2^l$, if Φ satisfies the Δ_2-condition for all u, for large u, respectively. Note that if $\Psi\in\Delta_2^l$, then Ψ is finitely valued and the condition (2) holds for each $x\in L_\Phi\setminus\{0\}$.

A function Φ is strictly convex ($\Phi\in(SC)$) if $\Phi((u+v)/2)<(\Phi(u)+\Phi(v))/2$ for all $u,v\in\mathbb{R},u\neq v$. Φ is uniformly convex for all arguments [for large arguments] if for any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ [there exists $u_0>0$ such that for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$] such that

$$\Phi\left(\frac{u+v}{2}\right)\leq(1-\delta)\,\frac{\Phi(u)+\Phi(v)}{2}$$

for all u,v satisfying $|u-v|\geq\varepsilon\max\{u,v\}$ $\big[|u-v|\geq\varepsilon\max\{u,v\}$ and $u,v\in[u_0,\infty)\big]$.

For more details we refer to [3] and [18].

Theorem 1. *Assume that* $\mu(T)<\infty$. *The following assertions are equivalent:*

(i) *An Orlicz space* L^o_Φ (L_Φ) *is uniformly noncreasy.*

(ii) Φ *and* Ψ *satisfy the* Δ_2*-condition for large* u *and one of the two conditions is fulfilled:*

  (a) Φ *is strictly convex and* Φ *is uniformly convex for large arguments.*

  (b) Ψ *is strictly convex and* Ψ *is uniformly convex for large arguments.*

(iii) *An Orlicz space* L^o_Φ (L_Φ) *is uniformly convex or uniformly smooth.*

Proof. We will prove only the case of the Orlicz norm, because for the Luxemburg norm the same ideas can be applicable.

(i) $\Longrightarrow$ (ii): If $L^o_\Phi \in (UNC)$, then L^o_Φ is reflexive (see [20]). Consequently $\Phi \in \Delta_2^l$ and $\Psi \in \Delta_2^l$, whence $(L^o_\Phi)^* = L_\Psi$ (see [3]). Suppose that both conditions (a) and (b) are not satisfied. We need to show that $L^o_\Phi \notin (UNC)$, i.e. there exist a number $\varepsilon_0 > 0$ and sequences $(x^*_n), (y^*_n)$ in $S(L_\Psi)$ such that

$$\|x^*_n - y^*_n\|_\Psi \geq \varepsilon_0 \tag{3}$$

and

$$\operatorname{diam} S(x^*_n, y^*_n, 1/n) > \varepsilon_0 \tag{4}$$

for each $n \in \mathbb{N}$.

We divide the proof into 4 parts.

A. Assume that $\Phi \notin (SC)$ and $\Psi \notin (SC)$. Then $\Phi \notin (SC)$ and p is not continuous. Thus $L^o_\Phi \notin (NC)$ by Theorem 3 in [4]. Note that the similar arguments as in case B can be applied in the proof.

B. Suppose that $\Phi \in (SC)$, $\Psi \in (SC)$ and Φ, Ψ are not uniformly convex for large arguments. Then, by Lemma 1.17 in [3], there exist positive numbers ε_1, ε_2 and sequences (u_n) and (v_n) of real numbers tending to infinity such that

$$p((1+\varepsilon_1)u_n) < (1+1/n)p(u_n), \tag{5}$$

$$q((1+\varepsilon_2)v_n) < (1+1/n)\,q(v_n). \tag{6}$$

Note that $\Phi, \Psi \in \Delta_2^l$, so Φ and Ψ are finitely valued. Hence, since $v_n \to \infty$ and $u_n \to \infty$, without loss of generality we may assume that there are disjoint measurable subsets T_1^n, T_2^n of T such that $\mu(T \setminus (T_1^n \cup T_2^n)) > \mu(T)/2$ and

$$\Psi[(1+\varepsilon_2)v_n]\,\mu(T_1^n) = \Psi[p((1+\varepsilon_1)u_n)]\,\mu(T_2^n) = \frac{1}{2} \tag{7}$$

for each $n \in \mathbb{N}$. We decompose each set T_i^n $(i = 1, 2)$ into two disjoint subsets T_{i1}^n and T_{i2}^n with

$$\mu(T_{i1}^n) = \mu(T_{i2}^n) \quad \text{for } i = 1, 2. \tag{8}$$

Take a_0 such that $\Psi\left(p(a_0)\right) > \max\left\{2/\mu(T), 1\right\}$. For each $n \in \mathbb{N}$ there is a measurable set $T_3^n \subset T \setminus \left(T_1^n \cup T_2^n\right)$ with

$$\begin{aligned} &\Psi\left[v_n\right]\mu\left(T_{11}^n\right) + \Psi\left[\left(1+\varepsilon_2\right)v_n\right]\mu\left(T_{12}^n\right) + \Psi\left[p\left(u_n\right)\right]\mu\left(T_{21}^n\right) \\ &\quad + \Psi\left[p\left(\left(1+\varepsilon_1\right)u_n\right)\right]\mu\left(T_{22}^n\right) + \Psi\left(p\left(a_0\right)\right)\mu\left(T_3^n\right) = 1. \end{aligned} \tag{9}$$

