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Coexistence problems for Hill's equations with three-step potentials

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Abstract. We study the coexistence of two linearly independent, periodic solutions of Hill's equation with a three-step potential. We give a simple, necessary and sufficient condition for the coexistence. Using this condition, we provide a formula for the number of nontrivial joints of the Arnold tongue of a family of Hill's equations with three-step potentials.

1. Introduction

The purpose of this paper is to give a simple, necessary and sufficient condition for Hill's equation with a three-step potential to admit two linearly independent, periodic solutions.

Given a subdivision

$$0 = t_0 < t_1 < t_2 < t_3 = 2\pi$$

of the interval $[0, 2\pi]$, we put

$$t = (t_1, t_2) \quad \text{and} \quad s_i = t_i - t_{i-1} \quad \text{for } i = 1, 2, 3.$$

For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, let $Q(a, t, \cdot\,) : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic step function such that

$$Q(a, t, \cdot\,) = a_i \quad \text{on} \quad [t_{i-1}, t_i) \quad \text{for } i = 1, 2, 3.$$

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We are concerned with Hill's equation of the form

$$-y''(x) + Q(a,t,x)y(x) = \lambda y(x) \text{ on } \mathbb{R}, \quad y, y' \in AC_{\mathrm{loc}}(\mathbb{R}), \tag{1}$$

where λ is a real parameter.

In order to formulate our claims, we recall from [5] some fundamental results and terminologies in the theory of Hill's equations. Let $y_1(a,t,\lambda,x)$ and $y_2(a,t,\lambda,x)$ be the solutions of the equation (1) subject to the initial conditions

$$y_1(a,t,\lambda,0) - 1 = y_1'(a,t,\lambda,0) = 0$$

and

$$y_2(a,t,\lambda,0) = y_2'(a,t,\lambda,0) - 1 = 0,$$

respectively. We introduce the discriminant of the equation (1):

$$D(a,t,\lambda) := y_1(a,t,\lambda,2\pi) + y_2'(a,t,\lambda,2\pi),$$

which is analytic in λ. Denoting by $\lambda_j(a,t)$ the jth root of the equation $D(a,t,\cdot)^2 - 4 = 0$ counted with multiplicity for each $j \in \mathbb{N}$, we have by the Liapounoff oscillation theorem (see [5, Theorem 2.1])

$$\lambda_1(a,t) < \lambda_2(a,t) \le \lambda_3(a,t) < \cdots < \lambda_{2k}(a,t) \le \lambda_{2k+1}(a,t) < \cdots. \tag{2}$$

This sequence also gives all the eigenvalues of (1) with the 4π-periodicity condition $y(\cdot + 4\pi) = y(\cdot)$ on $\mathbb{R}$ repeated according to multiplicity, while the subsequence

$$\lambda_1(a,t) < \lambda_4(a,t) \le \lambda_5(a,t) < \cdots < \lambda_{4k}(a,t) \le \lambda_{4k+1}(a,t) < \cdots$$

provides all the eigenvalues of (1) with the 2π-periodicity condition repeated according to multiplicity. If the equation (1) admits two linearly independent, periodic solutions of period 2π or 4π, we say that two such solutions coexist (see [5, Section 7.1]). Such coexistence is equivalent to the condition

$$\lambda = \lambda_{2k}(a,t) = \lambda_{2k+1}(a,t) \quad \text{for some } k \in \mathbb{N} = \{1,2,3,\dots\}.$$

The sequence (2) also characterizes the stability of the solutions of (1). Whenever all solutions of (1) are bounded on $\mathbb{R}$ we say that they are stable; otherwise we say that they are unstable. By the Liapounoff theorem, we see that the solutions of (1) are stable if and only if $\{\lambda\}$ is an interior point of the set

$$\bigcup_{k=1}^{\infty} [\lambda_{2k-1}(a,t), \lambda_{2k}(a,t)].$$

We call $(\lambda_{2k}(a,t), \lambda_{2k+1}(a,t))$ the kth instability interval for $k \in \mathbb{N}$. So the coexistence is also equivalent to the absence of the instability interval.

We define

$$p_i = p_i(a_i,\lambda) = \sqrt{\lambda - a_i}, \quad \arg p_i \in \left\{0, \frac{\pi}{2}\right\} \quad \text{for } i = 1,2,3.$$

Our main result is the following claim.

