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The cohomology of S-sets

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Abstract. The triple cohomology theory of semigroup acts is studied.

Introduction

The cohomology of S-sets, where S is any given monoid, can be approached in two ways.

In [4] we defined group coextensions of a right S-set A by an abelian group valued functor $\mathbb{G}$ on A, and showed that equivalence classes of group coextensions of A by $\mathbb{G}$ are the elements of an abelian group. If for instance S is commutative, this abelian group classifies the ways in which an arbitrary S-set can be constructed from an atransitive S-set and simply transitive group actions. This invites a general cohomology theory for S-sets, with abelian group valued functors for coefficients, whose second group would classify group coextensions.

In [2] Beck showed that every variety has a triple cohomology theory, with certain abelian group objects as coefficients, whose second group (called H^1 in [1],[2]) classifies certain extensions. This general construction yields a number of algebraic cohomology theories [2], [1], including the usual cohomology of groups, the Leech cohomology of monoids [5], [8], and commutative semigroup cohomology [3].

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For the variety of right S-sets we show in this article that the two approaches agree in dimension 2. Given a right S-set A we show in Section 1 that the abelian group objects which serve as coefficients in the triple cohomology of A may be identified with abelian group valued functors on A (up to an equivalence of categories). With this identification, we show in Section 2 that Beck extensions of an abelian group valued functor $\mathbb{G}$ by A may be identified with group coextensions of A by $\mathbb{G}$ (up to an isomorphism of categories); and we obtain in Section 3 a more concrete definition of the triple cohomology of right S-sets, along with its basic properties.

1. Abelian group objects

1. In what follows, S is a monoid and A is a given right S-set (a set A on which S acts so that $a1 = 1$ and $(as)t = a(st)$ for all $a \in A$ and $s, t \in S$). A *homomorphism* of right S-sets is a mapping f which preserves the action of S ($f(xs) = f(x)s$ for all x and s). Right S-sets and their homomorphisms are the objects and morphisms of a category $\mathcal{C}$.

A *right S-set over A* is a pair $\bar{X} = (X, \xi)$ of a right S-set X and an action preserving mapping $\xi : X \to A$; we use the exponential notation x^s for the action of S on X to distinguish it from forthcoming group actions. Equivalently, X is a right S-set which is a disjoint union $X = \bigcup_{a \in A} X_a$ in which $X_a^s \subseteq X_{as}$; then $\xi(x) = a$ when $x \in X_a$.

A *homomorphism* $f : \bar{X} \to \bar{Y} = (Y, \upsilon)$ of right S-sets over A is an action preserving mapping $f : X \to Y$ such that $\upsilon \circ f = \xi$; equivalently, $f(X_a) \subseteq Y_a$ for all $a \in A$. Right S-sets over A and their homomorphisms are the objects and morphisms of a category $\bar{\mathcal{C}}$.

An *abelian group object* of $\bar{\mathcal{C}}$ is a right S-set $\bar{G} = (G, \gamma)$ over A together with an "external" abelian group operation on every $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$, which we write additively, such that $\mathrm{Hom}_{\bar{\mathcal{C}}}(-, \bar{G})$ is a (contravariant) abelian group valued functor on $\bar{\mathcal{C}}$; equivalently, such that

$$(g + h) \circ f = (g \circ f) + (h \circ f)$$

whenever $f : \bar{X} \to \bar{Y}$ and $g, h : \bar{Y} \to \bar{G}$ are morphisms in $\bar{\mathcal{C}}$. Abelian group objects can also be defined by an "internal" addition $\bar{G} \times \bar{G} \to \bar{G}$, as in [6] or in Lemma 1.2 below.

A *homomorphism* $\varphi : \bar{G} \to \bar{H}$ of abelian group objects of $\bar{\mathcal{C}}$ is a morphism in $\bar{\mathcal{C}}$ such that $\mathrm{Hom}_{\bar{\mathcal{C}}}(-, \varphi)$ is a natural transformation; equivalently, such that

$$\varphi \circ (g + h) = (\varphi \circ g) + (\varphi \circ h)$$

whenever $g, h : \bar{X} \to \bar{G}$ are morphisms in $\bar{\mathcal{C}}$. Abelian group objects of $\bar{\mathcal{C}}$ and their homomorphisms are the objects and morphisms of a category.

2. To probe right S-sets over A we use the following construction.

Lemma 1.1. *Let S^S be the right S-set in which S acts on itself by right multiplication ($s^t = st$).*

(1) *For every right S-set X and $x \in X$ there is a unique homomorphism $x^* : S^S \to X$ such that $x^*(1) = x$, namely $x^*(s) = x^s$.*

(2) *For every $a \in A$, $\bar{S}_a = (S^S, a^*)$ is a right S-set over A.*

(3) *For every right S-set $\bar{X}$ over A and $x \in X_a$, x^* is the unique homomorphism $f : \bar{S}_a \to \bar{X}$ such that $f(1) = x$; hence $x \mapsto x^*$ and $f \mapsto f(1)$ are mutually inverse bijections between $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{X})$ and X_a.*

PROOF. x^* is action preserving since $x^*(s^t) = x^*(st) = x^{st} = (x^s)^t = x^*(s)^t$ for all $s, t \in S$. Then x^* is the unique action preserving mapping $S^S \to X$ such that $x^*(1) = x$, since $s = 1^s$. If $\bar{X} = (X, \xi)$ is a right S-set over A and $x \in X_a$, then $\xi \circ x^* = a^*$, since $\xi(x^*(1)) = \xi(x) = a = a^*(1)$, and $x^* \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{X})$. Then $x \mapsto x^*$ and $f \mapsto f(1)$ are mutually inverse bijections between $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{X})$ and X_a, by (1). □

Applying Lemma 1.1 to an abelian group object $\bar{G} = (G, \pi)$ of $\bar{\mathcal{C}}$ yields a partial addition on G.

Lemma 1.2. *When $\bar{G} = (G, \pi)$ is an abelian group object over A:*

(1) *π is surjective;*

(2) *For every $a \in A$ an abelian group addition on G_a is defined by*

$$g + h = (g^* + h^*)(1)$$

for all $g, h \in G_a$, and satisfies $(g + h)^* = g^* + h^*$;

(3) *$(g + h)^s = g^s + h^s$ for all $g, h \in G_a$ and $s \in S$;*

(4) *The addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ is pointwise for every $\bar{X}$.*

Proof. (1, 2): Let $g, h \in G_a$. Then g^*, $h^* \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$ by Lemma 1.1; g^*, h^* may be added in $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$; and $g + h \in G_a$ may be defined as in (2). Then

$$(g + h)^* = g^* + h^*,$$

since $(g + h)^*(1) = g + h = (g^* + h^*)(1)$. Then $g \mapsto g^*$ is an isomorphism of G_a onto $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$, by Lemma 1.1, and G_a is an abelian group under addition. In particular, $G_a \neq \emptyset$ for all a; hence π is surjective.

(4): Let $\bar{g}, \bar{h} : \bar{X} \to \bar{G}$. For every $x \in X_a$ we have $\bar{g} \circ x^* = \bar{g}(x)^*$, since $\bar{g}(x^*(1)) = \bar{g}(x)$. Hence

$$(\bar{g} + \bar{h}) \circ x^* = (\bar{g} \circ x^*) + (\bar{h} \circ x^*) = \bar{g}(x)^* + \bar{h}(x)^* = \big(\bar{g}(x) + \bar{h}(x)\big)^*;$$

evaluating at 1 yields $(\bar{g} + \bar{h})(x) = \bar{g}(x) + \bar{h}(x)$.

(3): Since addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$ is pointwise we have

$$(g + h)^s = (g + h)^*(s) = (g^* + h^*)(s) = g^*(s) + h^*(s) = g^s + h^s. \qquad \square$$

By Lemma 1.2, $g \mapsto g^s$ is a homomorphism $\gamma_{a,s} : G_a \to G_{as}$ for every a, s; $\gamma_{a,1}$ is the identity on G_a, and $\gamma_{as,t} \circ \gamma_{a,s} = \gamma_{a,st}$ for all a, s, t, since $(g^s)^t = g^{st}$.

