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A class of Kähler Einstein structures on the cotangent bundle

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Abstract. We use some natural lifts defined on the cotangent bundle T^*M of a Riemannian manifold (M, g) in order to construct an almost Hermitian structure (G, J) of diagonal type. The obtained almost complex structure J on T^*M is integrable if and only if the base manifold has constant sectional curvature and the coefficients as well as their derivatives, involved in its definition, do fulfill a certain algebraic relation. Next one obtains the condition that must be fulfilled in the case where the obtained almost Hermitian structure is almost Kählerian. Combining the obtained results we get a family of Kählerian structures on T^*M, depending on two essential parameters. Next we study three conditions under which the considered Kählerian structures are Einstein. In one of the obtained cases we get that (T^*M, G, J) has constant holomorphic curvature.

Introduction

In the study of the differential geometry of the cotangent bundle T^*M of a Riemannian manifold (M, g) one uses several Riemannian and semi-Riemannian metrics, induced from the Riemannian metric g on M. Among them, we may quote the metric of Sasaki type and the complete lift of the metric g. On the other hand, some notions similar to the natural lifts of g to the tangent bundle TM of M, will induce some new Riemannian and

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pseudo-Riemannian geometric structures with many nice geometric properties. Next, one can get from g some natural almost complex structures on T^*M. The study of the almost Hermitian structures induced from g on T^*M is an interesting problem in the differential geometry of the cotangent bundle.

In the present paper we study some classes of natural Kähler Einstein structures (G, J), of diagonal type induced on T^*M from the Riemannian metric g. They are obtained in a manner quite similar to that used in [11] (see also [15]) but the parametrization is a bit different. Namely, we adapt the situation presented in [9] to the case of the cotangent bundle, restricting ourselves to the case of the lifts of diagonal type. In fact we do not consider the most general situation due to the hard computations that must be done. However, in principle, the results obtained in the case of the general natural almost Hermitian structures on T^*M do not differ too much from that obtained in the case of the natural almost Hermitian structures of diagonal type. We consider the case where the vertical and horizontal distributions are orthogonal to each other but the dot products induced on them from G are not isomorphic (isometric). The family of the natural almost complex structures J on T^*M that interchange the vertical and horizontal distributions depends on two essential parameters a_1, b_1. These parameters are smooth real functions depending on the energy density t on the cotangent bundle. From the integrability condition for J it follows that the base manifold M must have constant curvature c and the second parameter b_1 must be expressed as a rational function depending on the first parameter a_1 and its derivative. Of course, in the obtained formula there are involved too the constant c and the energy density t.

A natural Riemannian metric G of diagonal type on T^*M is defined by four parameters c_1, c_2, d_1, d_2 which are smooth functions of t. From the condition for G to be Hermitian with respect to J we get two sets of proportionality relations, from which one obtains the parameters c_1, c_2, d_1, d_2 as functions depending on two new parameters λ, μ and the parameters a_1, b_1 involved in the expression of J. In the case where the fundamental 2-form ϕ, associated to the almost complex structure (G, J) is closed, one finds that $\mu = \lambda'$. If the the integrability condition for J is fulfilled, we get a Kählerian structure on T^*M and this structure depends on two essential parameters a_1 and λ.

In the case where the considered Kählerian structure is Einstein we get several situations in which the parameters a_1, λ are related by some algebraic relations. We have a general case, when (T^*M, G, J) has constant holomorphic curvature. In other two cases one obtains some simpler expressions for the components of the curvature tensor field on T^*M and, of course, we have some singularities. These cases will be discussed in some forthcoming papers.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices h, i, j, k, l, r, s being always$\{1, \dots, n\}$ (see [3], [13], [14]). We shall denote by $\Gamma(T^*M)$ the module of smooth vector fields on T^*M.

