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On topological ultraproducts

By MILOŠ S. KURILIĆ (Novi Sad)

Abstract. In "Ultraproducts in topology" [Gen. Topology Appl. 7 (1977) 283–308] Paul Bankston investigated ultraproducts of topological spaces and asked when is the quotient mapping $q : \square X_\alpha \to \square_{\mathcal{U}} X_\alpha$ closed. An answer for a wide class of spaces containing sequential Hausdorff spaces and Hausdorff spaces of cardinality $\leq \mathfrak{c}$ is obtained in [4]. Here we consider some classes of spaces that are not observed in [4].

1. Introduction

Let $(X_\alpha, \mathcal{O}_\alpha)$, $\alpha \in \kappa$, be a family of topological spaces. The box topology on the set $\prod X_\alpha$ is generated by the sets of the shape $\prod_{\alpha\in\kappa} O_\alpha$, where $O_\alpha \in \mathcal{O}_\alpha$ for all $\alpha \in \kappa$. Such a product is called the box product and denoted by $\square X_\alpha$. If $\mathcal{U} \subset P(\kappa)$ is an ultrafilter, the equivalence relation $\sim$ on the set $\prod X_\alpha$ given by $f \sim g$ iff $\{\alpha \in \kappa : f_\alpha = g_\alpha\} \in \mathcal{U}$ determines the quotient space $\square X_\alpha/\!\sim$, the ultraproduct of the given family of spaces, which will be denoted by $\square_{\mathcal{U}} X_\alpha$. The natural projection $q : \square X_\alpha \to \square_{\mathcal{U}} X_\alpha$ given by $q(f) = [f] = \{g \in \prod X_\alpha : g \sim f\}$ is always an open and continuous surjection.

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In [1] PAUL BANKSTON investigated ultraproducts of topological spaces and asked when the quotient map $q : \Box X_\alpha \to \Box_\mathcal{U} X_\alpha$ is closed (Problem 10.3). In [4], more general products-reduced products were considered and it was proved (in ZFC) that if the X_α-s belong to a wide class of spaces, then the mapping q is not closed. When the ultraproducts are in question, by Theorem 2 of [4] we have

Theorem 1. *Let κ be an infinite cardinal and $\mathcal{U} \subset P(\kappa)$ a non-principal ultrafilter. Let the spaces X_α, $\alpha \in \kappa$, contain closed subspaces Y_α containing closed sets I_α of ι many isolated (in Y_α) points and non-isolated points p_α of pseudocharacter $\psi_{Y_\alpha}(p_\alpha) = \nu$. If $\nu^\kappa \leq \iota^\kappa$, then the mapping $q : \Box X_\alpha \to \Box_\mathcal{U} X_\alpha$ is not closed.*

This theorem settles the question of Bankston if, for example, X_α are sequential Hausdorff spaces or ordinals $> \omega$ or Hausdorff spaces of cardinality $\leq \mathfrak{c}$. But it can not be applied if, for example, X_α-s have the cofinite topology. (Then each closed subspace of X_α is either finite (and discrete) or equal to X_α (and without isolated points)). In this paper, by methods different from those used in [4], we give some additional results concerning Bankston's question.

By $\prod_\mathcal{U} \omega$ we denote the ultrapower of structures $\langle \omega, < \rangle$ where $<$ is the natural order on ω, $\kappa = \omega$ and $\mathcal{U} \in \beta\omega \backslash \omega$. This ultrapower is a linearly ordered set again, where the relation $<_\mathcal{U}$ on ${}^\omega\omega/\sim$ is given by: $[f] <_\mathcal{U} [g]$ iff $\{n \in \omega : f_n < g_n\} \in \mathcal{U}$. The "ground set" of this structure has $\mathfrak{c}$ elements and the order type of $\prod_\mathcal{U} \omega$ is $\omega + Z\theta$, where Z is the order type of integers and θ is the order type of a dense linear ordering without end-points. ($\prod_\mathcal{U} \omega$ is in fact a non-standard model of the Peano arithmetic).