This suggests an abelian group valued functor on the *transitivity category* $\mathcal{T}(A)$ of A [4], whose objects are the elements of A and whose morphisms are all pairs $(a, s) \in A \times S$, with $(a, s) : a \to as$ and $(as, t) \circ (a, s) = (a, st)$; the identity on $a \in A$ is $(a, 1)$. An abelian group valued functor $\mathbb{G} = (G, \gamma)$ on A (actually, on $\mathcal{T}(A)$) assigns an abelian group G_a to each $a \in A$ and a homomorphism $\gamma_{a,s} : G_a \to G_{as}$ to each $(a, s) \in A \times S$, so that $\gamma_{a,1}$ is the identity on G_a and $\gamma_{as,t} \circ \gamma_{a,s} = \gamma_{a,st}$, for all $s, t \in S$ and $a \in A$. It is convenient to write

$$\gamma_{a,s}(g) = g^s \in G_{as} \quad \text{when} \quad g \in G_a$$

so that

$$(g + h)^s = g^s + h^s,\ g^1 = g, \quad \text{and} \quad (g^s)^t = g^{st}$$

for all $g, h \in G_a$ and all s, t. We call $\mathbb{G}$ *thin* when $\gamma_{a,s}$ depends only on a and as (when $as = at$ implies $\gamma_{a,s} = \gamma_{a,t}$).

When $\bar{G}$ is an abelian group object over A, then the functor (G, γ) constructed after Lemma 1.2 is an abelian group valued functor on A. We state this as part of:

Proposition 1.3. *When $\bar{G}$ is an abelian group object over A, then $\mathbf{F}\bar{G} = (G, \gamma)$ is an abelian group valued functor on A. When $\varphi : \bar{G} \to \bar{H}$ is a homomorphism of abelian group objects over A, then $\mathbf{F}\varphi = \big(a\varphi_{|G_a}\big)_{a\in A}$ is a natural transformation from $\mathbf{F}\bar{G}$ to $\mathbf{F}\bar{H}$.*

Proof. First, $\varphi(G_a) \subseteq H_a$, since φ is a homomorphism of right S-sets over A. Let $\varphi_a = \varphi_{|G_a}$ be the restriction of φ to G_a. For every $g, h \in G_a$,

$$\varphi \circ (g+h)^* = \varphi \circ (g^* + h^*) = (\varphi \circ g^*) + (\varphi \circ h^*);$$

evaluating at 1 yields $\varphi(g+h) = \varphi(g) + \varphi(h)$, so that every φ_a is a homomorphism of abelian groups. Finally let $\mathbf{F}\bar{G} = (G, \gamma)$ and $\mathbf{F}\bar{H} = (H, \delta)$. Every square

$$\begin{tikzcd} G_a \arrow[r, "\varphi_a"] \arrow[d, "\gamma_{a,s}"'] & H_a \arrow[d, "\delta_{a,s}"] \\ G_{as} \arrow[r, "\varphi_{as}"'] & H_{as} \end{tikzcd}$$

commutes: since φ preserves the action of S we have

$$\varphi_{as}\big(\gamma_{a,s}(g)\big) = \varphi(g^s) = \varphi(g)^s = \delta_{a,s}\big(\varphi_a(g)\big)$$

for all $g \in G_a$, and $\mathbf{F}\varphi$ is a natural transformation from $\mathbf{F}\bar{G}$ to $\mathbf{F}\bar{H}$. $\square$

3. The converse of Proposition 1.3 is:

Proposition 1.4. *Let $\mathbb{G} = (G, \gamma)$ be an abelian group valued functor on A. Let $\bar{G} = (G', \pi)$, where G' is the disjoint union $G' = \bigcup_{a\in A}(G_a \times \{a\})$, $(g,a)^s = (\gamma_{a,s}(g), as) = (g^s, as)$, and $\pi(g,a) = a$. With the addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ defined for every $\bar{X}$ by*

$$(\bar{g} + \bar{h})(x) = (g_x + h_x, a), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x, a),\ \bar{h}(x) = (h_x, a),$$

$\mathbf{O}\mathbb{G} = \bar{G}$ is an abelian group object of $\bar{\mathcal{C}}$. When $\varphi : \mathbb{G} \to \mathbb{H}$ is a natural transformation of abelian group valued functors on A, then $\mathbf{O}\varphi : (g,a) \mapsto (\varphi_a(g), a)$ is a homomorphism of abelian group objects of $\bar{\mathcal{C}}$.

PROOF. G' is a right S-set since $(g,a)^1 = (g,a)$ and $\big((g,a)^s\big)^t = (g,a)^{st}$ when $g \in G_a$. Moreover π is action preserving. Hence $\bar{G}$ is a right S-set over A.

Let $\bar{g}, \bar{h} : \bar{X} \to \bar{G}$ be morphisms in $\bar{\mathcal{C}}$. If $x \in X_a$, then $\bar{g}(x)$, $\bar{h}(x) \in G_a$ and $\bar{g}(x) = (g_x, a)$, $\bar{h}(x) = (h_x, a)$ for some g_x, $h_x \in G_a$; hence a mapping $\bar{g} + \bar{h} : X \to G'$ may be defined by $(\bar{g} + \bar{h})(x) = (g_x + h_x, a)$ as in the statement. Since $\bar{g}$ and $\bar{h}$ are homomorphisms of right S-sets over A we have $\bar{g}(x^s) = \bar{g}(x)^s = (g_x, a)^s = (g_x^s, as)$, $\bar{h}(x^s) = (h_x^s, as)$, and

$$(\bar{g} + \bar{h})(x^s) = \big(g_x^s + h_x^s, as\big) = (g_x + h_x, a)^s = \big((\bar{g} + \bar{h})(x)\big)^s;$$

thus $\bar{g} + \bar{h}$ is a homomorphism of right S-sets over A. Addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ is commutative and associative like the addition on every G_a. The identity element $\bar{z} : \bar{X} \to \bar{G}$ of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ is given by $\bar{z}(x) = (0, a)$ when $x \in X_a$: indeed

$$\bar{z}(x^s) = (0, as) = (0, a)^s = \bar{z}(x)^s,$$

so $\bar{z} \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$, and $\bar{z} + \bar{g} = \bar{g}$ for all $\bar{g} \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$. The opposite of $\bar{g} : \bar{X} \to \bar{G}$ is similarly defined by $(-\bar{g})(x) = (-g_x, a)$ when $x \in X_a$ and $\bar{g}(x) = (g_x, a)$; $-\bar{g}$ is a homomorphism since $\bar{g}(x^s) = (g_x^s, as)$ and

$$(-\bar{g})(x^s) = (-g_x^s, as) = ((-g_x)^s, as) = (-g_x, a)^s = (-\bar{g})(x)^s.$$

$\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ is now an abelian group. If $f : \bar{W} \to \bar{X}$ is a morphism in $\bar{\mathcal{C}}$, then $(\bar{g} + \bar{h}) \circ f = (\bar{g} \circ f) + (\bar{h} \circ f)$ for all $\bar{g}, \bar{h} : \bar{X} \to \bar{G}$; hence $\bar{G}$ is a abelian group object of $\bar{\mathcal{C}}$.

Let $\varphi : \mathbb{G} \to \mathbb{H} = (H, \eta)$ be a natural transformation of abelian group valued functors on A. Then $\bar{\varphi} : (g, a) \mapsto (\varphi_a(g), a)$ satisfies

$$\begin{aligned}\bar{\varphi}\big((g,a)^s\big) &= \bar{\varphi}(\gamma_{a,s}(g), as) = \big(\varphi_{as}(\gamma_{a,s}(g)), as\big) \\ &= \big(\eta_{a,s}(\varphi_a(g)), as\big) = \big(\varphi_a(g), a\big)^s = \big(\bar{\varphi}(g,k)\big)^s\end{aligned}$$

and $\bar{\varphi}$ is a homomorphism of right S-sets over A. Let $\bar{g}, \bar{h} : \bar{X} \to \bar{G}$ be morphisms in $\bar{\mathcal{C}}$. Let $x \in X_a$ and $\bar{g}(x) = (g_x, a)$, $\bar{h}(x) = (h_x, a)$, so that $(\bar{g} + \bar{h})(x) = (g_x + h_x, a)$. Then $\bar{\varphi}(\bar{g}(x)) = (\varphi_a(g_x), a)$, $\bar{\varphi}(\bar{h}(x)) = (\varphi_a(h_x), a)$, and

$$(\bar{\varphi} \circ \bar{g} + \bar{\varphi} \circ \bar{h})(x) = \big(\varphi_a(g_x) + \varphi_a(h_x), a\big) = \big(\varphi_a(g_x + h_x), a\big) = \bar{\varphi}\big((\bar{g} + \bar{h})(x)\big).$$

Thus $\mathbf{O}\varphi = \bar{\varphi}$ is a homomorphism of abelian group objects of $\bar{\mathcal{C}}$. $\square$

Proposition 1.5. *The functors* $\mathbf{F}$ *and* $\mathbf{O}$ *in Propositions 1.3 and 1.4 are equivalences of categories.*

PROOF. Let $\bar{G}$ be an abelian group object of $\bar{\mathcal{C}}$. Then $\mathbf{OF}\bar{G} = \bar{G}' = (G', \pi)$, where G' is the disjoint union $G' = \bigcup_{a\in A}\big(G_a \times \{a\}\big)$, $(g,a)^s = (\gamma_{a,s}(g), as)$, and $\pi(g,a) = a$, and the addition on every $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G}')$ is defined by

$$(\bar{g} + \bar{h})(x) = (g_x + h_x, a), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x, a),\ \bar{h}(x) = (h_x, a).$$

Define $\theta_G : \bar{G}' \to \bar{G}$ by $\theta_G(g,a) = g \in G_a$. Then θ_G is an isomorphism of right S-sets over A. If moreover $\varphi : \bar{G} \to \bar{H}$ is a homomorphism of abelian group objects, then $\bar{\varphi} = \mathbf{OF}\varphi$ sends (g,a) to $(\varphi(g), a)$ and we see that $\theta_H \circ \bar{\varphi} = \varphi \circ \theta_G$. Thus θ_G is natural in $\bar{G}$.