Theorem 1. *Let $k \in \mathbb{N}$. Assume that $a_m \neq a_n$ for $m \neq n$. Then the statements* (i) *and* (ii) *below are equivalent.*

(i) $\lambda = \lambda_{2k}(a,t) = \lambda_{2k+1}(a,t)$.

(ii) $s_1 p_1(a_1,\lambda) + s_2 p_2(a_2,\lambda) + s_3 p_3(a_3,\lambda) = k\pi$ *and* $s_i p_i(a_i,\lambda) \in \pi\mathbb{N}$ *for* $i = 1,2,3$.

Remark. For $k \in \mathbb{N}$, we claim by Theorem 1 that the kth instability interval is absent if and only if there exists $\lambda \in \mathbb{R}$ satisfying the statement (ii). In particular, both the first instability interval and the second instability interval are always present, provided that $a_m \neq a_n$ for $m \neq n$.

Next we give an application of the above theorem. We consider the family of Hill's equations

$$-y''(x) + \beta Q(a,t,x)y(x) = \lambda y(x) \text{ on } \mathbb{R}, \quad \lambda \in \mathbb{R},$$

indexed by $\beta \in \mathbb{R}$. For $k \in \mathbb{N}$, we define

$$R_k(a,t) = \{(\mu,\beta) \in \mathbb{R}^2 \mid \lambda_{2k}(\beta a,t) < \mu < \lambda_{2k+1}(\beta a,t)\}$$

and

$$P_k(a,t) = \{(\mu,\beta) \in \mathbb{R}^2 \mid \mu = \lambda_{2k}(\beta a,t) = \lambda_{2k+1}(\beta a,t),\ \beta \neq 0\}.$$

The set $R_k(a,t)$ is called the Arnord tongue or the instability region. The elements of $P_k(a,t)$ are called the resonance pockets, which are the nontrivial joints of $R_k(a,t)$. For a finite set K, we denote its cardinality by $\sharp K$. We note that $\sharp P_k(a,t)$ is equal to the number of bounded, connected components of $R_k(a,t)$. The following theorem gives a formula for $\sharp P_k(a,t)$.

Theorem 2. *Let $k \in \mathbb{N}$. Suppose that $a_m \neq a_n$ for $m \neq n$. Then it holds that*

$$\sharp P_k(a,t) = \sharp\Big\{(l_1,l_2,l_3) \in \mathbb{N}^3 \;\Big|\; \frac{a_2-a_3}{s_1^2}\, l_1^2 + \frac{a_3-a_1}{s_2^2}\, l_2^2 + \frac{a_1-a_2}{s_3^2}\, l_3^2 = 0,\ s_2 l_1 \neq s_1 l_2,\ l_1+l_2+l_3 = k\Big\}.$$

The coexistence problems for Hill's equations with *two-step* potentials have been studied in [2]–[4], and [6]. In order to review those results, we introduce needed notations. Given $0 < \kappa < 2\pi$ and $b = (b_1, b_2) \in \mathbb{R}^2$ with $b_1 \neq b_2$, let $W(b,\kappa,\cdot\,) : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $W(b,\kappa,\cdot\,) = b_1$ on $[0,\kappa)$ and that $W(b,\kappa,\cdot\,) = b_2$ on $[\kappa, 2\pi)$. E. Meissner [6] was the first to study the characteristic value problem

$$-z''(x) = \nu^2 W(b,\kappa,x) z(x) \text{ on } \mathbb{R}, \quad \nu > 0,$$

where $b_1, b_2 > 0$. He solved the coexistence problem for this equation in the case when $\kappa = \pi$. Furthermore, H. Hochstadt [4] investigated this problem for general κ. He proved that two linearly independent, periodic solutions to this equation can coexist for some ν if and only if the number $\sqrt{b_2/b_1}\,(2\pi - \kappa)/\kappa$ is rational. His method is based on a factorization of the function $\Delta(\nu) \pm 2$, where $\Delta(\nu)$ stands for the discriminant of this equation. Recently, Shaobo Gan and Meirong Zhang [2], [3] studied the eigenvalue problem

$$-z''(x) + W(b,\kappa,x) z(x) = \nu z(x) \text{ on } \mathbb{R}, \quad \nu \in \mathbb{R},$$

