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On harmonic sequences

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Abstract. A sequence $\{n_1, \dots, n_k\}$ of positive integers is called *harmonic* if there exist k integers $a_1, \dots, a_k$ such that for every integer b there exists at most one i with $b \equiv a_i \pmod{n_i}$. In this paper, for $k \le 14$, we prove that if $\{n_1, \dots, n_k\}$ is harmonic with $(n_i, n_j) \le k$ for all $1 \le i < j \le k$, then $(n_i, n_j) = k$ for all $1 \le i < j \le k$.

1. Introduction

For two integers a, n, let $a \pmod{n} = \{a + nt : t \in Z\}$. Let $n_1, \dots, n_k$ be k positive integers and $a_1, \dots, a_k$ be k integers. It is well known that if $(n_i, n_j) \mid a_i - a_j$ for all $1 \le i < j \le k$, then the system of congruences

$$x \equiv a_i \pmod{n_i}, \quad i = 1, \dots, k$$

has solutions. That is,

$$\bigcap_{i=1}^{k} a_i \pmod{n_i} \neq \phi.$$

When are $\{a_i \pmod{n_i}\}_{i=1}^{k}$ *disjoint, that is,*

$$a_i \pmod{n_i} \cap a_j \pmod{n_j} = \phi, \quad \textit{for all} \;\; 1 \le i < j \le k\,?$$

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A sequence $\{n_1, \dots, n_k\}$ of positive integers is called *harmonic* if there exist k integers $a_1, \dots, a_k$ such that $\{a_i \pmod{n_i}\}_{i=1}^k$ are disjoint, that is, $a_i \pmod{n_i} \cap a_j \pmod{n_j} = \phi$, for all $1 \le i < j \le k$. In 1982, A. P. Huhn and L. Megyesi [3] proved that if all $(n_i, n_j) > 1$ and (n_i, n_j) $(1 \le i < j \le k)$ are distinct, then $\{n_i\}_{i=1}^k$ is harmonic. Z. W. Sun [5] improved this by proving that if the number of pairs $\{i, j\}$ with $1 \le i < j \le k$ and $(n_i, n_j) = d$ is less than $\sqrt{(d+7)/8}$ for all $d \le 2^{k-2}$, then $\{n_i\}_{i=1}^k$ is harmonic. Y. G. Chen [1] proved that if the number of pairs $\{i, j\}$ with $1 \le i < j \le k$ and $(n_i, n_j) \le d$ is less than $\frac{1}{8}(d^2 + d) + \frac{1}{2}$ for all $d \le 2k - 2$, then $\{n_i\}_{i=1}^k$ is harmonic. Y. G. Chen [2] proved that if the number of pairs $\{i, j\}$ with $1 \le i < j \le k$ and $(n_i, n_j) \le d$ is less than d for all $d \le k - 1$, then $\{n_i\}_{i=1}^k$ is harmonic. The following conjecture appeared in M. R. Sun [4].

Conjecture. *If $\{n_i\}_{i=1}^k$ is harmonic with $(n_i, n_j) \le k$ for all $1 \le i < j \le k$, then $(n_i, n_j) = k$ for all $1 \le i < j \le k$.*

M. R. Sun [4] proved that the conjecture is true for $2 \le k \le 7$. In this paper, we prove that the conjecture is true for $8 \le k \le 14$.

Theorem. *Let $8 \le k \le 14$. If $\{n_i\}_{i=1}^k$ is harmonic with $(n_i, n_j) \le k$ for all $1 \le i < j \le k$, then $(n_i, n_j) = k$ for all $1 \le i < j \le k$.*

Remark. For given k, by $(n_i, n_j) \le k$ and Lemma 1 in Section 2 we may assume that every prime power divisor of n_i is less than k. Hence each n_i can take only finitely many possible values. We may also assume that $0 \le a_i < n_i$. Thus we change the problem into finite calculation. But it has many troubles except for several small k.

2. Preliminary lemmas

Lemma 1. *$a \pmod{m} \cap b \pmod{n} = \phi$ if and only if $(m, n) \nmid a - b$.*

Proof is clear. Thus, if $\{n_i\}_{i=1}^k$ is harmonic, then $(n_i, n_j) > 1$ for all i, j.