Conversely let $\mathbb{G} = (G, \gamma)$ be an abelian group valued functor on A; then $\mathbf{O}\mathbb{G} = \bar{G}' = (G', \pi)$, where G' is the disjoint union $G' = \bigcup_{a\in A} (G_a \times \{a\})$, $(g,a)^s = (\gamma_{a,s}(g), as)$, and $\pi(g,a) = a$, and the addition on every $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G}')$ is defined by

$$(\bar{g} + \bar{h})(x) = (g_x + h_x, a), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x, a),\ \bar{h}(x) = (h_x, a).$$

The induced addition on $G'_a = G_a \times \{a\}$ is given as before by

$$(g,a) + (h,a) = \big((g,a)^* + (h,a)^*\big)(1)$$

for all (g,a), $(h,a) \in G'_a$ using the mappings x^* in Lemma 1.1; that is,

$$(g,a) + (h,a) = \big((g,a)^* + (h,a)^*\big)(1) = (g+h, a)$$

since $(g,a)^*(1) = (g,a)$ and $(h,a)^*(1) = (h,a)$. Thus $\theta_a : (g,a) \mapsto g$ is an isomorphism of abelian groups of G'_a onto G_a. Moreover the homomorphism $\delta_{a,s}$ in $\mathbf{F}\bar{G}' = (G', \delta)$ are given by $\delta_{a,s}(g,a) = (g,a)^s = (\gamma_{a,s}(g), as)$, which show that $\theta = (\theta_a)_{a\in A}$ is an isomorphism from $\mathbf{F}\bar{G}'$ to $\mathbb{G}$. It is immediate that θ is natural in $\mathbb{G}$. $\square$

2. Beck extensions

1. In $\bar{\mathcal{C}}$, a *left action* of an abelian group object $\bar{G}$ on an object $\bar{E}$ assigns to every object $\bar{X}$ of $\bar{\mathcal{C}}$ a left group action $\boldsymbol{.}$ of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on

$\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ which is natural in $\bar{X}$; equivalently, such that

$$(\bar{g} \cdot \bar{e}) \circ f = (\bar{g} \circ f) \cdot (\bar{e} \circ f)$$

for all morphisms $\bar{g} : \bar{X} \to \bar{G}$, $\bar{e} : \bar{X} \to \bar{E}$, $f : \bar{W} \to \bar{X}$. This "external" group action can be replaced by an "internal" action $\bar{G} \times \bar{E} \to \bar{E}$ as in Lemma 2.1 below.

In $\bar{\mathcal{C}}$, a *Beck extension* of an abelian group object $\bar{G}$ by A is a right S-set $\bar{E} = (E, \pi)$ over A together with an action of $\bar{G}$ on $\bar{E}$ such that

(Be1) $\mathbb{U}\pi \circ \mu = 1_{\mathbb{U}A}$ for some $\mu : \mathbb{U}A \to \mathbb{U}E$, where $\mathbb{U} : \mathcal{C} \to$ Sets is the forgetful functor; in Sets this merely states that π is surjective;

(Be2) for every $\bar{X}$ the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ preserves projection to A : $\pi \circ (\bar{g} \cdot \bar{e}) = \pi \circ \bar{e}$ for every $\bar{g} : \bar{X} \to \bar{G}$ and $\bar{e} : \bar{X} \to \bar{E}$;

(Be3) for every $\bar{X}$ the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ is simply transitive: for every $\bar{e}, \bar{f} : \bar{X} \to \bar{E}$ there exists a unique $\bar{g} : \bar{X} \to \bar{G}$ such that $\bar{g} \cdot \bar{e} = \bar{f}$.

A *homomorphism* $\varphi : \bar{E} \to \bar{F}$ of Beck extensions of $\bar{G}$ by A is a morphism in $\bar{\mathcal{C}}$ which preserves the action of $\bar{G}$:

$$\varphi \circ (\bar{g} \cdot \bar{e}) = \bar{g} \cdot (\varphi \circ \bar{e})$$

for all $\bar{X}$ and morphisms $\bar{g} : \bar{X} \to \bar{G}$, $\bar{e} : \bar{X} \to \bar{E}$.

2. Applying Lemma 1.1 to a Beck extension $\bar{E}$ of $\bar{G}$ by A yields a partial action of G on E.

Lemma 2.1. *Let $\bar{E}$ be a Beck extension of $\bar{G}$ by A; let $\mathbf{F}\bar{G} = (G, \gamma)$.*

(1) *For every $a \in A$ a simply transitive group action of G_a on E_a is defined by*

$$g \cdot x = (g^* \cdot x^*)(1)$$

for all $g \in G_a$, $x \in E_a$, and satisfies $(g \cdot x)^* = g^* \cdot x^*$.

(2) *$(g \cdot x)^s = g^s \cdot x^s$ for all $g \in G_a$, $x \in X_a$, and $s \in S$.*

(3) *The action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ is pointwise, for every $\bar{X}$.*

PROOF. (1): When $g \in G_a$ and $x \in E_a$, then $g^* \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$, $x^* \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{E})$ by Lemma 1.1, and $g^* \cdot x^*$ is defined in $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{E})$, and $g \cdot x \in E_a$ may be defined by $g \cdot x = (g^* \cdot x^*)(1)$. Then

$$(g \cdot x)^* = g^* \cdot x^*,$$

since $(g \cdot x)^*(1) = g \cdot x = (g^* \cdot x^*)(1)$. The isomorphism $g \mapsto g^*$ and bijection $x \mapsto x^*$ take the action of G_a on E_a to the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{E})$; therefore the former is, like the latter, a simply transitive group action.

(3): Let $\bar{g} : \bar{X} \to \bar{G}$ and $\bar{e} : \bar{X} \to \bar{E}$. For every $x \in X_a$ we have $\bar{g} \circ x^* = \bar{g}(x)^*$, since $\bar{g}(x^*(1)) = \bar{g}(x)$, and $\bar{e} \circ x^* = \bar{e}(x)^*$. Hence

$$(\bar{g} \cdot \bar{e}) \circ x^* = (\bar{g} \circ x^*) \cdot (\bar{e} \circ x^*) = \bar{g}(x)^* \cdot \bar{e}(x)^* = \big(\bar{g}(x) \cdot \bar{e}(x)\big)^*;$$

evaluating at 1 yields $(\bar{g} \cdot \bar{e})(x) = \bar{g}(x) \cdot \bar{e}(x)$.

(2): Since the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{S}_a, \bar{E})$ is pointwise we have

$$(g \cdot x)^s = (g \cdot x)^*(s) = (g^* \cdot x^*)(s) = g^*(s) \cdot x^*(s) = g^s \cdot x^s. \qquad \square$$

This last equation can be rewritten

$$(g \cdot x)^s = \gamma_{a,s}(g) \cdot x$$

and implies that E is a group coextension of A by $\mathbf{F}\bar{G}$ as defined in [4]. Specifically, a *group coextension* $(E, \pi, .)$ of A by a group valued functor $\mathbb{G} = (G, \gamma)$ on A consists of a right S-set E, an action-preserving surjection $\pi : E \to A$, and, for every $a \in A$, a simply transitive action . of G_a on E_a such that

$$(g \cdot x)^s = \gamma_{a,s}(g) \cdot x^s = g^s \cdot x^s$$

for all $g \in G_a$, $x \in E_a$, $a \in A$, and $s \in S$. An *equivalence* $\theta : (E, \pi, .) \to (F, \rho, .)$ of group coextensions of A by $\mathbb{G}$ is a bijection $\theta : E \to F$ which preserves the action of S ($\theta(x^s) = \theta(x)^s$), projection to A ($\rho(\theta(x)) = \pi(x)$) and the action of $\mathbb{G}$ ($\theta(g \cdot x) = g \cdot \theta(x)$).