It is well-known that for every linearly ordered set $\langle L, < \rangle$ having no last element there exists the unique regular cardinal κ such that L contains a cofinal subset K of type κ. Then we write $\mathrm{cf}(\langle L, < \rangle) = \kappa$. The cofinality $\mathrm{cf}(\prod_\mathcal{U} \omega)$ will be used in our consideration. It is known that it depends on additional set-theoretic assumptions.

2. Two theorems

Here we give two theorems concerning ultraproducts of countably many spaces and leave generalizations to the reader. We will use the following elementary facts.

Fact 1 (Theorem 1.4.13 of [3]). A continuous mapping $f : X \to Y$ is closed iff for each $y \in Y$ and each $O \in \mathcal{O}_X$ such that $f^{-1}(\{y\}) \subset O$ there exists a neighbourhood V of y satisfying $f^{-1}(V) \subset O$.

Fact 2. Let $(X_\alpha, \mathcal{O}_\alpha)$, $\alpha \in \kappa$, be a family of topological spaces and $Y_\alpha \subset X_\alpha$, $\alpha \in \kappa$, nonempty closed sets. If the quotient mapping $q : \Box X_\alpha \to \Box_{\mathcal{U}} X_\alpha$ is closed, then the quotient mapping $q_1 : \Box Y_\alpha \to \Box_{\mathcal{U}} Y_\alpha$ is closed too.

Theorem 2. *Let $(X_n, \mathcal{O}_n)$, $n \in \omega$, be spaces containing countable non-discrete closed T_1-subspaces Y_n. If $\mathcal{U}$ is a non-principal ultrafilter on ω satisfying $\mathrm{cf}(\chi(\mathcal{U})) = \mathrm{cf}(\prod_{\mathcal{U}} \omega)$, then the mapping $q : \Box X_n \to \Box_{\mathcal{U}} X_n$ is not closed.*

Proof. Let X_n, Y_n and $\mathcal{U}$ be as supposed. By Fact 2, it is sufficient to prove that the mapping $q_1 : \Box Y_n \to \Box_{\mathcal{U}} Y_n$ is not closed.

For each $n \in \omega$ we pick a non-isolated point $y_n \in Y_n$. Clearly, we can suppose that $Y_n = \omega + 1$ and $y_n = \omega$. Since the spaces Y_n are T_1, the sets $(k, \omega]$ are open. In order to apply Fact 1 we will construct an open set $O \subset \Box(\omega + 1) = \Box Y_n$ such that

$$[\langle\omega\rangle] \subset O \quad \text{and} \quad [h] \not\subset O, \quad \text{for all } h \in {}^\omega\omega. \tag{1}$$

Firstly, let us suppose that such an O is constructed. Then $q_1^{-1}([\langle\omega\rangle]) = [\langle\omega\rangle] \subset O$ and it is sufficient to prove that $q_1^{-1}(V) \not\subset O$ for an arbitrary neighbourhood V of the point $[\langle\omega\rangle]$ of $\Box_{\mathcal{U}} Y_n$. Let $q_1(\prod_{n\in\omega} W_n)$ be a basic neighbourhood of $[\langle\omega\rangle]$ contained in V, where W_n is a neighbourhood of ω in Y_n, for $n \in \omega$. Since the point ω is non-isolated we choose $h_n \in W_n \setminus \{\omega\}$, $n \in \omega$, so, $h = \langle h_n : n \in \omega\rangle \in {}^\omega\omega \cap \prod W_n$. Now, $[h] = q_1^{-1}(\{q_1(h)\}) \subset q_1^{-1}(q_1(\prod W_n)) \subset q_1^{-1}(V)$, and since $[h] \not\subset O$ we have $q_1^{-1}(V) \not\subset O$. Thus $q_1^{-1}(V) \not\subset O$, for each neighbourhood V of $[\langle\omega\rangle]$ and, by Fact 1, the mapping q_1 is not closed.