Put

$$\begin{aligned} k_n = {} & v_n q(v_n)\mu\left(T_{11}^n\right) + (1+\varepsilon_2)v_n q\left(\left(1+\varepsilon_2\right)v_n\right)\mu\left(T_{12}^n\right) + u_n p\left(u_n\right)\mu\left(T_{21}^n\right) \\ & + \left(1+\varepsilon_1\right)u_n p\left(\left(1+\varepsilon_1\right)u_n\right)\mu\left(T_{22}^n\right) + a_0 p\left(a_0\right)\mu\left(T_3^n\right) \end{aligned} \tag{10}$$

and

$$x_n = (q(v_n)\chi_{T_{11}^n} + q((1+\varepsilon_2)v_n)\chi_{T_{12}^n} + u_n\chi_{T_{21}^n} + (1+\varepsilon_1)u_n\chi_{T_{22}^n} + a_0\chi_{T_3^n})/k_n,$$

$$y_n = (q(v_n)\chi_{T_{11}^n} + q((1+\varepsilon_2)v_n)\chi_{T_{12}^n} + (1+\varepsilon_1)u_n\chi_{T_{21}^n} + u_n\chi_{T_{22}^n} + a_0\chi_{T_3^n})/k_n,$$

$$x_n^* = v_n\chi_{T_{11}^n} + (1+\varepsilon_2)v_n\chi_{T_{12}^n} + p(u_n)\chi_{T_{21}^n} + p((1+\varepsilon_1)u_n)\chi_{T_{22}^n} + p(a_0)\chi_{T_3^n},$$

$$y_n^* = (1+\varepsilon_2)v_n\chi_{T_{11}^n} + v_n\chi_{T_{12}^n} + p(u_n)\chi_{T_{21}^n} + p((1+\varepsilon_1)u_n)\chi_{T_{22}^n} + p(a_0)\chi_{T_3^n}.$$

We will show that conditions (3) and (4) are satisfied by taking the above defined sequences. First note that, by (8) and (9), $I_\Psi(x_n^*) = I_\Psi(y_n^*) = 1$. Hence $x_n^*, y_n^* \in S\left(L_\Psi\right)$ for each $n \in \mathbb{N}$. Moreover, by $\Psi \in \Delta_2^l$, there exists $v_0 > 0$ and for $\alpha = \max\left\{2, 1+\varepsilon_2, 1/\varepsilon_2\right\}$ there is $k_\alpha > 2$ such that

$$\Psi\left(\alpha v\right) \le k_\alpha \Psi\left(v\right) \tag{11}$$

for every $|v| \ge v_0$ (see [3]). Consequently $\Psi\left(\varepsilon_2 v\right) \ge \beta\Psi\left(v\right)$ for each $|v| \ge v_0/\varepsilon_2$, where $\beta = 1/k_\alpha$. Since $v_n \to \infty$, without loss of generality, we assume that $v_n \ge v_0/\varepsilon_2$ for any $n \in \mathbb{N}$. Hence, by (7) and (11),

$$\begin{aligned} I_\Psi\left(2k_\alpha^2\left(x_n^* - y_n^*\right)\right) &\ge 2k_\alpha^2\Psi\left(\varepsilon_2 v_n\right)\mu\left(T_1^n\right) \ge 2k_\alpha^2\beta\Psi\left(v_n\right)\mu\left(T_1^n\right) \\ &\ge 2k_\alpha\Psi\left(\left(1+\varepsilon_2\right)v_n\right)\beta\mu\left(T_1^n\right) = 1. \end{aligned} \tag{12}$$

Then

$$\|x_n^* - y_n^*\|_\Psi \ge 1/2k_\alpha^2 \tag{13}$$

for all $n \in \mathbb{N}$. Now we will show that $x_n, y_n \in S\left(x_n^*, y_n^*, 1/n\right)$. Note that $v = p\left(q\left(v\right)\right)$ for any $v \ge 0$, since $\Psi \in (SC)$, so q is strictly increasing.