Lemma 2 (M. R. Sun [4])**.** *Let $l \geq 3$. If the conjecture is true for all $2 \leq k \leq l-1$ and $\{n_i\}_{i=1}^{l}$ is a harmonic sequence with $(n_i, n_j) \leq l$ for all $1 \leq i < j \leq l$, then there are at least three integers of $n_1, \dots, n_l$ which are divisible by l.*

Proof. Suppose that there are at most two integers of $n_1, \dots, n_l$ which are divisible by l. Without loss of generality, we may assume that $l \nmid n_i (1 \leq i \leq l-2)$. Then $(n_i, n_j) \leq l-1$ for all $1 \leq i < j \leq l$ and $1 \leq i \leq l-2$. Since the conjecture is true for $k = l-1$, by considering $\{n_i\}_{i=1}^{l-1}$ and $\{n_i\}_{i=1, i \neq l-1}^{l}$, we have $(n_i, n_j) = l-1$ for all $1 \leq i < j \leq l$ and $1 \leq i \leq l-2$. Thus $l-1 \mid n_{l-1}$ and $l-1 \mid n_l$. By $(n_{l-1}, n_l) \leq l$ we have $(n_{l-1}, n_l) = l-1$. Hence $(n_i, n_j) = l-1$ for all $1 \leq i < j \leq l$. Since $\{n_i\}_{i=1}^{l}$ is a harmonic sequence, by Lemma 1, there exist l integers $a_1, \dots, a_l$ such that $(n_i, n_j) \nmid a_i - a_j$ for all $1 \leq i < j \leq l$, that is, $a_1, \dots, a_l$ are incongruent each other modulo $l-1$, a contradiction. This completes the proof of Lemma 2. □

Lemma 3. *Let $d \geq 2$, $\{n_i\}_{i=1}^{dh}$ be a harmonic sequence with $(n_i, n_j) \leq dh$ for all $1 \leq i < j \leq dh-r$ and $h \mid n_i$ for $1 \leq i \leq dh-r$, and let $a_1, \dots, a_{dh-r}$ be integers with $(n_i, n_j) \nmid a_i - a_j$ for all $1 \leq i < j \leq dh-r$. If the conjecture is true for all $2 \leq k \leq d+1$, then there are at least $h-r$ residue classes modulo h in each of which there are exactly d integers of $a_1, \dots, a_{dh-r}$ and $dh \mid n_i$ when a_i are in these $h-r$ residue classes. Thus $dh \mid n_i$ for at least $d(h-r)$ of $i \in \{1, \dots, dh-r\}$.*

Proof. If there are $d+1$ integers of $a_1, \dots, a_{dh-r}$ in the same residue class modulo h, say $a_{i_j} = hb_{i_j} + s$, $j = 1, \dots, d+1$, then, by $(n_{i_u}, n_{i_v}) \nmid a_{i_u} - a_{i_v}$ we have

$$\left(\frac{n_{i_u}}{h}, \frac{n_{i_v}}{h}\right) \nmid b_{i_u} - b_{i_v}$$

for all $1 \leq u < v \leq d+1$. Thus $\{\frac{n_{i_j}}{h}\}_{j=1}^{d+1}$ is harmonic. By $(\frac{n_{i_u}}{h}, \frac{n_{i_v}}{h}) \leq d$ for all $1 \leq u < v \leq d+1$ and the conjecture being true for $k = d+1$, we have $(\frac{n_{i_u}}{h}, \frac{n_{i_v}}{h}) = d+1$ for all $1 \leq u < v \leq d+1$, a contradiction with $(\frac{n_{i_u}}{h}, \frac{n_{i_v}}{h}) \leq d$. Hence there are at most d integers of $a_1, \dots, a_{dh-r}$ in each residue class modulo h. By $(h-r-1)d + (r+1)(d-1) < dh-r$, there exist at least $h-r$ residue classes modulo h in each of which there are exactly d integers of $a_1, \dots, a_{dh-r}$.