where $b_1, b_2 \in \mathbb{R}$. They obtained a necessary and sufficient condition for the coexistence (see [2, Theorem 2.3] and [3, Proposition 3.1]). They also proved that the number of the resonance pockets in the nth instability region of the family of equations

$$-z''(x) + \alpha W(b,\kappa,x) z(x) = \nu z(x) \text{ on } \mathbb{R}, \quad \nu \in \mathbb{R},\ \alpha \in \mathbb{R}$$

is given by

$$\begin{cases} n-2 & \text{for } \dfrac{n\kappa}{2\pi} \in \mathbb{N}, \\ n-1 & \text{for } \dfrac{n\kappa}{2\pi} \notin \mathbb{N}. \end{cases}$$

Their method is based on a characterization of the eigenvalue by the rotation number of the Prüfer transform of the solution.

Our idea to prove Theorem 1 is different from the ones in [2]–[4], and [6]; we make effective use of the full components of the monodromy matrix. This enables us to reduce the problem to a simple arithmetic.

2. Proof of theorems

By $M(a,t,\lambda)$ we denote the monodromy matrix of (1):

$$M(a,t,\lambda) = \begin{pmatrix} y_1(a,t,\lambda,2\pi) & y_2(a,t,\lambda,2\pi) \\ y_1'(a,t,\lambda,2\pi) & y_2'(a,t,\lambda,2\pi) \end{pmatrix}.$$

Using $-y_j''(x) = (\lambda - a_i)y_j(x)$ on (t_{i-1}, t_i) for $i = 1,2,3$ and $j = 1,2$, we have, in the case when $p_1(a_1,\lambda)p_2(a_2,\lambda)p_3(a_3,\lambda) \neq 0$,

$$\begin{aligned} & y_1(a,t,\lambda,2\pi) \\ &\quad = \cos s_1p_1 \cos s_2p_2 \cos s_3p_3 - \frac{p_1}{p_2} \sin s_1p_1 \sin s_2p_2 \cos s_3p_3 \\ &\qquad - \frac{p_1}{p_3} \sin s_1p_1 \cos s_2p_2 \sin s_3p_3 - \frac{p_2}{p_3} \cos s_1p_1 \sin s_2p_2 \sin s_3p_3, \end{aligned} \tag{3}$$

$$\begin{aligned} & y_1'(a,t,\lambda,2\pi) \\ &\quad = -p_1 \sin s_1p_1 \cos s_2p_2 \cos s_3p_3 - p_2 \cos s_1p_1 \sin s_2p_2 \cos s_3p_3 \\ &\qquad - p_3 \cos s_1p_1 \cos s_2p_2 \sin s_3p_3 + \frac{p_1p_3}{p_2} \sin s_1p_1 \sin s_2p_2 \sin s_3p_3, \end{aligned} \tag{4}$$

$$\begin{aligned} & y_2(a,t,\lambda,2\pi) \\ &\quad = \frac{1}{p_1} \sin s_1p_1 \cos s_2p_2 \cos s_3p_3 + \frac{1}{p_2} \cos s_1p_1 \sin s_2p_2 \cos s_3p_3 \\ &\qquad + \frac{1}{p_3} \cos s_1p_1 \cos s_2p_2 \sin s_3p_3 - \frac{p_2}{p_1p_3} \sin s_1p_1 \sin s_2p_2 \sin s_3p_3, \end{aligned} \tag{5}$$

$$\begin{aligned} & y_2'(a,t,\lambda,2\pi) \\ &\quad = \cos s_1p_1 \cos s_2p_2 \cos s_3p_3 - \frac{p_2}{p_1} \sin s_1p_1 \sin s_2p_2 \cos s_3p_3 \\ &\qquad - \frac{p_3}{p_1} \sin s_1p_1 \cos s_2p_2 \sin s_3p_3 - \frac{p_3}{p_2} \cos s_1p_1 \sin s_2p_2 \sin s_3p_3. \end{aligned} \tag{6}$$

Notice that the statement (i) in Theorem 1 is equivalent to the condition

$$M(a,t,\lambda) = (-1)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda \in \{\lambda_{2k}(a,t),\ \lambda_{2k+1}(a,t)\} \tag{7}$$

(see the proof of [5, Lemma 2.4]). Let us demonstrate Theorem 1.