Hence $I_\Psi(p(k_n x_n)) = 1$ for any $n \in \mathbb{N}$, by (9). Thus, by (1), (2), (9) and (10), we get

$$\|x_n\|_\Phi^o = \frac{1 + I_\Phi(k_n x_n)}{k_n} = \frac{1 + \Phi(q(v_n))\,\mu(T_{11}^n) + \Phi(q((1+\varepsilon_2)v_n))\,\mu(T_{12}^n)}{k_n}$$
$$+ \frac{\Phi(u_n)\,\mu(T_{21}^n) + \Phi((1+\varepsilon_1)u_n)\,\mu(T_{22}^n) + \Phi(a_0)\,\mu(T_3^n)}{k_n}$$
$$= \frac{\{\Psi[v_n] + \Phi(q(v_n))\}\,\mu(T_{11}^n) + \{\Psi[(1+\varepsilon_2)v_n] + \Phi(q((1+\varepsilon_2)v_n))\}\,\mu(T_{12}^n)}{k_n}$$
$$+ \frac{\{\Psi[p(u_n)] + \Phi(u_n)\}\,\mu(T_{21}^n) + \{\Psi[p((1+\varepsilon_1)u_n)] + \Phi((1+\varepsilon_1)u_n)\}\,\mu(T_{22}^n)}{k_n}$$
$$+ \frac{\{\Psi(p(a_0)) + \Phi(a_0)\}\,\mu(T_3^n)}{k_n} = 1$$

for each $n \in \mathbb{N}$. Similarly $\|y_n\|_\Phi^o = 1$ for every $n \in \mathbb{N}$. Furthermore, by (10), $x_n^* x_n = 1$ for all $n \in \mathbb{N}$. Moreover, the inequalities (6), (8) and (10) yield

$$y_n^* x_n = \frac{(1+\varepsilon_2)v_n q(v_n)\mu(T_{11}^n) + v_n q((1+\varepsilon_2)v_n)\,\mu(T_{12}^n) + u_n p(u_n)\mu(T_{21}^n)}{k_n}$$
$$+ \frac{(1+\varepsilon_1)\,u_n p((1+\varepsilon_1)\,u_n)\,\mu(T_{22}^n) + a_0 p(a_0)\,\mu(T_3^n)}{k_n}$$
$$> \frac{(1+\varepsilon_2)v_n q((1+\varepsilon_2)v_n)\,\mu(T_{11}^n) + v_n q(v_n)\mu(T_{12}^n) + u_n p(u_n)\mu(T_{21}^n)}{k_n(1+1/n)}$$
$$+ \frac{(1+\varepsilon_1)\,u_n p((1+\varepsilon_1)\,u_n)\,\mu(T_{22}^n) + a_0 p(a_0)\,\mu(T_3^n)}{k_n}$$
$$> \frac{1}{1+1/n} > 1 - 1/n$$

for each $n \in \mathbb{N}$. Then $x_n \in S(x_n^*, y_n^*, 1/n)$ for every $n \in \mathbb{N}$. Analogously as above, applying inequalities (5), (8) and (10), one can easily get $x_n^* y_n > 1 - 1/n$ for every $n \in \mathbb{N}$. Similarly, using now inequalities (5), (6), (8) and (10), we obtain $y_n^* y_n > 1 - 1/n$ for all $n \in \mathbb{N}$. Thus $y_n \in S(x_n^*, y_n^*, 1/n)$ for every $n \in \mathbb{N}$. To finish the proof we evaluate the norm $\|x_n - y_n\|_\Phi^o$. From equality (7) we conclude $\Psi^{-1}\left(\frac{1}{2\mu(T_2^n)}\right) = p((1+\varepsilon_1)u_n)$, whence

$$\begin{aligned}\|x_n - y_n\|_\Phi^o &= \frac{\varepsilon_1 u_n}{k_n}\mu\left(T_2^n\right)\Psi^{-1}\left(\frac{1}{\mu\left(T_2^n\right)}\right) \\ &\geq \frac{\varepsilon_1 u_n}{k_n}\mu\left(T_2^n\right)p\left(\left(1+\varepsilon_1\right)u_n\right).\end{aligned} \tag{14}$$

Applying equality (1) we deduce

$$\left(1+\varepsilon_1\right)u_n p\left(\left(1+\varepsilon_1\right)u_n\right) \geq \Psi\left(p\left(\left(1+\varepsilon_1\right)u_n\right)\right), \quad n \in \mathbb{N}. \tag{15}$$