Now we will construct the set O satisfying (1). Let $\mathcal{B} = \{B_\alpha : \alpha < \lambda\}$ be a base for $\mathcal{U}$ satisfying $\lambda = \chi(\mathcal{U})$ and $\kappa = \mathrm{cf}(\lambda) = \mathrm{cf}(\prod_{\mathcal{U}} \omega)$ and let $\{[f_\beta] : \beta < \kappa\}$ be an increasing unbounded subset of $\prod_{\mathcal{U}} \omega$. There exists a non-decreasing unbounded function $\varphi : \lambda \to \kappa$. (If $\psi : \kappa \to \lambda$ is an unbounded function, then the function $\varphi : \lambda \to \kappa$ given by: $\varphi(\alpha) = \min\{\beta < \kappa : \alpha < \psi(\beta)\}$ is non-decreasing and unbounded). Let $O = \bigcup_{\alpha<\lambda} O_\alpha$ where the sets O_α, $\alpha < \lambda$, are defined by:

$$O_\alpha = \prod_{n \in B_\alpha} (f_{\varphi(\alpha)}(n), \omega] \times \prod_{n \in \omega \setminus B_\alpha} (\omega + 1).$$

For each $f \in [\langle\omega\rangle]$ we have $f^{-1}(\omega) \in \mathcal{U}$ and there is $\alpha < \lambda$ such that $B_\alpha \subset f^{-1}(\omega)$, hence $f \in O_\alpha$. So, $[\langle\omega\rangle] \subset O$ and O is an open subset of $\square(\omega + 1)$.

Suppose there is $h \in {}^\omega\omega$ such that $[h] \subset O$. The set $\{[f_{\varphi(\alpha)}] : \alpha < \lambda\}$ is unbounded in $\prod_{\mathcal{U}} \omega$ thus there exists $\alpha_0 < \lambda$ satisfying $[h] \leq_{\mathcal{U}} [f_{\varphi(\alpha_0)}]$.

Let $F \in \mathcal{U}$. We define $h_F \in [h]$ by

$$h_F(n) = \begin{cases} h(n) & \text{for } n \in F, \\ 0 & \text{for } n \in \omega \setminus F. \end{cases}$$

If $\alpha \geq \alpha_0$, then $\varphi(\alpha_0) \leq \varphi(\alpha)$ and $[f_{\varphi(\alpha_0)}] \leq_{\mathcal{U}} [f_{\varphi(\alpha)}]$ so we have $[h] \leq_{\mathcal{U}} [f_{\varphi(\alpha)}]$, that is $G = \{n \in \omega : h(n) \leq f_{\varphi(\alpha)}(n)\} \in \mathcal{U}$. Pick one $n \in F \cap G \cap B_\alpha$. Then $h_F(n) = h(n) \leq f_{\varphi(\alpha)}(n)$ and since $n \in B_\alpha$, there holds $h_F \notin O_\alpha$. Now, $h_F \notin O_\alpha$ for all $\alpha \geq \alpha_0$ so there is $\alpha < \alpha_0$ such that $h_F \in O_\alpha$. Thus

$$\forall F \in \mathcal{U} \quad \exists \alpha < \alpha_0 \quad h_F \in O_\alpha. \tag{2}$$

If $h_F \in O_\alpha$, then $B_\alpha \subset F$ (because $n \in B_\alpha \setminus F$ would imply $0 = h_F(n) \in (f_{\varphi(\alpha)}(n), \omega]$). From (2) it follows that $\forall F \in \mathcal{U}\ \exists \alpha < \alpha_0\ (B_\alpha \subset F)$. So, the family $\{B_\alpha : \alpha < \alpha_0\}$ is a base for $\mathcal{U}$ of cardinality $< \lambda$ which is impossible and (1) is proved. □