Proof of Theorem 1. It suffices to show that (ii) in Theorem 1 and (7) are equivalent.

Let us prove that (7) yields (ii). Assume that (7) holds. Our first task is to deduce that $\sin s_1p_1 \sin s_2p_2 \sin s_3p_3 = 0$ by contradiction. Suppose that $\sin s_1p_1 \sin s_2p_2 \sin s_3p_3 \neq 0$. We put $x_i = \cot s_ip_i$ for $i = 1, 2, 3$. Inserting (3) $\sim$ (6) into the equalities

$$y_1'(a,t,\lambda,2\pi) = 0, \quad y_2(a,t,\lambda,2\pi) = 0, \quad y_2'(a,t,\lambda,2\pi) - y_1(a,t,\lambda,2\pi) = 0,$$

and dividing those by $\sin s_1p_1 \sin s_2p_2 \sin s_3p_3$, we obtain

$$\frac{p_1p_3}{p_2} - p_1x_2x_3 - p_2x_1x_3 - p_3x_1x_2 = 0, \tag{8}$$

$$-\frac{p_2}{p_1p_3} + \frac{1}{p_1}x_2x_3 + \frac{1}{p_2}x_1x_3 + \frac{1}{p_3}x_1x_2 = 0, \tag{9}$$

$$x_3 = -\frac{(p_1^2 - p_3^2)p_2}{(p_1^2 - p_2^2)p_3}x_2 - \frac{(p_2^2 - p_3^2)p_1}{(p_1^2 - p_2^2)p_3}x_1. \tag{10}$$

We deduce from (8) and (9) that

$$(-p_1p_2^2 + p_1p_3^2)x_2x_3 + \left(-p_2^3 + \frac{p_1^2p_3^2}{p_2}\right)x_1x_3 + (-p_3p_2^2 + p_1^2p_3)x_1x_2 = 0. \tag{11}$$

Plugging (10) into (11), we have

$$(p_2^2 - p_3^2)(p_1^2 - p_3^2)p_1p_2x_2^2 + 2p_1^2(p_2^2 - p_3^2)^2x_1x_2 - \frac{p_1(p_1^2p_3^2 - p_2^4)(p_2^2 - p_3^2)}{p_2}x_1^2 = 0$$

and hence

$$x_2 = \left\{ -\frac{p_1(p_2^2 - p_3^2)}{p_2(p_1^2 - p_3^2)} \pm \frac{p_3(p_1^2 - p_2^2)}{p_2(p_1^2 - p_3^2)} \right\} x_1. \tag{12}$$

This together with (10) implies that

$$x_3 = \mp x_1. \tag{13}$$

Combining (8) with (12) and (13), we conclude that

$$x_1^2 = -1.$$

This violates the fact that $\cot z \neq \pm\sqrt{-1}$ for $z \in \mathbb{C}$. Thus we get $\sin s_1 p_1 \sin s_2 p_2 \sin s_3 p_3 = 0$.

Next we shall show that $p_1 p_2 p_3 \neq 0$. Let us first prove that $p_1 \neq 0$ by contradiction. Suppose that $p_1 = 0$. Noting $y_j''(x) = 0$ on (t_0, t_1) for $j = 1, 2$, we have

$$y_1(a, t, \lambda, 2\pi) = \cos s_2 p_2 \cos s_3 p_3 - \frac{p_2}{p_3} \sin s_2 p_2 \sin s_3 p_3, \tag{14}$$

$$\begin{aligned} y_2'(a, t, \lambda, 2\pi) = {} & \cos s_2 p_2 \cos s_3 p_3 - \frac{p_3}{p_2} \sin s_2 p_2 \sin s_3 p_3 \\ & - s_1 (p_2 \sin s_2 p_2 \cos s_3 p_3 + p_3 \cos s_2 p_2 \sin s_3 p_3), \end{aligned} \tag{15}$$

$$y_1'(a, t, \lambda, 2\pi) = -p_2 \sin s_2 p_2 \cos s_3 p_3 - p_3 \cos s_2 p_2 \sin s_3 p_3, \tag{16}$$

$$\begin{aligned} y_2(a, t, \lambda, 2\pi) = {} & s_1 \cos s_2 p_2 \cos s_3 p_3 - \frac{s_1 p_2}{p_3} \sin s_2 p_2 \sin s_3 p_3 \\ & + \frac{1}{p_2} \sin s_2 p_2 \cos s_3 p_3 + \frac{1}{p_3} \cos s_2 p_2 \sin s_3 p_3. \end{aligned} \tag{17}$$