From Proposition 1.6 in [3] we conclude

$$\frac{1}{2}vq\left(\frac{1}{2}v\right) \leq \Psi\left(v\right), \quad v > 0. \tag{16}$$

Then, applying (11), we get

$$\begin{gathered}\left(1+\varepsilon_2\right)v_n q\left(\left(1+\varepsilon_2\right)v_n\right) \leq \Psi\left(2\left(1+\varepsilon_2\right)v_n\right) \leq k_\alpha \Psi\left(\left(1+\varepsilon_2\right)v_n\right), \\ n \in \mathbb{N}.\end{gathered} \tag{17}$$

Note that $\mu\left(T_3^n\right) < \mu\left(T_3^n\right)\Psi\left(p\left(a_0\right)\right) \leq 1/2$, by (7), (8) and (9). Consequently, by (7), (10), (15) and (17) we get

$$\frac{k_n}{\left(1+\varepsilon_1\right)u_n p\left(\left(1+\varepsilon_1\right)u_n\right)\mu\left(T_{22}^n\right)}$$

$$\leq \frac{\left(1+\varepsilon_2\right)v_n q\left(\left(1+\varepsilon_2\right)v_n\right)\mu\left(T_1^n\right) + a_0 p\left(a_0\right)\mu\left(T_3^n\right)}{\left(1+\varepsilon_1\right)u_n p\left(\left(1+\varepsilon_1\right)u_n\right)\mu\left(T_{22}^n\right)}$$

$$+2 \leq \frac{k_\alpha \Psi\left(\left(1+\varepsilon_2\right)v_n\right)\mu\left(T_1^n\right) + a_0 p\left(a_0\right)}{\Psi\left(p\left(\left(1+\varepsilon_1\right)u_n\right)\right)\mu\left(T_{22}^n\right)} + 2 = \frac{\frac{k_\alpha}{2} + a_0 p\left(a_0\right)}{\frac{1}{4}} + 2.$$

Denote $\lambda = 2k_\alpha + 4a_0 p\left(a_0\right) + 2$. Hence the inequality (14) yields

$$\|x_n - y_n\|_\Phi^o \geq \frac{\varepsilon_1 u_n p\left(\left(1+\varepsilon_1\right)u_n\right)\mu\left(T_2^n\right)}{\lambda\left(1+\varepsilon_1\right)u_n p\left(\left(1+\varepsilon_1\right)u_n\right)\mu\left(T_{22}^n\right)} > \frac{2\varepsilon_1}{\lambda\left(1+\varepsilon_1\right)}.$$

Finally, denoting $\varepsilon_0 = \min\left\{\frac{2\varepsilon_1}{\lambda(1+\varepsilon_1)}, 1/2k_\alpha^2\right\}$, where $1/2k_\alpha^2$ is from (13), we conclude that conditions (3) and (4) are satisfied.

C. Suppose that $\Phi \notin (SC)$, $\Psi \in (SC)$ and Ψ is not uniformly convex for large arguments. Hence p is constant in some interval $[a,b]$, $0 < a < b$ and there exist positive number ε_2 and sequence (v_n) of real numbers tending to infinity such that $q\left((1+\varepsilon_2)\, v_n\right) < (1+1/n)\, q\,(v_n)$ for each n. Then we may step the same way as in case B by taking $u_n = a, (1+\varepsilon_1)\, u_n = b$, $n \in \mathbb{N}$ and $T_2^n = T_2$ of positive measure is chosen so that $\mu\,(T \backslash T_2) > 3\mu\,(T)\,/4$. Next for each $n \in \mathbb{N}$ we take a set T_1^n disjoint with T_2 such that $\mu\,(T \backslash \left(T_1^n \cup T_2\right)) > \mu\,(T)\,/2$ and $\Psi[(1+\varepsilon_2)v_n]\mu(T_1^n) = \frac{1}{2}$. Note that the second term in (7) disappears in this case. It may happen that $\Psi \in (SC)$ and Φ vanishes outside zero.

C.1. Suppose that $p(a) = p(b) = 0$. Then the third and fourth term in (9), (10) and in formulas of x_n^*, y_n^* disappear. However, the proof goes the same way. Note only that $\|x_n - y_n\|_{\Phi}^{o} = \frac{b-a}{k_n}\mu(T_2)\Psi^{-1}\left(\frac{1}{\mu(T_2)}\right)$ and one can analogously apply inequality (17) to finish the proof.

C.2. If $p\,(a) = p\,(b) > 0$, the proof is analogous. We need only additionally assume that the set T_2 is chosen to satisfy $0 < \Psi\,(p\,(b))\,\mu\,(T_2) \leq 1/2$.