Proposition 2.2. *Let $\bar{G}$ be an abelian group object of $\bar{\mathcal{C}}$ and $\mathbb{G} = \mathbf{F}\bar{G}$. When $\bar{E} = (E, \pi)$ is a Beck extension of $\bar{G}$ by A, then $\mathbf{C}\bar{E} = (E, \pi, .)$ is* *a group coextension of A by $\mathbb{G}$. When $\varphi : \bar{E} \to \bar{F}$ is a homomorphism of Beck extensions of $\bar{G}$ by A, then $\mathbf{C}\varphi = \varphi : \mathbf{C}\bar{E} \to \mathbf{C}\bar{F}$ is an equivalence of group coextensions.*

Proof. First, φ preserves projection to A ($\varphi(E_a) \subseteq F_a$) and the action of S, since φ is a homomorphism of right S-sets over A. For every $g \in G_a$ and $x \in E_a$,

$$\varphi \circ (g \cdot x)^* = \varphi \circ (g^* \cdot x^*) = g^* \cdot (\varphi \circ x^*);$$

evaluating at 1 yields $\varphi(g \cdot x) = g \cdot \varphi(x)$, so that φ preserves the action of $\mathbb{G}$. $\square$

3. The converse of Proposition 2.2 is:

Proposition 2.3. *Let $\bar{C} = (C, \pi, \mathbf{.})$ be a group coextension of A by an abelian group valued functor $\mathbb{G} = (G, \gamma)$ on A; let $\bar{G} = \mathbf{O}\mathbb{G}$. With the action $\mathbf{.}$ of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{C})$ defined for every $\bar{X}$ by*

$$(\bar{g} \cdot \bar{c})(x) = g_x \cdot \bar{c}(x), \quad \textit{where } x \in X_a,\ \bar{g}(x) = (g_x, a),$$

$\bar{C}$ is a Beck extension $\mathbf{E}\bar{C}$ of $\bar{G}$ by A. When $\theta : \bar{C} \to \bar{D}$ is an equivalence of group coextensions of A by $\mathbb{G}$, $\mathbf{E}\theta = \theta$ is a homomorphism of Beck extensions of $\bar{G}$ by A.

Proof. By definition, $\bar{G} = (G', \alpha)$, where G' is the disjoint union $G' = \bigcup_{a \in A} \big(G_a \times \{a\}\big)$, $(g, a)^s = (g^s, as)$, $\alpha(g, a) = a$, and addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ is given by

$$(\bar{g} + \bar{h})(x) = (g_x + h_x, a), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x, a),\ \bar{h}(x) = (h_x, a).$$

Let $\bar{g} : \bar{X} \to \bar{G}$ and $\bar{c} : \bar{X} \to \bar{C}$ be morphisms in $\bar{\mathcal{C}}$. If $x \in X_a$, then $\bar{c}(x) \in C_a$ and $\bar{g}(x) \in G'_a$, $\bar{g}(x) = (g_x, a)$ for some $g_x \in G_a$; hence a mapping $\bar{g} \cdot \bar{c} : X \to G'$ may be defined by $(\bar{g} \cdot \bar{c})(x) = g_x \cdot \bar{c}(x)$ as in the statement. Since $\bar{g}$ and $\bar{c}$ are homomorphisms of right S-sets over A we have $\bar{g}(x^s) = \bar{g}(x)^s = (g_x, a)^s = (g_x^s, as)$, $\bar{c}(x^s) = \bar{c}(x)^s$, and

$$(\bar{g} \cdot \bar{c})(x^s) = g_x^s \cdot \bar{c}(x)^s = \big(g_x \cdot \bar{c}(x)\big)^s = \big((\bar{g} \cdot \bar{c})(x)\big)^s;$$

thus $\bar{g} \cdot \bar{c}$ is a homomorphism of right S-sets over A. The definition of $\bar{g} \cdot \bar{c}$ shows that $(\bar{g} \cdot \bar{c}) \circ f = (\bar{g} \circ f) \cdot (\bar{c} \circ f)$ whenever $f : \bar{W} \to \bar{X}$ is a morphism in $\bar{\mathcal{C}}$. The action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X},\bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X},\bar{C})$ is a simple group action like the action of G_a on C_a. To show transitivity let $\bar{c},\bar{d} : \bar{X} \to \bar{C}$ be morphisms in $\bar{\mathcal{C}}$. Define a mapping $\bar{g} : X \to G'$ as follows: when $x \in X_a$, then $\bar{c}(x)$, $\bar{d}(x) \in C_a$ and there exists a unique $g_x \in G_a$ such that $g_x \cdot \bar{c}(x) = \bar{d}(x)$; let $\bar{g}(x) = (g_x, a)$. We have

$$\bar{d}(x^s) = \bar{d}(x)^s = \big(g_x \cdot \bar{c}(x)\big)^s = g_x^s \cdot \bar{c}(x)^s = g_x^s \cdot \bar{c}(x^s);$$

hence

$$\bar{g}(x^s) = (g_x^s, as) = (g_x, a)^s = \bar{g}(x)^s$$

and $\bar{g} : \bar{X} \to \bar{G}$ is a homomorphism of right S-sets over A. Also $\bar{g} \cdot \bar{c} = \bar{d}$ by definition. Hence $\bar{C}$ is a Beck extension $\mathbf{E}\bar{C}$ of $\bar{G}$ by A.

Let $\theta : \bar{C} \to \bar{D} = (D,\delta)$ be an equivalence of group coextensions of A by $\mathbb{G}$. Then θ is a morphism in $\bar{\mathcal{C}}$. Let $\bar{g} : \bar{X} \to \bar{G}$ and $\bar{c} : \bar{X} \to \bar{C}$ be homomorphisms of right S-sets over A. When $x \in X_a$ we have $\bar{g}(x) = (g_x, a)$ for some $g_x \in G_a$ and

$$\theta\big((\bar{g}\cdot\bar{c})(x) = \theta\big(g_x \cdot \bar{c}(x)\big) = g_x \cdot \theta(\bar{c}(x)) = \big(\bar{g}\cdot(\theta\circ\bar{c})\big)(x);$$

hence $\theta \circ (\bar{g}\cdot\bar{c}) = \bar{g}\cdot(\theta\circ\bar{c})$. Thus θ is a homomorphism of Beck extensions. □

Proposition 2.4. *The functors* $\mathbf{C}$ *and* $\mathbf{E}$ *in Propositions 2.2 and 2.3 are isomorphisms of categories.*

PROOF. Let $\bar{C} = (C,\pi,\boldsymbol{.})$ be a group coextension of A by $\mathbb{G} = (G,\gamma)$. Let $\bar{G} = \mathbf{O}\mathbb{G}$, so that $\bar{G} = (G',\alpha)$, where G' is the disjoint union $G' = \bigcup_{a\in A}\big(G_a \times \{a\}\big)$, $(g,a)^s = (\gamma_{a,s}(g), as)$, $\alpha(g,a) = a$, and addition on each $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X},\bar{G})$ is given by

$$(\bar{g}+\bar{h})(x) = (g_x + h_x, a), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x,a),\ \bar{h}(x) = (h_x,a).$$

Then $\mathbf{E}\bar{C} = \bar{C}$ is a Beck extension of $\bar{G}$ by A; the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X},\bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X},\bar{C})$ is given by

$$(\bar{g}\cdot\bar{c})(x) = g_x \cdot \bar{c}(x), \quad \text{where } x \in X_a,\ \bar{g}(x) = (g_x,a).$$

Next, $\mathbf{CE}\bar{C} = (C,\pi,\boldsymbol{.})$ is a group coextension of A by $\mathbf{F}\bar{G}$, in which G'_a acts on C_a by

$$(g,a)\cdot x = \big((g,a)^* \cdot x^*\big)(1)$$

for all $(g,a) \in G'_a$, $x \in C_a$; that is,

$$\big((g,a)^* \cdot x^*\big)(1) = g \cdot x^*(1) = g \cdot x,$$

since $(g,a)^*(1) = (g,a)$ and $x^*(1) = x$. Thus, up to the isomorphism $\mathbf{F}\bar{G} \cong \mathbb{G}$, the action of G'_a on C_a in $\mathbf{CE}\bar{C}$ coincides with the given action in $\bar{C}$, and $\mathbf{CE}\bar{C} = \bar{C}$.