Inserting (14) and (15) into $y_1(a, t, \lambda, 2\pi) - y_2'(a, t, \lambda, 2\pi) = 0$, and combining that with (16) and $y_1'(a, t, \lambda, 2\pi) = 0$, we obtain

$$\frac{p_2^2 - p_3^2}{p_2 p_3} \sin s_2 p_2 \sin s_3 p_3 = 0$$

and hence

$$\sin s_2 p_2 \sin s_3 p_3 = 0.$$

This together with $y_1(a, t, \lambda, 2\pi) = (-1)^k$ and (14) implies that

$$\cos s_2 p_2 \cos s_3 p_3 = (-1)^k$$

and thus $\sin s_2 p_2 = \sin s_3 p_3 = 0$. Therefore, we infer by (17) that

$$y_2(a, t, \lambda, 2\pi) = s_1 (-1)^k \neq 0$$

which is a contradiction. Hence we have $p_1 \neq 0$. Similarly we get $p_2 \neq 0$ and $p_3 \neq 0$.

Next we shall show that $\sin s_1p_1 = \sin s_2p_2 = \sin s_3p_3 = 0$. Because $\sin s_1p_1 \sin s_2p_2 \sin s_3p_3 = 0$, we have $\sin s_1p_1 = 0$ or $\sin s_2p_2 = 0$ or $\sin s_3p_3 = 0$. We first consider the case that $\sin s_1p_1 = 0$. By (3), (6), and $y_1(a,t,\lambda,2\pi) = y_2'(a,t,\lambda,2\pi) = \pm 1$, we obtain

$$\begin{aligned}&\cos s_2p_2 \cos s_3p_3 - \frac{p_2}{p_3}\sin s_2p_2 \sin s_3p_3 \\ &= \cos s_2p_2 \cos s_3p_3 - \frac{p_3}{p_2}\sin s_2p_2 \sin s_3p_3 = \pm 1.\end{aligned}$$

Thus we have $\sin s_1p_1 = \sin s_2p_2 = \sin s_3p_3 = 0$. This conclusion also follows from $\sin s_2p_2 = 0$ or $\sin s_3p_3 = 0$ in a similar manner.

Because $\sin s_1p_1 = \sin s_2p_2 = \sin s_3p_3 = 0$ and $p_1p_2p_3 \neq 0$, we have $s_ip_i \in \pi\mathbb{N}$ for $i = 1,2,3$. So we get

$$y_1(a,t,\lambda,x) = \begin{cases}\cos xp_1 & \text{for } x \in [0,t_1),\\ \cos(x-t_1)p_2 \cos s_1p_1 & \text{for } x \in [t_1,t_2),\\ \cos(x-t_2)p_3 \cos s_2p_2 \cos s_1p_1 & \text{for } x \in [t_2,2\pi).\end{cases}$$

Therefore we see that the number of zeros of $y_1(a,t,\lambda,\cdot)$ inside $[0,2\pi)$ is

$$(s_1p_1 + s_2p_2 + s_3p_3)/\pi.$$

Since

$$M(a,t,\lambda) = (-1)^k \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

we infer that $y_1(a,t,\lambda,x)$ is a periodic solution of (1) of period 2π or 4π. Because $\lambda \in \{\lambda_{2k}(a,t),\ \lambda_{2k+1}(a,t)\}$, the Haupt Theorem (see [1, Chapter 8, Theorem 3.1]) implies that $y_1(a,t,\lambda,\cdot)$ has exactly k zeros in $[0,2\pi)$. Thus it follows that $(s_1p_1 + s_2p_2 + s_3p_3)/\pi = k$. Hence we obtain (ii).

Finally we shall prove that (ii) implies (7). Suppose that (ii) holds. By (3) $\sim$ (6) we have

$$M(a,t,\lambda) = (-1)^k \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

As in the above observation, we see that $y_1(a,t,\lambda,x)$ is a periodic solution of (1) of period 2π or 4π and that the number of zeros of $y_1(a,t,\lambda,\cdot)$ inside

$[0, 2\pi)$ is k. Thus the Haupt theorem again implies that

$$\lambda \in \{\lambda_{2k}(a, t),\ \lambda_{2k+1}(a, t)\}.$$

Hence we obtain (7). □

We are now in a position to demonstrate Theorem 2.