Let $a_{w_j} = hb_{w_j} + t$, $j = 1, \dots, d$, then, by $(n_{w_u}, n_{w_v}) \nmid a_{w_u} - a_{w_v}$ we have

$$\left(\frac{n_{w_u}}{h}, \frac{n_{w_v}}{h}\right) \nmid b_{w_u} - b_{w_v}$$

for all $1 \le u < v \le d$. Thus $\left\{\frac{n_{w_j}}{h}\right\}_{j=1}^{d}$ is harmonic. By $(\frac{n_{w_u}}{h}, \frac{n_{w_v}}{h}) \le d$ for all $1 \le u < v \le d$ and the conjecture being true for $k = d$, we have $(\frac{n_{i_u}}{h}, \frac{n_{i_v}}{h}) = d$ for all $1 \le u < v \le d$. Hence $dh \mid n_i$ for at least $(h-r)d$ of $i \in \{1, \dots, dh-r\}$. This completes the proof of Lemma 3. □

Lemma 4. *Let p be a prime with $p \mid l+1$. If the conjecture is true for all $2 \le k \le l$, $\{n_i\}_{i=1}^{l+1}$ is harmonic with $(n_i, n_j) \le l+1$ for all $1 \le i < j \le l+1$ and $p \mid n_i$ for all $1 \le i \le l+1$, then $(n_i, n_j) = l+1$ for all $1 \le i < j \le l+1$.*

Proof follows from Lemma 3 immediately.

Lemma 5. *Let $\{n_i\}_{i=1}^{k}$ be harmonic with $(n_i, n_j) \le k$ for all $1 \le i < j \le k$. If $k \mid n_1, \dots, k \mid n_t$, then, either $(n_i, k) > 1$ for all $1 \le i \le k$ or $t \le \pi(k) - \omega(k)$, where $\pi(k)$ and $\omega(k)$ denote the number of primes not exceeding k and the number of distinct prime factors of k respectively. Furthermore, if $(n_j, k) = 1$, then n_j has at least t distinct prime factors which are less than k.*

Proof. Assume that $(n_{i_0}, k) = 1$ for some i_0. Since $(n_i, n_j) \le k$ for all $1 \le i < j \le t$, we have $(\frac{n_i}{k}, \frac{n_j}{k}) = 1$ for all $1 \le i < j \le t$. By $k \ge (n_{i_0}, \frac{n_i}{k}) = (n_{i_0}, n_i) > 1$ for all $1 \le i \le t$, we have n_{i_0} has at least t distinct prime factors which are less than k. Noting that $(n_{i_0}, k) = 1$, we have $t \le \omega(n_{i_0}) \le \pi(k) - \omega(k)$. This completes the proof of Lemma 5. □

Lemma 6. *Let d be a positive odd integer with $d \ge 3$ and r a positive integer. If the conjecture is true for $k < 2d$ and $\{n_i\}_{i=1}^{2d}$ is harmonic with $(n_i, n_j) \le 2d$ for all $1 \le i < j \le 2d$ such that $2 \mid n_i$ for $1 \le i \le 2d - r$ and $2 \nmid n_j$, $d \mid n_j$ for $2d - r + 1 \le j \le 2d$, then $r \ge 2$.*

Proof. Let $a_1, \dots, a_{2d}$ be integers with $(n_i, n_j) \nmid a_i - a_j$ for all $1 \le i < j \le 2d$. If $r = 1$, then by Lemma 3 there are d integers of $a_1, \dots, a_{2d-1}$

which are in the same residue class modulo 2, say $a_1, \dots, a_d$, and $2d \mid n_i$ for $1 \le i \le d$. Since $d \mid n_{2d}$ and $2d \nmid n_{2d}$, we have $(n_{2d}, n_i) = d$ by $(n_{2d}, n_i) \le 2d$ for $1 \le i \le d$. Thus $a_i \not\equiv a_{2d} \pmod d$ for $1 \le i \le d$. That is, a_i $(1 \le i \le d)$ are in $d-1$ residue classes modulo d. Hence there are two integers of a_i $(1 \le i \le d)$ which are congruent modulo $2d$, a contradiction. This completes the proof of Lemma 6. □