Conversely let $\bar{E}$ be a Beck extension of $\bar{G}$ by A. Then $\mathbf{C}\bar{E} = \bar{C} = (E, \pi, \boldsymbol{.})$ is a group coextension of A by $\mathbf{F}\bar{G}$, in which G_a acts on E_a by

$$g \cdot x = (g^* \cdot x^*)(1)$$

for all $g \in G_a$, $x \in E_a$; then the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ is pointwise. Next, $\mathbf{E}\bar{C} = \bar{C}$ is a Beck extension of $\mathbf{OF}\bar{G}$ by A. Now $\mathbf{OF}\bar{G} = \bar{G}' = (G', \alpha)$, where G' is the disjoint union $G' = \bigcup_{a \in A} \big(A_a \times \{a\}\big)$. The action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G}')$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{C})$ is given by

$$(\bar{g}' \cdot \bar{c})(x) = g_x \cdot \bar{c}(x), \quad \text{where } x \in X_a,\ \bar{g}'(x) = (g_x, a).$$

Let $\bar{g} \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$, $\bar{c} \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{C})$; when $x \in X_a$, then $\bar{g}'(x) = (\bar{g}(x), a)$ defines $\bar{g}' \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G}')$, and

$$(\bar{g}' \cdot \bar{c})(x) = \bar{g}(x) \cdot \bar{c}(x) = (\bar{g} \cdot \bar{c})(x),$$

since the action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$ is pointwise. Thus, up to the isomorphism $\mathbf{OF}\bar{G} \cong \bar{G}$, $\bar{E}$ and $\mathbf{EC}\bar{E}$ have the same action of $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{G})$ on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{X}, \bar{E})$, and $\mathbf{EC}\bar{E} = \bar{E}$. □

3. Cohomology

1. The ingredients of triple cohomology are: categories $\mathcal{Z}$ and $\mathcal{C}$; a functor $\mathbb{U} : \mathcal{C} \to \mathcal{Z}$ with a left adjoint $\mathbb{F} : \mathcal{Z} \to \mathcal{C}$, providing natural transformations $\eta : 1_{\mathcal{A}} \to \mathbb{UF}$ and $\epsilon : \mathbb{FU} \to 1_{\mathcal{C}}$; an object A of $\mathcal{C}$; and an abelian group object $\bar{G}$ in the category $\bar{\mathcal{C}}$ of objects of $\mathcal{C}$ over A.

The adjunction $(\mathbb{F}, \mathbb{U}, \eta, \epsilon)$ lifts to an adjunction $(\bar{\mathbb{F}}, \bar{\mathbb{U}}, \bar{\eta}, \bar{\epsilon})$ between $\bar{\mathcal{C}}$ and the category $\bar{\mathcal{Z}}$ of objects of $\mathcal{Z}$ over $\mathbb{U}A$; when $\zeta : Z \to \mathbb{U}A$ and $\rho : C \to A$, then

$$\bar{\mathbb{F}}(Z, \zeta) = (\mathbb{F}Z, \bar{\zeta}),\ \bar{\mathbb{U}}(C, \rho) = (\mathbb{U}C, \mathbb{U}\rho),\ \bar{\eta}_{(Z,\zeta)} = \eta_Z,\ \bar{\epsilon}_{(C,\rho)} = \epsilon_C,$$

where $\bar{\zeta} : \mathbb{F}Z \to A$ is the unique morphism such that $\mathbb{U}\bar{\zeta} \circ \eta_Z = \zeta$ (equivalently, $\bar{\zeta} = \epsilon_A \circ \mathbb{F}\zeta$). Let $\mathbb{V} = \bar{\mathbb{F}}\bar{\mathbb{U}}$. When $\bar{C}$ is an object of $\bar{\mathcal{C}}$ and $n \geq 1$,

$$C^n(\bar{C}, \bar{G}) = \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}^n \bar{C}, \bar{G})$$

is an abelian group. The coboundary $\delta^n : C^n(\bar{C}, \bar{G}) \to C^{n+1}(\bar{C}, \bar{G})$ is

$$\delta^n(\varphi) = \sum_{0 \leq i_n} (-1)^i \varphi \circ \epsilon_C^{n,i}$$

for every $\varphi : \mathbb{V}^n \bar{C} \to \bar{G}$, where

$$\epsilon_C^{n,i} = \mathbb{V}^{n-i} \epsilon_{\mathbb{V}^i \bar{C}} : \mathbb{V}^{n+1} \bar{C} \to \mathbb{V}^n \bar{C};$$

also $\delta 0 = 0 : 0 \to C^1(\bar{C}, \bar{G})$. A standard argument, using the identity $\epsilon^{n,j} \circ \epsilon^{n+1,i} = \epsilon^{n,i} \circ \epsilon^{n+1,j+1}$ which holds for all $i, j = 0, 1, \ldots, n$, yields $\delta^{n+1} \circ \delta^n = 0$. Hence

$$B^n(\bar{C}, \bar{G}) = \mathrm{Im}\, \delta^{n-1} \subseteq \mathrm{Ker}\, \delta^n = Z^n(\bar{C}, \bar{G})$$

for all $n \geq 1$. By definition

$$H^n(\bar{C}, \bar{G}) = Z^n(\bar{C}, \bar{G}) / B^n(\bar{C}, \bar{G})$$

for all $n \geq 1$. In particular,

$$H^n(A, \bar{G}) = H^n(\bar{A}, \bar{G}),$$

where $\bar{A} = (A, 1_A)$. In [2], [1], $H^n(\bar{C}, \bar{G})$ is called $H^{n-1}(\bar{C}, \bar{G})$; here we use a more traditional numbering.

For this cohomology, Beck proved the following properties ([2], Theorems 2 and 6).

Theorem A. *$H^n(\bar{\mathbb{F}}\bar{X}, \bar{G}) = 0$ for all $n \geq 2$, and $H^1(\mathbb{V}\bar{C}, \bar{G}) \cong C^1(\bar{C}, \bar{G})$.*

A sequence $\bar{G} \to \bar{G}' \to \bar{G}''$ of abelian group objects and morphisms is *short* $\mathbb{V}$*-exact* when

$$0 \to \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}\bar{C}, \bar{G}) \to \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}\bar{C}, \bar{G}') \to \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}\bar{C}, \bar{G}'') \to 0$$

is short exact for every $\bar{C}$.

Theorem B. *Every short* $\mathbb{V}$*-exact sequence* $\mathcal{E} : \bar{G} \to \bar{G}' \to \bar{G}''$ *of abelian group objects of* $\bar{\mathcal{C}}$ *induces an exact sequence*

$$\cdots H^n(\bar{C}, \bar{G}) \to H^n(\bar{C}, \bar{G}') \to H^n(\bar{C}, \bar{G}'') \to H^{n+1}(\bar{C}, \bar{G}) \cdots$$

which is natural in $\mathcal{E}$ *and* $\bar{C}$.

Theorem C. *When* $\mathcal{C}$ *is tripleable over* $\mathcal{Z}$, *there is a one-to-one correspondence between elements of* $H^2(A, \bar{G})$ *and isomorphy classes of Beck extensions of* $\bar{G}$ *by* A.

Up to natural isomorphism, $H^n(\bar{C}, -)$ is the only abelian group valued functor for which Theorems A and B hold [1]; [1] has a similar characterization of $H^n(-, \bar{G})$.

2. Now let S be a monoid, $\mathcal{C}$ be the category of right S-sets and action preserving mapping s, A be a fixed right S-set, and $\bar{\mathcal{C}}$ be the category of right S-sets over A and their homomorphism s, as in Sections 1 and 2; $\mathbb{U} : \mathcal{C} \to$ Sets is the forgetful functor to the category Sets of sets and mappings, which strips right S-sets of the action of S.

Every set Z has a free right S-set $\mathbb{F}Z = Z \times S$, in which $(z, s)^t = (z, st)$; when $f : Z \to T$ is a mapping, then $\mathbb{F}f : (z, s) \mapsto (f(z), s)$ is action preserving. The mapping $\eta_Z : z \mapsto (z, 1)$ has the requisite universal property: for every mapping $f : Z \to Y$ of Z into a right S-set Y there is a unique action preserving mapping $g : \mathbb{F}Z \to Y$ such that $g \circ \eta_Z = f$, namely, $g(z, s) = f(z)^s$. Thus $\mathbb{F} = - \times S$ is a left adjoint of $\mathbb{U}$. In this adjunction, $\epsilon_X : X \times S = \mathbb{F}\mathbb{U}X \to X$ is the action of S: indeed ϵ_X is the unique action preserving mapping such that $\mathbb{U}\epsilon_X \circ \eta_X = 1_{\mathbb{U}X}$; hence $\epsilon_X(x, 1) = x$ and $\epsilon_X(x, s) = x^s$.

The adjunction $(\mathbb{F}, \mathbb{U}, \eta, \epsilon)$ lifts to an adjunction $(\bar{\mathbb{F}}, \bar{\mathbb{U}}, \bar{\eta}, \bar{\epsilon})$ between $\bar{\mathcal{C}}$ and the category of sets over $\mathbb{U}A$, as follows. When (Z, f) is a set over A (where $f : Z \to A$ is a mapping), $\bar{\mathbb{F}}(Z, f) = (\mathbb{F}Z, \bar{f}) = (Z \times S, \bar{f})$, where $\bar{f} : Z \times S \to A$ is the unique morphism such that $\mathbb{U}\bar{f} \circ \eta_X = \pi$:

$$\bar{f}(z, s) = f(z)^s,$$

equivalently, $\bar{f} = \epsilon_A \circ \mathbb{F}f$. For any right S-set $\bar{X} = (X, \pi)$ over A, $\bar{\mathbb{U}}\bar{X} = (\mathbb{U}X, \mathbb{U}\pi) = (X, \pi)$ strips $\bar{X}$ of the action of S; $\bar{\eta}_{(X,f)} = \eta_X$; and $\bar{\epsilon}_{(X,\pi)} = \epsilon_X$.

