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On the 1953 Barthel connection of a Finsler space and its physical aspect

By ROMAN S. INGARDEN (Toruń) and MAKOTO MATSUMOTO (Neyagawa)

In a recent paper [2] the importance of the Barthel connection, defined by W. BARTHEL [1] in 1953, has been discussed from the viewpoint of physics. We shall give a precise formulation to this connection based on the theory of connections in fibre bundles and consider its future applications to physics.

§1. Finsler connection

We consider a geometrical structure $(M, F\Gamma)$, a differentiable n-manifold M equipped with a Finsler connection $F\Gamma$ [3]. The Finsler bundle $F(M)$ of M is the induced bundle from the linear frame bundle $L(M)$ by the projection of the tangent bundle $T(M)$. A Finsler connection $F\Gamma$ of M is defined as a pair (Γ, N) of a connection Γ in $F(M)$ and a nonlinear connection N in $T(M)$. Then two kinds of convariant derivatives $\nabla^h X$ and $\nabla^v X$ of a Finsler vector field $X(x,y)$ are given as follows:

$$\begin{aligned}\nabla^h X : X^i{}_{|j} &= \partial X^i/\partial x^j - \left(\partial X^i/\partial y^r\right) N^r{}_j + X^r F_r{}^i{}_j,\\ \nabla^v X : X^i|_j &= \partial X^i/\partial y^j \qquad\qquad\qquad\qquad + X^r V_r{}^i{}_j.\end{aligned}$$

The functions $F_j{}^i{}_k(x,y)$, $N^i{}_j(x,y)$ and $V_j{}^i{}_k(x,y)$, are connection coefficients and $F\Gamma$ denotes the triad $(F_j{}^i{}_k,\ N^i{}_j,\ V_j{}^i{}_k)$.

In particular we are concerned with $F\Gamma$ satisfying the D-condition $\left(y^j F_j{}^i{}_k = N^i_k\right)$ and the V_1-condition $\left(y^j V_j{}^i{}_k = 0\right)$. Then the absolute differential Dy^i of y^i is written as $Dy^i = dy^i + N^i{}_j(x,y)dx^j$ and we have for a Finsler vector field $X^i(x,y)$

$$(1)\qquad DX^i = dX^i + X^j \left(F_j{}^i{}_k(x,y)dx^k + V_j{}^i{}_k(x,y)Dy^k\right).$$

The detailed version of this paper will be published in Reports on Math. Phys.

§2. Linear Y-connection

We consider a structure $(M, F\Gamma, Y(x))$, a structure $(M, F\Gamma)$ having a tangent vector field $Y(x)$. Since $Y(x)$ is regarded as a cross-section $M \to T$, we get a mapping $\eta : T \to F$ by attaching a frame to $Y(x)$ and the dual η^* of the differential η'. Then the connection form ω of the connection Γ of $F\Gamma$ gives rise to the induced connection form $\underline{\omega} = \eta^*(\omega)$ on $L(M)$, and we obtain the connection $\Gamma(Y)$ corresponding to $\underline{\omega}$. This $\Gamma(Y)$ is called the *linear Y-connection associated to $F\Gamma$ by $Y(x)$* [4]. Connection coefficients $\underline{\Gamma}_j{}^i{}_k(x)$ of $\Gamma(Y)$ are written as

$$\underline{\Gamma}_j{}^i{}_k(x) = F_j{}^i{}_k(x, Y) + V_j{}^i{}_r(x, Y) Y^r_k(x), \tag{2}$$

where we put $Y^r_k(x) = \partial Y^r / \partial x^k + N^r{}_k(x, Y)$. Thus $\Gamma(Y)$ has the torsion tensor $\underline{T}$, the components being

$$\underline{T}_j{}^i{}_k(x) = T_j{}^i{}_k(x, Y) + V_j{}^i{}_r(x, Y) Y^r_k(x) - V_k{}^i{}_r(x, Y) Y^r_j(x), \tag{3}$$

where $T_j{}^i{}_k(x, y) = F_j{}^i{}_k(x, y) - F_k{}^i{}_j(x, y)$ are components of the $(h)h$-torsion tensor T of $F\Gamma$.

Given a Finsler tensor field $K(x, y)$ we get an ordinary tensor field $\underline{K}(x) = K(x, Y)$, which is called the *$Y$-tensor field.* With respect to $\Gamma(Y)$ we have the covariant derivative $\underline{\nabla}\underline{K}$ given by

$$\underline{\nabla}\underline{K} : \underline{K}^i{}_{j|k} = K^i{}_{j|k}(x, Y) + K^i{}_j|_r Y^r_k(x). \tag{4}$$