Lemma 7. *Let p be an old prime and r, s be positive integers. Suppose that the conjecture is true for all $2 \le k < 2p$ and $\{n_i\}_{i=1}^{2p}$ is harmonic with $(n_i, n_j) \le 2p$ for all $1 \le i < j \le 2p$,*

$$2 \mid n_i,\ p \nmid n_i\ (1 \le i \le s), \quad 2p \mid n_j (s+1 \le j \le 2p-r),$$

$$p \mid n_u,\ 2 \nmid n_u (2p - r + 1 \le u \le 2p).$$

Then each n_i $(1 \le i \le s)$ has at least $\max\{r, p-s+r-1\}+1$ *distinct prime factors which are less than $2p$ and*

$$\max\{r, p-s+r-1\} \le \pi(2p) - 2.$$

Proof. By $(n_u, n_{u'}) \le 2p$ we have $(n_u, n_{u'}) = p$ for $2p - r + 1 \le u < u' \le 2p$. Hence, all a_u are incongruent modulo p and then $r \le p$. Since $(n_j, n_u) = p$, we have that all a_j are in at most $p - r$ residue classes modulo p. Given $1 \le i \le s$. If there are $p-r+2$ of n_j which have no prime factors of n_i beyond 2, then there are $p - r + 1$ of n_j with $4 \nmid n_j$ which have no prime factors of n_i beyond 2. For these n_j, we have $(n_i, n_j) = 2$ and then $a_j \not\equiv a_i \pmod 2$. Thus, these corresponding a_j are congruent modulo 2. Since these a_j are in at most $p - r$ residue classes modulo p, there exist j, j' with $a_j \equiv a_{j'} \pmod{2p}$, a contradiction. Hence there are at most $p - r + 1$ of n_j which have no prime factors of n_i beyond 2. By $(n_i, n_u) > 1$, $2 \nmid (n_i, n_u)$ and $p \nmid (n_i, n_u)$, we have that each n_u has at least one prime factor of n_i beyond 2. Since

$$\left(\frac{n_j}{p}, \frac{n_{j'}}{p}\right) = 2, \ \left(\frac{n_u}{p}, \frac{n_{u'}}{p}\right) = 1, \ \left(\frac{n_j}{p}, \frac{n_u}{p}\right) = 1,$$

we have that each n_i has at least $r + \max\{0, 2p - r - s - (p - r + 1)\} + 1 = \max\{r, p - s + r - 1\} + 1$ distinct prime factors which are less than $2p$ and then

$$\max\{r, p-s+r-1\} \le \pi(2p)-2.$$

This completes the proof of Lemma 7. □

3. Proof of the theorem

Let $\{n_i\}_{i=1}^k$ be a harmonic sequence with $(n_i, n_j) \le k$ for all $1 \le i < j \le k$. Let $a_1, \dots, a_k$ be integers with $(n_i, n_j) \nmid a_i - a_j$ for all $1 \le i < j \le k$. For $k \ge 3$, by Lemma 2, we may assume that $k \mid n_1$, $k \mid n_2$ and $k \mid n_3$. It is enough to prove that $k \mid n_i$ for all i. For $d \ge 2$, let

$$A_d = \{i : 1 \le i \le k,\ d \mid n_i\}$$

and

$$\overline{A_d} = \{i : 1 \le i \le k,\ d \nmid n_i\}.$$

By Lemma 4 we may assume that $|\overline{A_d}| \ge 1$ and $|A_d| \le k-1$.

For $2 \le k \le 7$, M. R. Sun [4] proved that the conjecture is true. In fact, the cases $k = 2, 3$ are clear.

Case $k = 4$: By Lemma 5 we have $(4, n_i) > 1$. Thus $2 \mid n_i$ $(1 \le i \le 4)$. Then, by Lemma 4 we obtain a proof.

Case $k = 5$: By Lemma 5 we have $5 \mid n_i$ $(1 \le i \le 5)$.

Case $k = 6$: By Lemma 5 we have $(6, n_i) > 1$ $(i = 4, 5, 6)$. By Lemmas 4 and 6 we may assume that $2 \nmid n_4$, $2 \nmid n_5$ and $3 \nmid n_6$. By Lemma 7 we have $\max\{2, 3-1+2-1\} \le \pi(6)-2$, a contradiction.