D. Assume that $\Psi \notin (SC)$, $\Phi \in (SC)$ and Φ is not uniformly convex for large arguments. Hence q is constant in an interval $[c,d]$, $0 < c < d$ and there exist positive number ε_1 and sequence (u_n) of real numbers tending to infinity such that $p\left((1+\varepsilon_1)\, u_n\right) < (1+1/n)\, p\,(u_n)$, $n \in \mathbb{N}$. We consider two cases.

D.1. Suppose that Ψ vanishes outside zero and $q(c) = q(d) = \Psi(c) = \Psi(d) = 0$. Then we may step similarly as in case B with $v_n = c$, $(1+\varepsilon_2)v_n = d$, $n \in \mathbb{N}$ and $T_1^n = T_1$ is chosen so that $0 < \mu(T_1) < \mu(T)/4$. Then $\|x_n^* - y_n^*\|_{\Psi} = \|(d-c)\chi_{T_1}\|_{\Psi} > 0$. Clearly, the first term in (7), the first and second term in (9), (10) and in formulas of x_n, y_n disappear.

D.2. Assume that Ψ vanishes only at zero. Then we assume that the interval $[c,d]$ is a structural affine interval of Ψ (i.e. Ψ is affine on $[c,d]$ and it is not affine on either $[c-\delta, d]$ nor $[c, d+\delta]$ for any $\delta > 0$). Hence we find $v_n < c$ such that $q(d) < (1+1/n)q\,(v_n)$ and $p\,(q\,(v_n)) = v_n$. Taking $T_1^n = T_1$, where $0 < \Psi\,(d)\,\mu\,(T_1) \leq 1/2$ and replacing $(1+\varepsilon_2)\, v_n$ by d in (9), (10) and in the formulas for x_n, y_n, x_n^*, y_n^* we may step similarly as in case B. Then $I_{\Psi}\left(s\left(x_n^* - y_n^*\right)\right) \geq s\Psi\,(d-c)\,\mu\,(T_1) \geq 1$, where $s = \max\left\{1, (\Psi(d-c)\mu(T_1))^{-1}\right\}$, so $\|x_n^* - y_n^*\|_{\Psi} \geq 1/s$. We do not need to use the Δ_2^l condition for Ψ in order to obtain the estimation of the norm $\|x_n - y_n\|_\Phi^o$. We simply get

$$\frac{k_n}{(1+\varepsilon_1)\, u_n p\left((1+\varepsilon_1)\, u_n\right) \mu\left(T_{22}^n\right)} \leq \frac{dq\,(d)\,\mu\,(T_1) + a_0 p\,(a_0)}{1/4} + 2$$

and it is easy to finish the proof.

(ii) $\implies$ (iii): By the assumptions we have $L_\Phi^o \in (UC)$ or $L_\Psi \in (UC)$ (see [11]). But $(L_\Psi)^* = L_\Phi^o$, because $\Psi \in \Delta_2^l$ (see [3]). Moreover, a Banach space $X \in (UC)$ iff $X^* \in (US)$ (see [5]). Thus $L_\Phi^o \in (UC)$ or $L_\Phi^o \in (US)$.

(iii) $\implies$ (i): This follows immediately from [20]. □

Using the same methods as in the proof of Theorem 1, one can show the respective result for an infinite nonatomic measure (the required proof is even much simpler than in the case of finite measure). Namely

Theorem 2. *Assume that $\mu(T) = \infty$. The following assertions are equivalent:*

(i) *An Orlicz space L_Φ^o (L_Φ) is uniformly noncreasy.*

(ii) *Φ and Ψ satisfy the Δ_2-condition for all u and one of the two conditions are fulfilled:*

  (a) *Φ is uniformly convex for all arguments.*

  (b) *Ψ is uniformly convex for all arguments.*

(iii) *An Orlicz space L_Φ^o (L_Φ) is uniformly convex or uniformly smooth.*

The sketch of the proof. Only the implication (i) $\implies$ (ii) needs to be discussed. The existence of sequences (u_n) and (v_n) satisfying (5) and (6) follows analogously. However, we have to consider three cases for each of these sequences, namely $u_n \to 0$, $u_n \to \infty$ and $u_n \to u_0$ for some $u_0 > 0$, similarly for (v_n). Hence we divide the proof into three cases.

I. If $u_n \to 0$ or $u_n \to \infty$, $v_n \to 0$ or $v_n \to \infty$ and $\Phi, \Psi \in (SC)$, we follow analogously as in case B of Theorem 1.

