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A note on generalized inverses and a block-rank equation

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Abstract. In this paper we study the rank equation $\operatorname{rank}\left[\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right] = \operatorname{rank}(A)$ and find the necessary and sufficient conditions when $X = A^{(1,2)}$ and $X = A^d$ are the solutions of that equation. In both cases we give a explicit form of matrices B and C.

1. Introduction

Let $C^{m\times n}$ denote the set of complex $m \times n$ matrices. I_n denotes the unit matrix of order n. By A^*, $R(A)$, $\operatorname{rank}(A)$ and $N(A)$ we denote the conjugate transpose, the range, the rank and the null space of $A \in C^{n\times m}$. The symbol A^- stands for an arbitrary generalized inner inverse of A, i.e. A^- satisfies $AA^-A = A$. By $A^\dagger$ we denote the Moore–Penrose inverse of A, i.e. the unique matrix $A^\dagger$ satisfying

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

For $A \in C^{n\times n}$ the smallest nonnegative integer k such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$ is called the index of A and denoted by $\operatorname{ind}(A)$. If $A \in C^{n\times n}$, with $\operatorname{ind}(A) = k$, then the matrix $X \in C^{n\times n}$ which satisfies the following

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conditions

$$A^kXA = A^k,\ XAX = X,\ AX = XA,$$

is called the Drazin inverse of A and it is denoted by A^d. When $\operatorname{ind}(A) = 1$ then the Drazin inverse A^d is called the group inverse and it is denoted by $A^\#$. Also, the matrix X which satisfies

$$AXA = A \quad \text{and} \quad XAX = X$$

is called the reflexive inverse of A and it is denoted by $A^{(1,2)}$. For other important properties of generalized inverses see [1] and [3].

In this paper we will consider the rank equation

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = \operatorname{rank}(A), \tag{1}$$

for arbitrary $A \in C^{n\times n}$. First, we give a necessary and sufficient conditions such that $X = A^{(1,2)}$ is the solution of equation (1) and all possible matrices B and C are described. As a corollary we obtain the result of J. Gross [8] and N. Thome and Y. Wei [7]. Moreover, we consider when $X = A^d$ is the solution of the equation (1), for an arbitrary matrix A with $\operatorname{ind}(A) = k \geq 1$ and we obtain some interesting corollaries.

2. Main results

We start this section with some well-known results. The following lemma was proved in [4], [5] and [6].

Lemma 2.1. *Let $A \in C^{n\times n}$, $B \in C^{n\times m}$, $C \in C^{m\times n}$ and $X \in C^{m\times m}$. Then*

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(L) + \operatorname{rank}(M) + \operatorname{rank}(W),$$

where $S = I_n - A^-A$, $L = CS$, $M = SB$ and $W = (I_m - LL^-)(X - CA^-B)(I_m - M^-M)$.

The following theorem, which is proved by J. Gross [8], gives a characterization of the existence of the solution of the equation (1) by means of geometrical conditions.

Theorem 2.1. *Let $A \in C^{m\times n}, B \in C^{m\times m}$ and $C \in C^{n\times n}$. Then there exists a solution $X \in C^{n\times m}$ of the equation* (1) *if and only if $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$, in which case $X = CA^\dagger B$.*

Notice that the conditions $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$ are equivalent to $AA^\dagger B = B$ and $CA^\dagger A = C$. Also, the matrix product CA^-B is invariant with respect to the choice of generalized inverse A^- of A if and only if $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$.

First we consider a necessary and sufficient conditions such that $X = A^{(1,2)}$ is the solution of the equation (1) and in this case we find the explicit form for B and C.

Matrix $A \in C^{m\times n}$ such that $\text{rank}(A) = r$ can be decomposed by

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q, \tag{2}$$

where $P \in C^{m\times m}$, $Q \in C^{n\times n}$ and $D \in C^{r\times r}$ are invertible matrices. Given that decomposition arbitrary reflexive generalized inverse of A has the following form

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & U \\ V & VDU \end{bmatrix} P^{-1} \tag{3}$$

where U and V are arbitrary matrices of suitable size (see [2]).

The following theorem gives a sufficient and necessary conditions such that $X = A^{(1,2)}$ is the solution of the equation (1).