Case $k = 7$: If there are four of n_i which are divisible by 7, then by Lemma 5 we have $7 \mid n_i$ for all $1 \le i \le 7$. Now assume that $7 \nmid n_i$ $(4 \le i \le 7)$. By Lemma 5 each of n_i $(4 \le i \le 7)$ has at least three distinct prime factors which are less than 7. This means that $30 \mid n_i$ $(4 \le i \le 7)$, a contradiction with $(n_i, n_j) \le 7$.

Case $k = 8$: By Lemma 5 we know that if $2 \nmid n_i$, then n_i has at least three prime factors which are less than 8. Hence n_i must be divisible by 3, 5 and 7. Thus there are at most one n_i with $2 \nmid n_i$. By Lemma 3 there are at least four of n_i with $8 \mid n_i$. By $4 > \pi(8) - \omega(8)$ and Lemma 5 we have $2 \mid n_i$ for all $1 \le i \le 8$. By Lemma 4 we have $8 \mid n_i$ for all i.

Case $k = 9$: By Lemma 5 we know that if $3 \nmid n_i$, then n_i has at least three prime factors which are less than 9. Hence n_i must be divisible by 2, 5 and 7. Thus there are at most one n_i with $3 \nmid n_i$. By Lemma 3 there are at least six of n_i with $9 \mid n_i$. By $6 > \pi(9) - \omega(9)$ and Lemma 5 we have $3 \mid n_i$ for all $1 \le i \le 9$. By Lemma 4 we have $9 \mid n_i$ for all i.

Case $k = 10$: By $3 > \pi(10) - \omega(10)$ and Lemma 5 we have $(10, n_i) > 1$ for all i. By Lemmas 4 and 6 we have $s \ge 1$, $r \ge 2$, where r, s are as in Lemma 7. By Lemma 7 we have that if $5 \nmid n_i$, then n_i has at least 3 distinct prime factors which are less than 10 and thus $42 \mid n_i$. Hence $s = 1$. By Lemma 7, $5 - 1 + r - 1 \le \pi(10) - 2$, a contradiction.

Case $k = 11$: By Lemma 5, if $i \in \overline{A_{11}}$, then n_i must be divisible by at least three of 2, 3, 5 and 7. Noting that $(n_i, n_j) \le 11$, we have $|\overline{A_{11}}| \le 4$. That is, $|A_{11}| \ge 7$. By Lemma 5 we have $|A_{11}| = 11$.

Case $k = 12$: Subcase 12.1: $|A_2 \cup A_3| \le 11$. If $i \notin A_2 \cup A_3$ and $j \in A_3$, then $(n_i, n_j) = 5, 7, 11$. Since $|A_3| \ge 3$ and each of $5, 7$ and 11 divides at most one of n_j with $j \in A_3$ by $(n_j, n_{j'}) \le 12$, we have $A_3 = \{1, 2, 3\}$ and n_i must be divisible by $5 \times 7 \times 11$. Thus $|A_2 \cup A_3| = 11$. By $\{1, 2, 3\} \subseteq A_2$ we have $|A_2| = 11$. By Lemma 3 we have $|A_{12}| \ge 6$. This contradicts with $|A_3| = 3$.

Subcase 12.2: $|A_2 \cup A_3| = 12$ and $|A_2| = 11$. By Lemma 3, without loss of generality, we may assume that a_i $(1 \le i \le 6)$ are in one residue class modulo 2 and $12 \mid n_i$ $(1 \le i \le 6)$. Since $|A_9 \cap A_2| \le 1$, we may assume that $9 \nmid n_i$ $(1 \le i \le 5)$. Let $j \notin A_2$. Then $(n_i, n_j) = 3$ $(1 \le i \le 5)$. Thus $a_i \not\equiv a_j \pmod 3$ $(1 \le i \le 5)$. That is, a_i $(1 \le i \le 5)$ are in two residue classes modulo 3. Hence a_i $(1 \le i \le 5)$ are in four residue classes modulo 12, a contradiction.