II. If $u_n \to u_0$ for some $u_0 > 0$, $v_n \to 0$ or $v_n \to \infty$, $\Psi \in (SC)$, then Φ is affine on the interval $[u_0, (1+\varepsilon_1)\, u_0]$. Since $\Phi \in \Delta_2^a$, Φ vanishes only at zero and consequently we step as in case C.2.

III. If $v_n \to v_0$ for some $v_0 > 0$, $\Phi \in (SC)$, the situation is analogous to the case D.2.

3. Köthe spaces

In this section we would like to draw a comparison between the *uniform noncreaseness* and the *orthogonal uniform convexity*(each of them implies the fixed point property). But first we recall the necessary terminology.

Let $E = (E, \leq, \|\cdot\|_E)$ be a Banach function lattice over the measure space (T, Σ, μ) (with a σ-finite and complete measure μ), where $\leq$ is semi-order relation in the space L^0 and $(E, \|\cdot\|_E)$ is a Banach function space, i.e. E is linear subspace of L^0, norm $\|\cdot\|_E$ is complete in E and the following two conditions are satisfied:

(i) if $x \in E$, $y \in L^0$, $|y| \leq |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists function x in E that is positive on the whole T (see [17]).

We will call the space E the *Köthe space.* In particular, if we consider the space E over a non-atomic measure μ, then we will say that E is a *Köthe function space.* If we replace the measure space (T, Σ, μ) by a counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, m\right)$, then we will say that E is a *Köthe sequence space.*

A Köthe space E is said to be *strictly monotone* ($E \in (SM)$) if for every $0 \leq y \leq x$ with $y \neq x$ we have $\|y\|_E < \|x\|_E$. We say that a Köthe space E is *uniformly monotone* ($E \in (UM)$) if for every $q \in (0,1)$ there exists $p \in (0,1)$ such that for all $0 \leq y \leq x$ satisfying $\|x\|_E \leq 1$ and $\|y\|_E \geq q$ we have $\|x - y\|_E \leq 1 - p$ (see [1]).

A Köthe space E is called *order continuous* ($E \in (OC)$) if for every $x \in E$ and each sequence $(x_m) \in E$ such that $0 \leq x_m \leq |x|$ and $x_m \to 0$ we have $\|x_m\|_E \to 0$ (see [17]).

The geometry of Banach spaces is strictly connected with the fixed point theory (see [7]). The *orthogonal uniform convexity* is a geometric property related to the fixed point property. It was introduced in [13] during studies on property (β) of Rolewicz.

The notation $r \vee s = \max\{r, s\}$ for any $r, s \in \mathbb{R}$ and $A \div B = (A \setminus B) \cup (B \setminus A)$ for every $A, B \in \Sigma$ will be used.

Definition 2. A Köthe space $(E, \|\cdot\|_E)$ is orthogonally uniformly convex $\left(E \in \left(UC^{\perp}\right)\right)$, if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in B(E)$ the inequality $\left\|x\chi_{A_{xy}}\right\|_E \vee \left\|y\chi_{A_{xy}}\right\|_E \geq \varepsilon$ implies $\|(x+y)/2\|_E \leq 1 - \delta$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$.

Obviously, if $E \in (UC)$, then $E \in (UC^{\perp})$. It is known that uniformly convex Köthe spaces are uniformly monotone (see [9]). Moreover,

Lemma 1 (Lemma 3 in [13]). *If* $E \in \left(UC^{\perp}\right)$, *then* $E \in (UM)$.

The converse of Lemma 1 is not true as examples L^1, l^1 show.

It is known that orthogonal uniform convexity implies the fixed point property (see [14]). It is also connected with property (β) (property (β) has been introduced by Rolewicz during studies on well-posed problems in optimization theory – see [19]). Namely, $UC \overset{(1)}{\Rightarrow} UC^{\perp} \overset{(2)}{\Rightarrow} \beta$ in every Köthe sequence space, $UC \Rightarrow \beta \overset{(3)}{\Rightarrow} UC^{\perp}$ in every Köthe function space and the converse of implications (1), (2) and (3) is not true in general (see [13], [14] and [15]).

It can be deduced by Theorems 1, 2 and Theorem 4 from [14] that $UNC \Rightarrow UC^{\perp}$ in Orlicz function spaces with the Luxemburg norm. Furthermore, the Example 1.10 from [8] of Orlicz function space shows that this implication can not be reversed in general. However, one can pose a natural question whether the implication $UNC \Rightarrow UC^{\perp}$ can be extended to all Köthe spaces. The next example shows that this is not the case.