Proof of Theorem 2. Notice that $s_i p_i(\beta a_i, \lambda) \in \pi\mathbb{N}$ if and only if there exists $m_i \in \mathbb{N}$ such that

$$\lambda - \beta a_i = \frac{\pi^2 m_i^2}{s_i^2}.$$

Thus we see by Theorem 1 that, for $\beta \in \mathbb{R} - \{0\}$, the equality

$$\lambda_{2k}(\beta a, t) = \lambda_{2k+1}(\beta a, t)$$

holds if and only if there exist $(l_1, l_2, l_3) \in \mathbb{N}^3$ and $\lambda \in \mathbb{R}$ satisfying the conditions (18) $\sim$ (21) below.

$$\lambda - \beta a_1 = \frac{\pi^2 l_1^2}{s_1^2}. \tag{18}$$

$$\lambda - \beta a_2 = \frac{\pi^2 l_2^2}{s_2^2}. \tag{19}$$

$$\lambda - \beta a_3 = \frac{\pi^2 l_3^2}{s_3^2}. \tag{20}$$

$$l_1 + l_2 + l_3 = k. \tag{21}$$

We find that the existence of such $(l_1, l_2, l_3) \in \mathbb{N}^3$ and $\lambda \in \mathbb{R}$ is unique, since the function $\mathbb{R} \ni t \mapsto t - \beta a_i \in \mathbb{R}$ is strictly monotone increasing. Both (18) and (19) hold if and only if both

$$\beta = \frac{\pi^2}{a_2 - a_1}\left(\frac{l_1^2}{s_1^2} - \frac{l_2^2}{s_2^2}\right) \quad \text{and} \quad \lambda = \frac{\pi^2}{a_2 - a_1}\left(\frac{a_2 l_1^2}{s_1^2} - \frac{a_1 l_2^2}{s_2^2}\right)$$

are valid. Plugging these into (20) and $\beta \neq 0$, we obtain

$$\frac{a_2 - a_3}{s_1^2} l_1^2 + \frac{a_3 - a_1}{s_2^2} l_2^2 + \frac{a_1 - a_2}{s_3^2} l_3^2 = 0 \quad \text{and} \quad s_2 l_1 \neq s_1 l_2,$$

respectively. Thus we get the assertion of Theorem 2. □

Remark. For Hill's equation with a four-step potential, there is no analogy to Theorem 1. To see this we give a counterexample. We put

$$t_0 = 0, \quad t_1 = \frac{\pi}{6}, \quad t_2 = \frac{\pi}{2}, \quad t_3 = \frac{3}{2}\pi, \quad t_4 = 2\pi,$$

$$s_j = t_j - t_{j-1}, \qquad a_j = -\frac{\pi^2}{4s_j^2} \quad \text{for } j = 1, 2, 3, 4.$$

Let $V : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that

$$V(\,\cdot\,) = a_j \text{ on } [t_{j-1}, t_j) \quad \text{for } j = 1, 2, 3, 4.$$

We consider the equation of the form

$$-y''(x) + V(x)y(x) = 0 \quad \text{on } \mathbb{R}.$$

We define

$$T_j = \begin{pmatrix} \cos s_j\sqrt{-a_j} & \frac{1}{\sqrt{-a_j}} \sin s_j\sqrt{-a_j} \\ -\sqrt{-a_j}\sin s_j\sqrt{-a_j} & \cos s_j\sqrt{-a_j} \end{pmatrix} \quad \text{for } j = 1, 2, 3, 4.$$

We notice that the equation (22) admits two linearly independent, periodic solutions of period 2π, because its monodromy matrix is given by

$$T_4T_3T_2T_1 = \begin{pmatrix} \dfrac{s_2s_4}{s_1s_3} & 0 \\ 0 & \dfrac{s_1s_3}{s_2s_4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, we have

$$s_j\sqrt{-a_j} = \frac{\pi}{2} \notin \pi\mathbb{N} \quad \text{for } j = 1, 2, 3, 4.$$

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