Theorem 2.2. *Let $A \in C^{m\times n}$, $B \in C^{m\times m}$ $C \in C^{n\times n}$ and $X \in C^{n\times m}$ and let the matrix A and its reflexive generalized inverse be given by* (2) *and* (3) *respectively. Then $X = A^{(1,2)}$ is the solution of the equation* (1) *if and only if*

$$B = P \begin{bmatrix} DL & (DLD)U \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ VL^{-1} & 0 \end{bmatrix} Q \tag{4}$$

for some nonsingular matrix $L \in C^{r\times r}$.

Proof. Suppose that $X = A^{(1,2)}$ is the solution of the equation (1). Then there exist matrices $G \in C^{n\times m}$ and $F \in C^{n\times m}$ such that $B = AG$,

$C = FA$ and $CA^{-}B = A^{(1,2)}$. Let

$$QGP = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad QFP = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

Hence,

$$B = AG = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P^{-1} = P \begin{bmatrix} DG_1 & DG_2 \\ 0 & 0 \end{bmatrix} P^{-1} \tag{5}$$

and

$$C = FA = Q^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q = Q^{-1} \begin{bmatrix} F_1 D & 0 \\ F_3 D & 0 \end{bmatrix} Q. \tag{6}$$

Also,

$$A^{(1,2)} = FAG = Q^{-1} \begin{bmatrix} F_1 DG_1 & F_1 DG_2 \\ F_3 DG_1 & F_3 DG_2 \end{bmatrix} P^{-1}.$$

Now, from (3) we have that

$$F_1 DG_1 = D^{-1}, \quad F_1 DG_2 = U, \quad F_3 DG_1 = V.$$

From the first equation we obtain that F_1, G_1 are invertible matrices and $F_1 D = D^{-1} G_1^{-1}$. Now, $DG_2 = F_1^{-1} U = DG_1 DU$ and $F_3 D = VG_1^{-1}$. If we replace that in (5) and (6) and put $G_1 = L$, we obtain (4).

Now, suppose that (4) holds. Then $AA^{-}B = B$ and $C = CA^{-}A$, for generalized inner inverse A^{-} of A, which is given by

$$A^{-} = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

So by Theorem 2.1 there exists a solution $X = CA^{-}B$ of the equation (1). By (4) we can easily check that $X = CA^{-}B = A^{(1,2)}$. □

Remark that when we consider the special reflexive inverse of A,

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \tag{7}$$

for $U = V = 0$, we obtain the ([7], Theorem 3).

Corollary 2.1. *Let $A \in C^{m\times n}$, $B \in C^{m\times m}$, $C \in C^{n\times n}$ and $X \in C^{n\times m}$. Let the matrix A and one its reflexive generalized inverse be given by* (2) *and* (7) *respectively. Then $X = A^{(1,2)}$ is the solution of the equation* (1) *if and only if*

$$B = P \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q, \tag{8}$$

for some nonsingular matrix $L \in C^{r\times r}$.

Now, we consider the singular value decomposition of $A \in C^{m\times n}$ such that $\operatorname{rank}(A) = r$

$$A = M \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} N^*, \tag{9}$$

where $M \in C^{m\times m}$ and $N \in C^{n\times n}$ are unitary and $D \in C^{r\times r}$ is a real positive definite diagonal matrix. By Theorem 2.2 we obtain ([8], Theorem 2).

Corollary 2.2. *Let $A \in C^{m\times n}$, $B \in C^{m\times m}$, $C \in C^{n\times n}$ and $X \in C^{n\times m}$. Let the matrix A be given by* (9). *Then $X = A^\dagger$ is the solution of the equation* (1) *if and only if*

$$B = M \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} M^* \quad \text{and} \quad C = N \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} N^*, \tag{10}$$

for some nonsingular matrix $L \in C^{r\times r}$.

PROOF. Taking $P = M$ and $Q = N^*$ in (2), we obtain that the matrix A has the representation (9) and in that case $A^\dagger = N \left[\begin{smallmatrix} D^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] M^*$, which has the form (7). Hence, the result follows from Corollary 2.1. □

In the rest of the paper, we consider the following question: When $X = A^d$ is the solution of the equation (1)?

First, let $A \in C^{n\times n}$ and $\operatorname{ind}(A) = 1$. Using the Jordan canonical form of A, there exist nonsingular matrices $P \in C^{n\times n}$ and $D \in C^{r\times r}$ such that

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{-1}. \tag{11}$$

We obtain the result of N. THOME and Y. WEI ([7], Theorem 2).