Subcase 12.3: $|A_2 \cup A_3| = 12$ and $|A_3| = 11$. By Lemma 3, without loss of generality, we may assume that a_i $(1 \le i \le 8)$ are in two residue classes modulo 3 and $12 \mid n_i$ $(1 \le i \le 8)$. Since $|A_3 \cap A_8| \le 1$ and $|A_3 \cap A_5| \le 1$, we may assume that $8 \nmid n_i$ $(1 \le i \le 8,\ i \ne i_0)$ and $5 \nmid n_i$ $(1 \le i \le 8,\ i \ne j_0)$.

Let $j \notin A_3$. If $4 \nmid n_j$, then $(n_i, n_j) = 2$ $(1 \le i \le 8,\ i \ne j_0)$. Thus $a_i \not\equiv a_j \pmod 2$ $(1 \le i \le 8,\ i \ne j_0)$. That is, a_i $(1 \le i \le 8,\ i \ne j_0)$ are in one residue class modulo 2. Hence a_i $(1 \le i \le 8,\ i \ne j_0)$ are in four residue classes modulo 12, a contradiction. If $4 \mid n_j$, then $(n_i, n_j) = 4$ $(1 \le i \le 8,\ i \ne i_0)$. Thus $a_i \not\equiv a_j \pmod 4$ $(1 \le i \le 8,\ i \ne i_0)$. That is, a_i $(1 \le i \le 8,\ i \ne i_0)$ are in three residue classes modulo 4. Hence a_i $(1 \le i \le 8,\ i \ne i_0)$ are in six residue classes modulo 12, a contradiction.

Subcase 12.4: $|A_2 \cup A_3| = 12$, $|A_2| \le 10$ and $|A_3| \le 10$. For $i \notin A_2$ and $j \notin A_3$, we have $(n_i, n_j) = 5, 7, 11$ and $i \in A_3$, $j \in A_2$. Since each of 5, 7 and 11 divides at most one of n_i with $i \in A_3$, we have that if $|\overline{A_2}| \ge 3$, then n_j must be divisible by $5 \times 7 \times 11$, a contradiction with $|\overline{A_3}| \ge 2$ and $(n_j, n_{j'}) \le 12$. Hence $|\overline{A_2}| = 2$. Thus n_j must be divisible by at least two of $5, 7, 11$. Since for $j, j' \notin A_3$, $(n_j, n_{j'}) \le 12$, we have $|\overline{A_3}| = 2$. Without loss of generality, we may assume that

$$6 \mid n_i \ (1 \le i \le 8), \quad 2 \mid n_9, \quad 2 \mid n_{10}, \quad 3 \nmid n_9,$$

$$3 \nmid n_{10}, \quad 2 \nmid n_{11}, \quad 2 \nmid n_{12}, \quad 3 \mid n_{11}, \quad 3 \mid_{12} .$$

By $(n_i, n_j) = 5, 7$ and 11 $(i = 9, 10; j = 11, 12)$, without loss of generality, we may assume that $5 \times 7 \mid n_9$, $5 \times 11 \mid n_{10}$, $5 \mid n_{11}$ and $7 \times 11 \mid n_{12}$. Further, we may assume that $4 \nmid n_9$. Thus

$$(n_i, 5 \times 7 \times 11) = 1, \quad (n_i, n_9) = 2 \quad (1 \le i \le 8).$$

So $a_i \not\equiv a_9 \pmod 2$ $(1 \le i \le 8)$. Hence there are $1 \le i < i' \le 8$ with $a_i \equiv a_{i'} \pmod{12}$, a contradiction.

Case $k = 13$: By Lemma 5, for $i \notin A_{13}$, n_i must be divisible by at least three of 2, 3, 5, 7 and 11. Since $|A_{77}| \le 1$, $|A_{55}| \le 1$ and $|A_{35}| \le 1$, we have $|\overline{A_{13}}| \le 6$. Thus $|A_{13}| \ge 7$. By Lemma 5 we have $13 \mid n_i$ for all i.