Example 1. Let $X_\beta = \left(l^2, \|\cdot\|\right)$, where $\|x\| = \max\{\|x\|_{l^2}, \beta\|x\|_{l^\infty}\}$ for $\beta \in (1, \sqrt{2}]$. Then $X_\beta \in (UNC)$ by Theorem 4 from [20]. Let $x = \frac{1}{\beta}e_1$ and $y = \frac{1}{\beta}e_1 + \sqrt{1-\frac{1}{\beta^2}}\, e_2$, where $e_i = (0,\dots,0,1,0,\dots)$ is the ith unit vector. Then $\|x\| = 1$ and $\|y\| = 1$, by $\beta\sqrt{1-\frac{1}{\beta^2}} \le 1$. Since $x \le y$ and $x \ne y$, so $X_\beta \notin (SM)$. Thus Lemma 1 yields $X_\beta \notin \left(UC^{\perp}\right)$.

The previous example suggests the following question. If a uniformly noncreasy and uniformly monotone Köthe space is orthogonally uniformly convex. However, this is also not true.

Example 2. Given Banach spaces X_0 and X_1, by $(X_0 \times X_1)_\infty$ we denote the product $X_0 \times X_1$ with the norm $\|(x_0, x_1)\|_\infty = \max\{\|x_0\|, \|x_1\|\}$, where $x_0 \in X_0$ and $x_1 \in X_1$. Analogously $(X_0 \times X_1)_1$ stands for the product $X_0 \times X_1$ with the norm $\|(x_0, x_1)\|_1 = \|x_0\| + \|x_1\|$. Take $Y = (Z \times Z)_1$, where Z is a Köthe space which is both uniformly convex and uniformly smooth. Then $Y \in (UNC)$, by Proposition 1 in [20]. Note that Y is a Köthe–Bochner space $E(X)$ with $X = Z$ and $E = l^1_2$ (two-dimensional l^1).

Since $E, X \in (UM)$, so $Y \in (UM)$ (see [2]). On the other hand, since $l_2^1 \notin \left(UC^{\perp}\right)$ and E, X are isometrically embedded into $E(X)$, so $Y \notin \left(UC^{\perp}\right)$ (see also [16]).

Note that for the space Y from the previous example we have $Y^* = E^*\left(X^*\right)$ with $E^* = l_2^{\infty}$. Since $l_2^{\infty} \notin (UM)$, so $Y^* \notin (UM)$. This suggests the following

Question. Do conditions $E, E^* \in (UM)$ and $E \in (UNC)$ imply $E \in (UC^{\perp})$?

Theorem 3. *Suppose that E, $E^* \in (UM)$. Then $E \in \left(UC^{\perp}\right)$.*

Proof. Assume for the contrary that $E, E^* \in (UM)$ and $E \notin \left(UC^{\perp}\right)$. Then there are $\varepsilon_1 > 0$ and sequences $(x_n), (y_n)$ in $B(E)$ such that

$$\|x_n \chi_{A_n}\|_E \vee \|y_n \chi_{A_n}\|_E \geq \varepsilon_1 \quad \text{and} \quad \|x_n + y_n\|_E \to 2, \tag{18}$$

where $A_n = \operatorname{supp} x_n \div \operatorname{supp} y_n$. In view of the inequality $\|x_n + y_n\|_E \leq \|\,|x_n| + |y_n|\,\|_E$ we may assume that x_n, y_n are nonnegative. Applying Hahn–Banach Theorem we find a sequence $(x_n^*)_{n=1}^{\infty}$ in $S(E^*)$ with $x_n^*((x_n + y_n)/2) = \|(x_n + y_n)/2\|_E$, $n \in \mathbb{N}$. Applying (18) we conclude

$$x_n^* x_n \to 1 \quad \text{and} \quad x_n^* y_n \to 1. \tag{19}$$

Recall that a *Köthe dual* $E^{'}$ of E is defined by

$$E^{'} = \left\{ h \in L^0 : \|h\|_{E'} = \sup \left\{ \int_T |h(t) g(t)|\, d\mu : g \in E,\ \|g\|_E \leq 1 \right\} < \infty \right\}.$$