Theorem 2.3. *Let* $A \in C^{n\times n}$ *with* $\operatorname{ind}(A) = 1$ *and* $\operatorname{rank}(A) = r$ *be given by* (11) *and* $B, C, X \in C^{n\times n}$. *Then* $X = A^{\#}$ *is the solution of the equation* (1) *if and only if*

$$B = P\begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = P\begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \tag{12}$$

for some nonsingular matrix $L \in C^{r\times r}$.

Proof. If the matrix A is given by (11), then $A^{\#} = P\left[\begin{smallmatrix} D^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P^{-1}$. Hence, the result follows from Corollary 2.1 taking $Q = P^{-1}$ and noticing that $A^{\#}$ is given by (7). □

Now, we consider a more general case when $A \in C^{n\times n}$ is such that $\operatorname{ind}(A) = k \geq 1$ and $\operatorname{rank}(A) = r$. Then the matrix A can be written as

$$A = P^{-1}\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P, \tag{13}$$

where $P \in C^{n\times n}$, $M \in C^{r\times r}$ are nonsingular matrices and $N \in C^{(n-r)\times(n-r)}$ is nilpotent, that is $N^k = 0$. In this case

$$A^d = P^{-1}\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P.$$

Theorem 2.4. *Let* $A \in C^{n\times n}$, *with index*$(A) = k$, *be represented by* (13) *and* $B, C, X \in C^{n\times n}$.*Then* $X = A^d$ *is the solution of the equation* (1) *if and only if there exist* $G_1, F_1 \in C^{r\times r}$, $G_2, F_2 \in C^{r\times(n-r)}$, $G_3, F_3 \in C^{(n-r)\times r}$ *and* $G_4, F_4 \in C^{(n-r)\times(n-r)}$ *such that*

$$B = P^{-1}\begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \text{ and } C = P^{-1}\begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P \tag{14}$$

and

$$\begin{aligned} F_1MG_1 + F_2NG_3 &= M^{-1}, \\ F_1MG_2 + F_2NG_4 &= 0, \\ F_3MG_1 + F_4NG_3 &= 0, \\ F_3MG_2 + F_4NG_4 &= 0. \end{aligned} \tag{15}$$

Proof. Suppose that $X = A^d$ is the solution of the equation (1). From Theorem 2.1 we have that $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$, so there exist matrices G and F such that $B = AG$ and $C = FA$ and $A^d = CA^\dagger B$. Let

$$PGP^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad PFP^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

It follows that

$$B = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P$$

and

$$C = P^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P = P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P.$$

Since the matrix A has the form (13), it follows that

$$A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \quad \text{and} \quad A^\dagger = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} P.$$

Hence,

$$\begin{aligned} A^d &= P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} F_1MG_1 + F_2NG_3 & F_1MG_2 + F_2NG_4 \\ F_3MG_1 + F_4NG_3 & F_3MG_2 + F_4NG_4 \end{bmatrix} P. \end{aligned}$$

We obtain the following system

$$\begin{aligned} F_1MG_1 + F_2NG_3 &= M^{-1}, \\ F_1MG_2 + F_2NG_4 &= 0, \\ F_3MG_1 + F_4NG_3 &= 0, \\ F_3MG_2 + F_4NG_4 &= 0. \end{aligned}$$

Conversely, suppose that the matrices B and C satisfied (14). Then we see that $AA^\dagger B = B$ and $C = CA^\dagger A$. From Theorem 2.1 we have that there exists a solution $X = CA^\dagger B$ of the equation (1). Now, from the system (15) it follows that $CA^\dagger B = A^d$, so $X = A^d$ is the solution of the equation (1). □

Notice that Theorem 2.4 is a generalization of Theorem 2.3.

Now, we state some interesting results.

Theorem 2.5. *Let $A \in C^{n\times n}$, with $\operatorname{ind}(A) = k$ has the form (13), let p, m, n be positive integers and $m, n \geq k$. Then $X = A^d$ is the solution of the equation*

$$\operatorname{rank}\begin{bmatrix} A^p & A^n \\ A^m & X \end{bmatrix} = \operatorname{rank}(A^p), \tag{16}$$

if and only if $M^{m+n-p} = M^{-1}$.