When $\bar{X} = (X, \pi)$ is a right S-set over A,

$$\mathbb{V}\bar{X} = \bar{\mathbb{F}}\bar{\mathbb{U}}(X, \pi) = (X \times S, \bar{\pi}),$$

where $(x, s)^t = (x, st)$ and $\bar{\pi}(x, s) = \pi(x)^s$; if $f : \bar{X} \to \bar{Y}$ is a homomorphism of right S-sets over A, then $\mathbb{V}f = f \times S : (x, s) \mapsto (f(x), s)$.

We identify $(X \times S^{n-1}) \times S$ with $X \times S^n$, and $\big((x, s_1, \dots, s_{n-1}), s_n\big)$ with $(x, s_1, \dots, s_{n-1}, s_n)$. When $n \geq 1$,

$$\mathbb{V}^n \bar{X} = (X \times S^n, \pi_n);$$

S acts on $\mathbb{V}^n \bar{X}$ by

$$(x, s_1, \dots, s_{n-1}, s_n)^s = (x, s_1, \dots, s_{n-1}, s_n s);$$

and $\pi_n = \bar{\pi}_{n-1}$ is found by induction:

$$\begin{aligned} \pi_n(x, s_1, \dots, s_{n-1}, s_n) &= \pi_{n-1}(x, s_1, \dots, s_{n-1})^{s_n} \\ &= \pi_{n-2}(x, s_1, \dots, s_{n-2})^{s_{n-1}, s_n} = \cdots \\ &= \pi(x)^{s_1 \dots s_{n-1} s_n}. \end{aligned}$$

A similar induction yields $\epsilon_X^{n,i} = \mathbb{V}^{n-i} \epsilon_{\mathbb{V}^i \bar{X}} : \mathbb{V}^{n+1} \bar{X} \to \mathbb{V}^n \bar{X}$. First, $\epsilon_X^{n,n} = \epsilon_{\mathbb{V}^n \bar{X}}$:

$$\begin{aligned} \epsilon_X^{n,n}(x, s_1, \dots, s_n, s_{n+1}) &= (x, s_1, \dots, s_{n-1}, s_n)^{s_{n+1}} \\ &= (x, s_1, \dots, s_{n-1}, s_n s_{n+1}). \end{aligned}$$

If $0 < i < n$, then $\epsilon_X^{n,i} = \mathbb{V} \epsilon_X^{n-1,i}$ and

$$\begin{aligned} \epsilon_X^{n,i}(x, s_1, \dots, s_n, s_{n+1}) &= \big(\epsilon_X^{n-1,i}(x, s_1, \dots, s_n), s_{n+1}\big) \\ &= \big(\epsilon_X^{n-2,i}(x, s_1, \dots, s_{n-1}) s_n, s_{n+1}\big) = \cdots \\ &= \big(\epsilon_X^{i,i}(x, s_1, \dots, s_i, s_{i+1}), s_{i+2}, \dots, s_n, s_{n+1}\big) \\ &= (x, s_1, \dots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \dots, s_{n+1}). \end{aligned}$$

Similarly, $\epsilon_X^{n,0} = \mathbb{V}^n \epsilon_X$ and

$$\epsilon_X^{n,0}(x, s_1, \dots, s_n, s_{n+1}) = (x^{s_1}, s_2, \dots, s_n, s_{n+1})$$

since $\epsilon_X^{0,0}(x, s_1) = \epsilon_X(x, s_1) = x^{s_1}$.

3. We now turn to cochains.

Lemma 3.1. *Let $\bar{X} = (X, \pi)$ be a right S-set over A, $\bar{G}$ be an abelian group object over A, and $\mathbb{G} = (G, \gamma) = \mathbf{F}\bar{G}$. There is an isomorphism*

$$\Theta : C^n(\bar{X}, \bar{G}) \to \prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}$$

which is natural in $\bar{X}$ and $\bar{G}$. When $v \in C^n(\bar{X}, \bar{G})$,

$$\Theta(v) = \big(v(x, s_1, \dots, s_{n-1}, 1)\big)_{(x, s_1, \dots, s_{n-1}) \in \mathbb{V}^{n-1}\bar{X}};$$

when $u = \big(u(c)\big)_{c \in \mathbb{V}^{n-1}\bar{X}}$, $v = \Theta^{-1}(u)$ is given for all $(x, s_1, \dots, s_n) \in \mathbb{V}^n\bar{X}$ by

$$v(x, s_1, \dots, s_{n-1}, s_n) = u(x, s_1, \dots, s_{n-1})^{s_n}.$$

Proof. When $u = \big(u(c)\big)_{c \in \mathbb{V}^{n-1}\bar{X}} \in \prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}$, define $v = \Xi(u) : \mathbb{V}^n\bar{X} \to G$ for all $(x, s_1, \dots, s_n) \in \mathbb{V}^n\bar{X}$ by

$$v(x, s_1, \dots, s_{n-1}, s_n) = u(x, s_1, \dots, s_{n-1})^{s_n}$$

as calculated in $\bar{G}$; $\Xi(u)$ will be $\Theta^{-1}(u)$. If $a = \pi_{n-1}(x, s_1, \dots, s_{n-1})$, then $u(x, s_1, \dots, s_{n-1}) \in G_a$ and

$$v(x, s_1, \dots, s_{n-1}, s_n) = u(x, s_1, \dots, s_{n-1})^{s_n} \in G_{as_n};$$

thus v preserves projection to A. Also

$$v\big((x, s_1, \dots, s_{n-1}, s_n)^s\big) = v\big((x, s_1, \dots, s_{n-1}, s_n s)\big)$$

$$= u(x, s_1, \dots, s_{n-1})^{s_n s} = \big(v(x, s_1, \dots, s_{n-1}, s_n)^s\big).$$

Therefore v is a homomorphism of right S-sets over A and

$$v \in \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}^n\bar{X}, \bar{G}) = C^n(\bar{X}, \bar{G}).$$

We see that $\Theta(v) = u$.

Conversely, let v be an n-cochain $v \in C^n(\bar{X}, \bar{G}) = \mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}^n\bar{X}, \bar{G})$. If $a = \pi_{n-1}(x, s_1, \dots, s_{n-1})$, then $v(x, s_1, \dots, s_{n-1}, 1) \in G_a$ and $u = \Theta(v) \in \prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}$, since $u(c) \in G_a$ when $c = (x, s_1, \dots, s_{n-1})$ and $\pi_{n-1}c = a$. The calculation above shows that $v = \Xi(u)$. Thus Θ and Ξ are mutually inverse bijections. Since the addition on $\prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}$ is componentwise, and the addition on $\mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}^n\bar{X}, \bar{A})$ is pointwise, Θ is in fact an isomorphism. Naturality is immediate. $\square$

4. Lemma 3.1 suggests that we define

$$C^n(\bar{X}, \mathbb{G}) = \prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}.$$

Note that the latter depends only on $\mathbb{G}$. Then $C^n(\bar{X}, \bar{G}) \cong C^n(\bar{X}, \mathbb{G})$ when $\mathbb{G} = \mathbf{F}\bar{G}$. The coboundary becomes:

Lemma 3.2. *Up to the natural isomorphisms* $C^n(\bar{X}, \bar{G}) \cong C^n(\bar{X}, \mathbb{G})$,

$$\begin{aligned}(\delta^n u)(x, s_1, \ldots, s_n) = u(x^{s_1}, s_2, \ldots, s_n) \\ + \sum_{0<i<n} (-1)^i u(x, s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_n) \\ + (-1)^n u(x, s_1, \ldots, s_{n-1})^{s_n}\end{aligned}$$

for all $u \in C^n(\bar{X}, \mathbb{G})$.