It is known that $E^{'}$ is a Banach function lattice. Recall that if $E \in (UM)$, then $E \in (OC)$ (see Proposition 2.1 in [6]). Moreover $E^* = E^{'}$ iff $E \in (OC)$ (see [17]). Let $A_n^1 = \operatorname{supp} x_n \setminus \operatorname{supp} y_n$. In virtue of (18), without loss of generality, passing to a subsequence, if necessary, we may assume that $\left\|x_n \chi_{A_n^1}\right\|_E \geq \varepsilon_1$ for any $n \in \mathbb{N}$. Recall that $E \in (UM)$ iff for any $\sigma \in (0, 1)$ there is $\eta_E(\sigma) > 0$ such that for any $x \in E$, $x \geq 0$ with $\|x\|_E = 1$ and for any $A \in \Sigma$

$$\text{if } \|x \chi_A\|_E \geq \sigma \text{ then } \left\|x \chi_{T \setminus A}\right\|_E \leq 1 - \eta_E(\sigma) \quad \text{(Theorem 6 in [9]).} \tag{20}$$

Consequently condition (20) with $\eta_1 = \eta_E(\varepsilon_1)$ implies $\left\|x_n \chi_{T \setminus A_n^1}\right\|_E \leq 1 - \eta_1$, whence $\left|x_n^* x_n \chi_{T \setminus A_n^1}\right| \leq 1 - \eta_1$. Hence, by (19), $\left|x_n^* x_n \chi_{A_n^1}\right| \geq \eta_1/2$

for sufficiently large n. Consequently

$$\left\|x_n^* \chi_{A_n^1}\right\|_{E^*} \geq \eta_1/2 \tag{21}$$

for infinitely many $n \in \mathbb{N}$. Consequently, since $E^* \in (UM)$, again condition (20) with $\eta_2 = \eta_{E^*}(\eta_1/2)$ yields $\left\|x_n^* \chi_{T \setminus A_n^1}\right\|_{E^*} \leq 1 - \eta_2$ for infinitely many $n \in \mathbb{N}$. On the other hand, by (19), $\left\|x_n^* \chi_{T \setminus A_n^1}\right\|_{E^*} \geq x_n^* \chi_{T \setminus A_n^1} y_n = x_n^* y_n \to 1$. This contradiction proves the theorem. □

Taking into account that every orthogonally uniformly convex Köthe space is reflexive (Corollary 1 in [14]), applying Lemma 1 and Theorem 3, we conclude

Corollary 1. *$E \in (UM)$ and $E^* \in \left(UC^{\perp}\right)$ if and only if $E^* \in (UM)$ and $E \in \left(UC^{\perp}\right)$.*

Note that uniform noncreaseness is self-dual property (i.e. $X \in (UNC)$ iff $X^* \in (UNC)$, see [20]).

Remark 1. Orthogonal uniform convexity is not self-dual property (equivalently the converse of Theorem 3 is not true). Indeed, let $E = l_\varphi$ be the Musielak–Orlicz sequence space with the Luxemburg norm generated by a Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^\infty$, where

$$\varphi_1(u) = \begin{cases} u/2 & \text{if } 0 \leq u \leq 1, \\ \dfrac{1}{4}u^2 + \dfrac{1}{4} & \text{if } u > 1 \end{cases} \qquad \text{and}$$

$$\varphi_i(u) = u^2 \text{ for } u \geq 0,\ i = 2, 3, \ldots .$$

One can compute a function $\varphi^* = (\varphi_i^*)_{i=1}^\infty$ which is complementary to $\varphi = (\varphi_i)_{i=1}^\infty$. Then

$$\varphi_1^*(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 1/2, \\ v^2 - \dfrac{1}{4} & \text{if } v > 1/2 \end{cases} \qquad \text{and}$$

$$\varphi_i^*(v) = v^2 \quad \text{for } v \geq 0,\ i = 2, 3, \ldots .$$

Hence φ and φ^* satisfy the δ_2-condition. It is also easy to check that φ fulfills the condition $(*)$. Moreover, $(\varphi_i)_{i \geq 2}$ is uniformly convex in 1-neighborhood of zero. We refer to [12] to the respective definitions of conditions δ_2, $(*)$ and uniform convexity of (φ_i). Then $l_\varphi \in \left(UC^\perp\right)$ by Theorem 6 in [15]. Since $\varphi \in \delta_2$, $E^* = l^o_{\varphi^*}$ (The Musielak–Orlicz space ℓ_{φ^*} with the Orlicz norm). But one can easy show that $l^o_{\varphi^*} \notin (SM)$. It is enough to take $x = (1/2e_1 + 1/2e_2)/k$ and $y = (1/4e_1 + 1/2e_2)/k$ with $k = \frac{5}{4}$. Then $y \leq x$, $y \neq x$ and $\|x\|^o_{\varphi^*} = \|y\|^o_{\varphi^*} = 1$.

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