Proof. Suppose that $X = A^d$ is the solution of the equation (16). Then $A^d = A^m(A^p)^- A^n$. Hence,

$$\begin{aligned} &P^{-1}\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \\ &\quad = P^{-1}\begin{bmatrix} M^m & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} M^{-p} & 0 \\ 0 & (N^p)^- \end{bmatrix}\begin{bmatrix} M^n & 0 \\ 0 & 0 \end{bmatrix} P \\ &\quad = P^{-1}\begin{bmatrix} M^{(m+n-p)} & 0 \\ 0 & 0 \end{bmatrix} P, \end{aligned}$$

i.e. $M^{m+n-p} = M^{-1}$.

On the contrary, suppose that $M^{m+n-p} = M^{-1}$. First, we show that there exists a solution X of the equation (16), i.e. that $R(A^n) \subseteq R(A^p)$ and $N(A^p) \subseteq N(A^m)$.

If $y \in R(A^n)$, then there exists x such that $y = A^n x$, i.e.

$$y = P^{-1}\begin{bmatrix} M^n z_1 \\ 0 \end{bmatrix}, \quad \text{where} \quad Px = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Now,

$$y = P^{-1}\begin{bmatrix} M^p & 0 \\ 0 & N^p \end{bmatrix}\begin{bmatrix} M^{(p-n)}z_1 \\ 0 \end{bmatrix},$$

implying that $y = A^p x'$, where $x' = P^{-1}\left[\begin{smallmatrix} M^{(p-n)}z_1 \\ 0 \end{smallmatrix}\right]$. Hence, $R(A^n) \subseteq R(A^p)$ and analogously $N(A^p) \subseteq N(A^m)$. Using the same computation as in the first part, we obtain that $X = A^d$ is the solution of the equation (16).

□

Remark 1. Notice that Theorem 2.5 is also valid if we put $f(n)$ and $g(m)$ instead of n, m, where f, g are arbitrary positive functions.

Corollary 2.3. *Let* $A \in C^{n\times n}$, *with* $\mathrm{ind}(A) = k$ *has the form* (13), *let* p, m, n *be positive integers such that* $m, n \geq k$ *and* $m + n = p - 1$. *Then* $X = A^d$ *is the solution of the equation* (16).

Corollary 2.4. *Let* $A \in C^{n\times n}$, *then*

$$\operatorname{rank}\begin{bmatrix} A^{(2l+1)} & A^l \\ A^l & A^d \end{bmatrix} = \operatorname{rank} A^{(2l+1)},$$

for arbitrary integer $l \geq \mathrm{ind}(A)$.

Corollary 2.5. *Let* $A \in C^{n\times n}$ *and* $\mathrm{ind}(A) = 1$, *then*

$$\operatorname{rank}\begin{bmatrix} A^3 & A \\ A & A^{\#} \end{bmatrix} = \operatorname{rank}(A^3).$$

References

[1] A. Ben-Israel and T. N. E. Greville, Generalized inverses: Theory and Applications, *Springer-Verlag*, *New York*, 2003.

[2] R. Bru and N. Thome, Group inverse and group involutory matrices, *Lin. Multilin. Alg.* **45** (2–3) (1998), 207–218.

[3] S. R. Caradus, Generalized Inverses and Operator Theory, Queen's Paper in Pure and Applied Mathematics, *Queen's University*, *Kingston, Ontario*, 1978.

[4] G. Marsaglia and G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Lin. Multilin. Alg.* **2** (1974), 269–292.

[5] C. D. Meyer Jr, Generalized inverses and ranks of block matrices, *SIAM Journal on Applied Mathematics* **25** (1973), 597–602.

[6] D. V. QUELLETTE, Schur complements and statistics, *Linear Algebra Appl.* **36** (1981), 187–295.

[7] N. THOME and Y. WEI, Generalized inverses and a block-rank equation, *Appl. Math. Comput.* **141** (2003), 471–476.

[8] J. GROSS, Solution to a rank equation, *Linear Algebra Appl.* **289** (1999), 127–130.

[9] Y. WEI, A characterization and representation of the Drazin inverse, *SIAM J. Matrix. Anal. Appl.* **17** (1996), 744–747.

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