The absolute differential $\underline{D}X$ of a tangent vector field $X(x)$ with respect to $\Gamma(Y)$ is, of course, written as $\underline{D}X^i = dX^i + X^j \underline{\Gamma}_j{}^i{}_k dx^k$. If the $F\Gamma$ satisfies the D– and V_1 –conditions, then $\underline{D}Y$ of $Y(x)$ is given by

$$\underline{D}Y^i = dY^i + N^i{}_j(x, Y) dx^j = Y^i_j(x) dx^j. \tag{5}$$

§3. $\Gamma(Y)$ associated to the Cartan connection

We consider a structure $(M, L(x, y), Y(x))$, *a Finsler space* $(M, L(x, y))$ *having a tangent vector field* $Y(x)$. In the following we restrict our consideration to a open subspace of M where $Y(x)$ does not vanish. From the fundamental function $L(x, y)$ we get the fundamental tensor $g(x, y)$ having the components $g_{ij} = \partial^2 \left(L^2/2\right) / \partial y^i \partial y^j$ and the C-tensor $C(x, y)$ having the components $C_{ijk} = \partial\left(g_{ij}/2\right)/\partial y^k$. $g(x, y)$ gives rise to the *Y-Riemannian metric* $\underline{g}(x) = g(x, Y)$.

We deal with the linear Y-connection associated to the well-known Cartan connection $C\Gamma\left(\Gamma^{*}{}_j{}^i{}_k, G^i{}_j, C_j{}^i{}_k\right)$ [3]. Since $C\Gamma$ is h– and v–mertical,

$\underline{\Gamma}(Y)$ is metrical ($\underline{\nabla} \underline{g} = 0$) from (4) and it follows from (2) that the connection coefficients $\underline{\Gamma}_j{}^i{}_k$ are written as

$$\underline{\Gamma}_j{}^i{}_k = \Gamma^*_j{}^i{}_k(x, Y) + C_j{}^i{}_r(x, Y) Y^r_k(x). \tag{6}$$

Thus the torsion tensor $\underline{T}$ of $\underline{\Gamma}(Y)$ is written as

$$\underline{T}_j{}^i{}_k(x) = C_j{}^i{}_r(x, Y) Y^r_k(x) - C_k{}^i{}_r(x, Y) Y^r_j(x). \tag{7}$$

We shall pay attention to the absolute differential $\underline{D}Y$ given by (5). For $C\Gamma$ we have $N^i{}_j = G^i{}_j$. Thus we have

$$N^i{}_j(x, Y) = Y^h \left\{ \gamma_h{}^i{}_j(x, Y) - C_j{}^i{}_r(x, Y) \gamma_s{}^r{}_h(x, Y) Y^s \right\}, \tag{8}$$

where $\gamma_h{}^i{}_j(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to x. Therefore $\underline{D}Y^i = 0$ just coincides with the equation given in the last line of Table I of the paper [2]. Consequently we have

Theorem. *The Barthel connection is the linear Y-connection associated to the Cartan connection by a nonzero tangent vector field $Y(x)$. It is metrical with respect to the Y-Riemannian metric and has the torsion tensor given by (7).*

§4. Physical remark

We shall pay attention to the connection coefficients $\underline{\Gamma}_j{}^i{}_k$ given by (6), which is not linear in Y. The nonlinearity and the dependence on a physical field Y are interesting for physicists. We may distinguish two kinds of spaces; point Finsler space and Finsler phase space (space of linear elements). The unique and *true* (i.e., metrical) connection in the former is the *Barthel connection*, while the one in the latter is the *Cartan connection.*

Adopting this philosophy we obtain clear physical applications. For instance, in thermodynamics we have two principal spaces; space of states (characterized by n independent thermodynamical parameters) and thermodynamical phase–space with $2n$ parameters (thermodynamical parameters and their conjugates). Then we need in physics a unique connection. Up to now this was not the case, and just therefore Finsler spaces were not interesting for most physicists.

The close relation between the Cartan connection and the Barthel connection shown in the present paper will be expected to give better appreciation of Finsler geometry to physicists.

References

[1] W. Barthel, Über eine Parallelverschiebung mit Längeninvarianz in lokal-Minkowskischen Räumen I, II, *Arch. Math.* **4** (1953), 346–365.

[2] R. S. Ingarden, On physical interpretations of Finsler and Kawaguchi geometries and the Barthel nonlinear connection, *Tensor, N.S.* **46** (1987), 354–360.
[3] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press, Saikawa, Ōtsu, Japan*, 1986.
[4] M. Matsumoto, Theory of Y-extremal and minimal hypersurfaces in a Finsler spaces, *J. Math. Kyoto Univ.* **26** (1986), 647–665.
[5] M. Matsumoto, Contributions of prescribed supporting element and the Cartan Y-connection, *Tensor, N.S.* **49** (1990), 9–17.

ROMAN S. INGARDEN
INST. OF PHYSICS
N. COPERNICUS UNIV.
TORUŃ, POLAND

MAKOTO MATSUMOTO
INST. OF MATHEMATICS
SETSUNAN UNIV.
NEYAGAWA, JAPAN

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