If the spaces X_n do not contain T_1-subspaces, then the previous theorem and the results of [4] do not work. We will say that a space $(X, \mathcal{O})$ is rightly-open of type $\kappa + 1$ (where κ is an infinite cardinal) iff there is a (bijective) enumeration $X = \{x_\alpha : \alpha \leq \kappa\}$ such that $\{x_\kappa\} \notin \mathcal{O}$ and

$\{x_\alpha : \beta < \alpha \leq \kappa\} \in \mathcal{O}$ for each $\beta < \kappa$. (For example, for each cardinal $\kappa \geq \omega$, the space $\kappa + 1$ with the order topology is rightly-open of type $\kappa + 1$.)

Theorem 3. *Let $(X_n, \mathcal{O}_n)$, $n \in \omega$, be rightly-open spaces of type $\kappa+1$ and $\mathcal{U} \in \beta\omega\backslash\omega$ where $\mathrm{cf}(\chi(\mathcal{U})) = \kappa > \omega$. Then the mapping $q : \square X_n \to \square_{\mathcal{U}} X_n$ is not closed.*

Proof. Clearly, we can assume $X_n = \kappa + 1$ and $(\beta, \kappa] \in \mathcal{O}_n$ for each $\beta < \kappa$ and $n \in \omega$, where $(\beta, \kappa] = \{\alpha \in \kappa + 1 : \alpha > \beta\}$.

Let $\mathcal{B} = \{B_\alpha : \alpha < \lambda\}$ be a base for $\mathcal{U}$, where $\lambda = \chi(\mathcal{U})$ and let $\varphi : \lambda \to \kappa$ be a non-decreasing cofinal mapping. Let $O = \bigcup_{\alpha<\lambda} O_\alpha$, where the sets O_α, $\alpha < \lambda$ are defined by

$$O_\alpha = \prod_{n\in B_\alpha} (\varphi(\alpha), \kappa] \times \prod_{n\in\omega\backslash B_\alpha} (\kappa + 1).$$

The proof that $[\langle\kappa\rangle] \subset O$ and $[h] \not\subset O$ for all $h \in {}^\omega\kappa$ is analogous to the corresponding part of the proof of the preceding theorem (here we use the fact that each $h \in {}^\omega\kappa$ is dominated by some constant function since $\kappa > \omega$ is regular). The proof that $q^{-1}(V) \not\subset O$ for each neighbourhood V of $[\langle\kappa\rangle]$ is a copy of the corresponding part of the proof of the preceding theorem again. By Fact 1, the mapping q is not closed. $\square$

Example 1. Let $(X_n, \mathcal{O}_n)$, $n \in \omega$ be countable spaces with the cofinite topology. If $\mathcal{U}$ is a non-principal ultrafilter on ω satisfying $\mathrm{cf}(\chi(\mathcal{U})) = \mathrm{cf}(\prod_{\mathcal{U}} \omega)$, then, by Theorem 2, the mapping $q : \square X_n \to \square_{\mathcal{U}} X_n$ is not closed.

Example 2. Let $X_n = \omega_1 + 1$ and $\mathcal{O}_n = \{\omega_1 + 1\} \cup \{(\alpha, \omega_1] : \alpha < \omega_1\}$. If $\mathcal{U} \in \beta\omega\backslash\omega$ where $\mathrm{cf}(\chi(\mathcal{U})) = \omega_1$, then by Theorem 3, the mapping $q : \square X_n \to \square_{\mathcal{U}} X_n$ is not closed.