Proof. The coboundary $C^n(\bar{X}, \mathbb{G}) \to C^{n+1}(\bar{X}, \mathbb{G})$ is really $\Theta \circ \delta^n \circ \Theta^{-1}$, where $\Theta : C^n(\bar{X}, \bar{G}) \to C^n(\bar{X}, \mathbb{G})$ is the natural isomorphism in Lemma 3.1. When $u \in C^n(\bar{X}, \mathbb{G})$, $v = \Theta^{-1}(u) : \mathbb{V}^n \bar{X} \to \bar{G}$ is given by

$$v(x, s_1, \ldots, s_{n-1}, s_n) = u(x, s_1, \ldots, s_{n-1})^{s_n}.$$

Then $w = \delta^n(v) = \sum_{0 \le i \le n} (-1)^i v \circ \epsilon_X^{n,i} : \mathbb{V}^{n+1}\bar{X} \to \bar{G}$ is given by

$$\begin{aligned}w(x, s_1, \ldots, s_n, s_{n+1}) &= \sum_{0 \le i \le n} (-1)^i v\big(\epsilon_X^{n,i}(x, s_1, \ldots, s_n, s_{n+1})\big) \\ &= v(x^{s_1}, s_2, \ldots, s_n, s_{n+1}) \\ &\quad + \sum_{0<i<n} (-1)^i v(x, s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_{n+1}) \\ &\quad + (-1)^n v(x, s_1, \ldots, s_{n-1}, s_n s_{n+1})\end{aligned}$$

Hence

$$\begin{aligned}\Theta(w)(x, s_1, \ldots, s_n) &= w(x, s_1, \ldots, s_n, 1) \\ &= v(x^{s_1}, s_2, \ldots, s_n, 1) \\ &\quad + \sum_{0<i<n} (-1)^i v(x, s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_n, 1)\end{aligned}$$

$$
\begin{aligned}
&+ (-1)^n v(x, s_1, \ldots, s_{n-1}, s_n) \\
&= u(x^{s_1}, s_2, \ldots, s_n) \\
&\quad + \sum_{0<i<n} (-1)^i u(x, s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_n) \\
&\quad + (-1)^n u(x, s_1, \ldots, s_{n-1})^{s_n}. \qquad \square
\end{aligned}
$$

In particular, a 1-cochain $u \in C^1(\bar{X}, \mathbb{G}) = \prod_{c \in \mathbb{V}^0 \bar{X}} G_{\pi_0 c} = \prod_{x \in X} G_{\pi x}$ is a family $u = \big(u(x)\big)_{x \in X}$ such that $u(x) \in G_{\pi x}$ for all x, equivalently $u(x) \in G_a$ for all $x \in X_a$. Its coboundary is

$$(\delta u)(x, s) = u(x^s) - u(x)^s.$$

A 1-cocycle is a 1-cochain z such that $z(x^s) = z(x)^s = \gamma_{a,s}\big(z(x)\big)$ for all $x \in X_a$.

A 2-cochain $u \in \prod_{c \in \mathbb{V}^1 \bar{X}} G_{\pi_1 c} = \prod_{(x,s) \in X \times S} G_{\pi(x)s}$ is a family $u = \big(u(x, s)\big)_{(x,s) \in X \times S}$ such that $u(x, s) \in G_{\pi(x)s}$ for all s, x; equivalently $u(x, s) \in G_{as}$ when $x \in X_a$. Its coboundary is

$$(\delta u)(x, s, t) = u(x^s, t) - u(x, st) + u(x, s)^t.$$

A 2-cocycle is a 2-cochain z such that $z(x, st) = z(x^s, t) + z(x, s)^t$ for all s, t, x. A 2-coboundary is a 2-cochain b of the form $b(x, s) = u(x^s) - u(x)^s$ for some 1-cochain u.

In general, Lemma 3.2 yields:

Theorem 3.3. *Let $\bar{X} = (X, \pi)$ be a right S-set over A, $\bar{G}$ be an abelian group object over A, and $\mathbb{G} = (G, \gamma) = \mathbf{F}\bar{G}$. Up to an isomorphism which is natural in $\bar{X}$ and $\bar{G}$, the triple cohomology groups of $\bar{X}$ with coefficients in $\bar{G}$ are the homology groups $H^n(\bar{X}, \mathbb{G})$ of the complex*

$$0 \to C^1(\bar{X}, \mathbb{G}) \to \cdots \to C^n(\bar{X}, \mathbb{G}) \xrightarrow{\delta^n} C^{n+1}(\bar{X}, \mathbb{G}) \to \cdots$$

in Lemma 3.2.

In other words, $H^n(\bar{X}, \bar{G}) \cong H^n(\bar{X}, \mathbb{G}) = Z^n(\bar{X}, \mathbb{G})/B^n(\bar{X}, \mathbb{G})$, where $Z^n(\bar{X}, \mathbb{G}) = \operatorname{Ker} \delta^n$, $B^n(\bar{X}, \mathbb{G}) = \operatorname{Im} \delta^{n-1}$ if $n \geq 2$, and $B^1(\bar{X}, \mathbb{G}) = 0$. Replacing $\bar{X} = (X, \pi)$ by $\bar{A} = (A, 1_A)$ in Theorem 3.3 yields the cohomology of A.

5. Theorems A, B, and C yield basic properties of triple cohomology.

Theorem 3.4. *If A is a free right S-set, then for every abelian group valued functor $\mathbb{G}$ on A we have $H^n(A, \mathbb{G}) = 0$ for all $n \geq 2$.*

This follows from Theorem A.

Theorem 3.5. *Every short exact sequence $\mathcal{E} : 0 \to \mathbb{G} \to \mathbb{G}' \to \mathbb{G}'' \to 0$ of abelian group valued functors on A induces an exact sequence*

$$\cdots H^n(A, \mathbb{G}) \to H^n(A, \mathbb{G}') \to H^n(A, \mathbb{G}'') \to H^{n+1}(A, \mathbb{G}) \cdots$$

which is natural in $\mathcal{E}$.

Proof. This follows from Theorem B, applied to $\bar{X} = (A, 1_A)$. Exactness in the abelian category of abelian group valued functors on A is pointwise [7]: $0 \to \mathbb{G} \to \mathbb{G}' \to \mathbb{G}'' \to 0$ is exact if and only if $0 \to G_a \to G'_a \to G''_a \to 0$ is exact for every $a \in A$. When $\mathbb{G}$ is identified with the corresponding abelian group object $\mathbf{O}\mathbb{G}$, Lemma 3.1 provides for any $\bar{X} = (X, \xi) \in \bar{\mathcal{C}}$ a natural isomorphism

$$\mathrm{Hom}_{\bar{\mathcal{C}}}(\mathbb{V}\bar{X}, \mathbb{G}) \cong C^1(\bar{X}, \mathbb{G}) = \prod_{x \in X} G_{\xi x}.$$

Now

$$0 \to \prod_{x \in X} G_{\xi x} \to \prod_{x \in X} G'_{\xi x} \to \prod_{x \in X} G''_{\xi x} \to 0$$

is exact. Hence $\mathbb{G} \to \mathbb{G}' \to \mathbb{G}''$ is short $\mathbb{V}$-exact, and Theorem 3.5 follows from Theorem B. $\square$

Theorem 3.6. *There is a one-to-one correspondence between elements of $H^2(A, \mathbb{G})$ and equivalence classes of group coextensions of A by $\mathbb{G}$.*

This follows either from Theorem C and Proposition 2.4, or from [4] and the above descriptions of 2-cocycles and 2-coboundaries.

6. We prove one more property. As in [3] we show that the cohomology of A is that of a projective complex in the category $\mathcal{F}$ of abelian group valued functors on A. This provides a more direct proof of Theorem 3.5.

For each $n \geq 1$ and $a \in A$ let

$$C_n(a) = \{(x, s_1, \dots, s_n) \in A \times S^n \mid xs_1 \dots s_n = a\}.$$

Let $C_n(A)_a$ be the free abelian group on $C_n(a)$. For each $s \in S$ there is a unique homomorphism $\kappa_{a,s} : C_n(A)_a \to C_n(A)_{as}$ such that

$$\kappa_{a,s}(x, s_1, \dots, s_n) = (x, s_1, \dots, s_{n-1}, s_n s).$$

Lemma 3.7. *For every $n \geq 1$:*

(1) $\mathbb{C}_n(A) = (C_n(A), \kappa)$ *is an abelian group valued functor on A;*

(2) *there is an isomorphism* $\mathrm{Hom}_{\mathcal{F}}(\mathbb{C}_n(A), \mathbb{G}) \cong C^n(A, \mathbb{G})$ *which is natural in $\mathbb{G}$;*

(3) $\mathbb{C}_n(A)$ *is projective in $\mathcal{F}$.*

Proof. (1): $\kappa_{a,1}$ is the identity on $C_n(A)_a$, since it leaves fixed every generator of $C_n(A)_a$; $\kappa_{as,t} \circ \kappa_{a,s} = \kappa_{a,st}$ for all $s, t \in S$, since

$$\begin{aligned}\kappa_{as,t}\big(\kappa_{a,s}(x, s_1, \dots, s_n)\big) &= (x, s_1, \dots, s_{n-1}, s_n st) \\ &= \kappa_{a,st}(x, s_1, \dots, s_n)\end{aligned}$$

for every generator of $C_n(A)_a$.