Case $k = 14$: Subcase 14.1: $|\overline{A_2} \bigcap \overline{A_7}| \ge 2$. We may assume that $(n_{13}, 14) = 1$, $(n_{14}, 14) = 1$. By Lemma 5 we have that n_{13}, n_{14} must be divisible by at least three of 3, 5, 11 and 13. Thus $(n_{13}, n_{14}) \ge 15$. This contradicts with $(n_{13}, n_{14}) \le 14$.

Subcase 14.2: $|\overline{A_2} \bigcap \overline{A_7}| = 1$. We may assume that $(n_{14}, 14) = 1$, $(n_i, 14) > 1, 1 \le i \le 13$, then $(n_i, n_{14}) = 3, 5, 9, 11$ and 13 $(1 \le i \le 13)$.

If 7 divides five of n_i with $1 \le i \le 13$, then one of 3, 5, 11 and 13 must divide at least two of the five integers. Thus the largest common divisor of the two integers is not less than 21, a contradiction. So $|A_7| \le 4$. By Lemma 2 there are at least three n_i with $i \in A_7$ which must be divisible by 2. Thus there are at least twelve even integers in $n_1, \dots, n_{13}$. If $n_1, \dots, n_{13}$ are all even integers, then by Lemma 3 we know that 14 must divide at least seven integers in $n_1, \dots, n_{13}$, a contradiction. Now we may assume that

$$14 \mid n_i \ (i = 1, 2, 3),\ 7 \mid n_4,\ 2 \nmid n_4,\ 2|n_j,\ 7 \nmid n_j \ (5 \le j \le 13).$$

Since $(n_{14}, n_i) = 3, 5, 9, 11, 13$ $(1 \le i \le 4)$ and each of 3, 5, 11 and 13 divides at most one of n_1, n_2, n_3 and n_4 otherwise $(n_i, n_j) \ge 21$ for some $i \ne j$, we have that $3 \times 5 \times 11 \times 13 \mid n_{14}$ and may assume that $p_i \mid n_i$ $(i = 1, 2, 3, 4)$, where p_1, p_2, p_3 and p_4 is a permutation of 3, 5, 11 and 13. By $(n_4, n_i) = 3, 5, 9, 11$ and 13 $(5 \le i \le 13)$, we have $p_4 \mid n_i$ $(5 \le i \le 13)$. Thus $(p_1p_2p_3, n_i) = 1$ $(5 \le i \le 13)$ otherwise $(n_i, n_{14}) \ge 15$. Since at most one of n_1, n_2 and n_3 is divisible by 4, we may assume that $4 \nmid n_1$. Then $(n_i, n_1) = 2$ and $a_i \not\equiv a_1 \pmod 2$ $(5 \le i \le 13)$. Let $a_i = 2b_i + a_1 + 1$ $(5 \le i \le 13)$. Then

$$\left(\frac{n_i}{2}, \frac{n_j}{2}\right) \nmid b_i - b_j, \quad \left(\frac{n_i}{2}, \frac{n_j}{2}\right) \le 7, \quad 5 \le i < j \le 13.$$

This contradicts Case $k = 9$.

Subcase 14.3: $|\overline{A_2} \bigcap \overline{A_7}| = 0$. That is $(n_i, 14) > 1$, $1 \le i \le 14$. Let

$$2 \mid n_i,\ 7 \nmid n_i \ (1 \le i \le s), \quad 14 \mid n_j \ (s+1 \le j \le 14 - r),$$

$$7 \mid n_u,\ 2 \nmid n_u \ (15 - r \le u \le 14).$$

By Lemma 4 we may assume that $s \ge 1$ and $r \ge 1$. By Lemma 2 we have $s + r \le 11$. By Lemma 7, each n_i $(1 \le i \le s)$ must be divisible by at least $\max\{r, 6 - s + r\}$ of 3, 5, 11 and 13, and $\max\{r, 6 - s + r\} \le 4$. By Lemma 6 we have $r \ge 2$. Thus $s \ge 2 + r \ge 4$. Since each of 11 and 13 divides at most one of $n_1, \dots, n_s$, there are at least two of $n_1, \dots, n_s$ which must be divisible by 3×5, a contradiction. This completes the proof of Case $k = 14$.

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