Finally, a few words about the condition $\mathrm{cf}(\chi(\mathcal{U})) = \mathrm{cf}(\prod_{\mathcal{U}} \omega)$. The cardinal $\mathrm{cf}(\prod_{\mathcal{U}} \omega)$ is bounded by some "small" uncountable cardinals. Namely, let $\leq^*$ be the relation on ${}^\omega\omega$ given by: $f \leq^* g$ iff there is $n_0 \in \omega$ such that $f_n \leq g_n$ for all $n \geq n_0$. A family $\mathcal{B} \subset {}^\omega\omega$ is *unbounded* iff there is no $g \in {}^\omega\omega$ such that $b \leq^* g$ for all $b \in \mathcal{B}$. A family $\mathcal{D} \subset {}^\omega\omega$ is *dominating* iff for each $f \in {}^\omega\omega$ there exists $d \in \mathcal{D}$ satisfying $f \leq^* d$. The cardinals $\mathfrak{b}$, $\mathfrak{d}$ and $\mathfrak{u}$ are defined by:

$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subset {}^\omega\omega$ is an unbounded family$\}$,

$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subset {}^\omega\omega$ is a dominating family$\}$ and

$\mathfrak{u} = \min\{|\mathcal{V}| : \mathcal{V} \subset [\omega]^\omega$ is a base for a $\mathcal{U} \in \beta\omega\backslash\omega\}$.

For more information see [2] and [5].

Fact 3. (a) $\mathfrak{b}$ is a regular cardinal.

(b) $\mathfrak{b} \leq \mathfrak{u} \leq \mathfrak{c}$.

(c) $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

(d) $\mathfrak{b} \leq \mathrm{cf}(\prod_\mathcal{U} \omega) \leq \mathfrak{d}$, for each $\mathcal{U} \in \beta\omega\backslash\omega$.

Proof. For (a), (b) and (c) see [2] and we prove (d). Since $f \leq^* g$ implies $[f] \leq_\mathcal{U} [g]$, we see that if $\{[f_\alpha] : \alpha < \kappa\}$ is a cofinal family in $\prod_\mathcal{U} \omega$, the family $\{f_\alpha : \alpha < \kappa\}$ must be unbounded in ${}^\omega\omega$. On the other hand, if $\{d_\alpha : \alpha < \kappa\}$ is a dominating family, then the family $\{[d_\alpha] : \alpha < \kappa\}$ is cofinal in $\prod_\mathcal{U} \omega$. □

For example, the equality $\mathfrak{b} = \mathfrak{c}$ implies $\mathrm{cf}(\chi(\mathcal{U})) = \mathrm{cf}(\prod_\mathcal{U} \omega)$ for each $\mathcal{U} \in \beta\omega\backslash\omega$. Namely, by Fact 3, $\mathfrak{b} = \mathfrak{c}$ gives $\mathfrak{b} = \mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ and $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$. Then clearly $\chi(\mathcal{U}) = \mathrm{cf}(\chi(\mathcal{U})) = \mathfrak{c}$ and $\mathrm{cf}(\prod_\mathcal{U} \omega) = \mathfrak{c}$. Specially, this equality holds under CH or MA.

But the equality $\mathrm{cf}(\chi(\mathcal{U})) = \mathrm{cf}(\prod_\mathcal{U} \omega)$ is not a theorem of ZFC. Moreover, it is consistent with ZFC that for each $\mathcal{U} \in \beta\omega\backslash\omega$ there holds $\mathrm{cf}(\prod_\mathcal{U} \omega) < \mathrm{cf}(\chi(\mathcal{U}))$. For example, if we add ω_2 many random reals to a model of GCH, we obtain a model of $\mathfrak{d} = \omega_1 < \omega_2 = \mathfrak{c} = \mathfrak{u}$. In this model for each $\mathcal{U} \in \beta\omega\backslash\omega$ we have $\mathrm{cf}(\prod_\mathcal{U} \omega) = \omega_1 < \omega_2 = \chi(\mathcal{U}) = \mathrm{cf}(\chi(\mathcal{U}))$.

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MILOŠ S. KURILIĆ
DEPARTMENT OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD
SERBIA AND MONTENEGRO

E-mail: milos@im.ns.ac.yu

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