(2): Let $\varphi = (\varphi_a)_{a \in A}$ be a natural transformation $\varphi : \mathbb{C}_n(A) \to \mathbb{G} = (G, \gamma)$, so that $\gamma_{a,s} \circ \varphi_a = \varphi_{as} \circ \kappa_{a,s}$ for all a, s. For every $(x, s_1, \dots, s_n) \in A \times S^n$,

$$\begin{aligned}\varphi_a(x, s_1, \dots, s_n) &= \varphi_a\big(\kappa_{b,s_n}(x, s_1, \dots, s_{n-1}, 1)\big) \\ &= \gamma_{b,s_n}\big(\varphi_b(x, s_1, \dots, s_{n-1}, 1)\big)\end{aligned}$$

where $a = xs_1 \dots s_n$, $b = xs_1 \dots s_{n-1}$. Therefore φ is uniquely determined by the n-cochain $u = \Theta(\varphi)$ defined by

$$u(x, s_1, \dots, s_{n-1}) = \varphi_b(x, s_1, \dots, s_{n-1}, 1) \in G_b,$$

where $b = xs_1 \dots s_{n-1} = \pi_{n-1}(x, s_1, \dots, s_{n-1})$. In other words, the additive homomorphism Θ is injective.

Conversely let $u \in C^n(A, \mathbb{G}) = \prod_{c \in \mathbb{V}^{n-1}\bar{X}} G_{\pi_{n-1}c}$. For every $a \in A$ there is a unique homomorphism $\varphi_a : C_n(A)_a \to G_a$ such that

$$\varphi_a(x, s_1, \dots, s_n) = \gamma_{b,s_n}\big(u(x, s_1, \dots, s_{n-1})\big)$$

whenever $xs_1 \dots s_n = a$, where $b = xs_1 \dots s_{n-1}$. Then

$$\begin{aligned}\varphi_{as}\big(\kappa_{a,s}(x, s_1, \dots, s_n)\big) &= \varphi_{as}(x, s_1, \dots, s_{n-1}, s_n s) \\ &= \gamma_{b,s_n s}\big(u(x, s_1, \dots, s_{n-1})\big) \\ &= \gamma_{a,s}\Big(\gamma_{b,s_n}\big(u(x, s_1, \dots, s_{n-1})\big)\Big) \\ &= \gamma_{a,s}\big(\varphi_a(x, s_1, \dots, s_n)\big)\end{aligned}$$

whenever $xs_1 \dots s_n = a$, so that $\gamma_{a,s} \circ \varphi_a = \varphi_{as} \circ \kappa_{a,s}$ and $\varphi = (\varphi_a)_{a \in A}$ is a natural transformation $\varphi = \Phi(u) : \mathbb{C}_n(A) \to \mathbb{G}$. We have $\Theta(\varphi) = u$: indeed

$$\big(\Theta(\varphi)\big)(x, s_1, \dots, s_{n-1}) = \varphi_a(x, s_1, \dots, s_{n-1}, 1) = u(x, s_1, \dots, s_{n-1})$$

where $a = xs_1 \dots s_{n-1}$, since $\gamma_{a,1}$ is the identity. If conversely $\varphi : \mathbb{C}_n(A) \to \mathbb{G}$ is a natural transformation, then

$$\begin{aligned}\Phi\big(\Theta(\varphi)\big)_a(x, s_1, \dots, s_n) &= \gamma_{b,s_n}\big(\Theta(\varphi)(x, s_1, \dots, s_{n-1})\big) \\ &= \gamma_{b,s_n}\big(\varphi_b(x, s_1, \dots, s_{n-1}, 1)\big) \\ &= \varphi_{bs_n}\big(\kappa_{b,s_n}(x, s_1, \dots, s_{n-1}, 1)\big) \\ &= \varphi_a(x, s_1, \dots, s_n)\end{aligned}$$

whenever $xs_1 \dots s_n = a$, with $b = xs_1 \dots s_{n-1}$ as before. Therefore $\Phi\big(\Theta(\varphi)\big)\bar{r} = \varphi$. Thus Θ and Φ are mutually inverse isomorphisms. Naturality is immediate.

(3): Epimorphisms in $\mathcal{F}$ are pointwise [7]. If $\sigma : \mathbb{G} \to \mathbb{H}$ is an epimorphism, then every $\sigma_a : G_a \to H_a$ is surjective, and so is the induced homomorphism $\sigma^* : C^n(A, \mathbb{G}) \to C^n(A, \mathbb{H})$: indeed σ^* is given by

$$\big(\sigma^*(u)\big)(c) = \sigma_{\pi c}\big(u(c)\big)$$

for all $c \in C_n(A)$; given $v \in C^n(A, \mathbb{H})$ there is for every $v(c) \in H_{\pi c}$ some $u(c) \in G_{\pi c}$ such that $\sigma_{\pi c}\big(u(c)\big) = v(c)$, and then $u \in C^n(A, \mathbb{G})$ satisfies $\sigma^*(u) = v$. Then $\mathrm{Hom}_{\mathcal{F}}(\mathbb{C}_n(A), \mathbb{G}) \to \mathrm{Hom}_{\mathcal{F}}(\mathbb{C}_n(A), \mathbb{H})$ is an epimorphism, by (2), showing that $\mathbb{C}_n(A)$ is projective in $\mathcal{F}$. (This also follows from [7].) □

Proposition 3.8. *Up to natural isomorphisms, $H^n(A, \mathbb{G})$ is the cohomology of the projective complex*

$$0 \leftarrow \mathbb{C}_1(A) \leftarrow \cdots \leftarrow \mathbb{C}_n(A) \leftarrow \mathbb{C}_{n+1}(A) \leftarrow \cdots$$

where $\partial : \mathbb{C}_n(A) \to \mathbb{C}_{n-1}(A)$ *is given for all* $n \geq 2$ *by*

$$\partial_a(x, s_1, \dots, s_n) = (xs_1, s_2, \dots, s_n) + \sum_{0<i<n} (-1)^i (x, s_1, \dots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \dots, s_n)$$

whenever $xs_1 \dots s_n = a$.

Proof. For every $u \in C^n(A, \mathbb{G})$

$$\begin{array}{ccc} \mathbb{C}_{n-1}(A) & \xleftarrow{\partial} & \mathbb{C}_n(A) \\ {\scriptstyle \Phi(u)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \mathbb{G} & & \mathbb{G} \end{array}$$

we show that $\varphi = \Phi(u) \circ \partial$ satisfies $\Theta(\varphi) = \delta u$, where Φ and Θ are the natural isomorphisms in the proof of Lemma 3.7. We have

$$\begin{aligned} \varphi_a(x, s_1, \dots, s_n) &= \big(\Phi(u)\big)_a\big(\partial(x, s_1, \dots, s_n)\big) \\ &= \big(\Phi(u)\big)_a(xs_1, s_2, \dots, s_n) \\ &\quad + \sum_{0<i<n-1} (-1)^i \big(\Phi(u)\big)_a(x, s_1, \dots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \dots, s_n) \\ &\quad + (-1)^{n-1}\big(\Phi(u)\big)_a(x, s_1, \dots, s_{n-2}, s_{n-1}s_n) \\ &= \gamma_{b,s_n}\big(u(xs_1, s_2, \dots, s_{n-1})\big) \\ &\quad + \sum_{0<i<n-1} (-1)^i \gamma_{b,s_n}\big(u(x, s_1, \dots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \dots, s_{n-1})\big) \\ &\quad + (-1)^{n-1}\gamma_{c,s_{n-1}s_n}\big(u(x, s_1, \dots, s_{n-2})\big) \end{aligned}$$

whenever $xs_1 \dots s_n = a$, with $b = xs_1 \dots s_{n-1}$ and $c = xs_1 \dots s_{n-2}$. Then

$$\begin{aligned} \big(\Theta(\varphi)\big)(x, s_1, \dots, s_{n-1}) &= \varphi_b(x, s_1, \dots, s_{n-1}, 1) \\ &= \big(u(xs_1, s_2, \dots, s_{n-1})\big) \\ &\quad + \sum_{0<i<n-1} (-1)^i \big(u(x, s_1, \dots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \dots, s_{n-1})\big) \\ &\quad + (-1)^{n-1}\gamma_{c,s_{n-1}}\big(u(x, s_1, \dots, s_{n-2})\big) = (\delta u)(x, s_1, \dots, s_{n-1}), \end{aligned}$$

since $\gamma_{b,1}$ is the identity and

$$u(x, s_1, \ldots, s_{n-2})^{s_{n-1}} = \gamma_{c,s_{n-1}}\big(u(x, s_1, \ldots, s_{n-2})\big). \qquad \square$$

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