1. Natural almost complex structures of diagonal type on T^*M

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \longrightarrow M$. Recall that there is a structure of a $2n$-dimensional smooth manifold on T^*M, induced from the structure of smooth n-dimensional manifold of M. From every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$, on T^*M, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \dots, q^n$ are the local coordinates $x^1, \dots, x^n$ of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi$, $i = 1, \dots n$). The last n local coordinates $p_1, \dots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the natural basis $(dx^1_{\pi(p)}, \dots, dx^n_{\pi(p)})$, defined by the local chart (U, φ), i.e. $p = p_i dx^i_{\pi(p)}$. Due to this special structure of differentiable manifold for T^*M it is possible to introduce the concept of M-tensor field on it. An M-tensor field of type (r, s) on T^*M is defined by sets of n^{r+s} components (functions depending on q^i and p_i), with r upper indices and s lower indices, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on T^*M, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r, s) on the base manifold M, when a change of local charts on M (and hence on T^*M) is performed (see [5] for further details in the case of the tangent bundle); e.g., the components $p_i,\ i = 1, \ldots, n$, corresponding to the last n local coordinates of a cotangent vector p, assigned to an induced local chart $(\pi^{-1}(U), \Phi)$ define an M-tensor field of type $(0, 1)$ on T^*M. A usual tensor field of type (r, s) on M may be thought of as an M-tensor field of type (r, s) on T^*M. If the considered tensor field on M is covariant only, the corresponding M-tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M. Some useful M-tensor fields on T^*M may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a smooth function and let $\|p\|^2 = g^{-1}_{\pi(p)}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ (g^{-1} is the tensor field of type $(2, 0)$ having as components the entries $g^{ij}(x)$ of the inverse of the matrix $(g_{ij}(x))$ defined by the components of g in the local chart (U, φ)). If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|p\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on T^*M. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U, φ), then the components $u(\|p\|^2)g_{ij}(\pi(p))$ define a symmetric M-tensor field of type $(0, 2)$ on T^*M. The components $g^{0i} = p_h g^{hi}$, as well as $u(\|p\|^2)g^{0i}$ define M-tensor fields of type $(1, 0)$ on T^*M. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

We shall use the horizontal distribution HT^*M, defined by the Levi Civita connection $\dot\nabla$ of g, in order to define some first order natural lifts to T^*M of the Riemannian metric g on M. Denote by $VT^*M = \mathrm{Ker}\ \pi_* \subset TT^*M$ the vertical distribution on T^*M. Then we have the direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local chart on T^*M, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$

and the local vector fields $\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih}\frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k\Gamma^k_{ih}$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n})$ defines a local frame on T^*M, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial p_i} = (dx^i)^V, \quad \frac{\delta}{\delta q^i} = \left(\frac{\partial}{\partial x^i}\right)^H,$$

where θ^V denotes the vertical lift to T^*M of the 1-form θ on M and X^H denotes the horizontal lift to T^*M of the vector field X on M.

Now we shall present the following auxiliary result.

Lemma 1. *If $n > 1$ and u, v are smooth functions on T^*M such that*

$$ug_{ij} + vp_ip_j = 0, \quad p \in \pi^{-1}(U)$$

on the domain of any induced local chart on T^*M, then $u = 0$, $v = 0$.

The proof is obtained easily by transvecting the given relation with components g^{ij} of the tensor field g^{-1} and g^{0j} (Recall that the functions $g^{ij}(x)$ are the components of the inverse of the matrix $(g_{ij}(x))$, associated to g in the local chart (U, φ) on M).

Remark. From the relations of the type

$$ug^{ij} + vg^{0i}g^{0j} = 0, \qquad p \in \pi^{-1}(U),$$

$$u\delta^i_j + vg^{0i}p_j = 0, \qquad p \in \pi^{-1}(U),$$

it is obtained, in a similar way, $u = v = 0$. We have used the notation $g^{0i} = p_h g^{hi}$.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\pi^{-1}(U), \Phi)$ on T^*M, we shall use the following simpler notations

$$\frac{\partial}{\partial p_i} = \partial^i, \quad \frac{\delta}{\delta q^i} = \delta_i.$$

Denote by

$$t=\frac{1}{2}\|p\|^2=\frac{1}{2}g^{-1}_{\pi(p)}(p,p)=\frac{1}{2}g^{ik}(x)p_ip_k, \quad p\in\pi^{-1}(U) \tag{2}$$

the energy density defined by g in the cotangent vector p. We have $t\in[0,\infty)$ for all $p\in T^*M$. For a vector field X on M we shall denote by g_X the 1-form on M defined by $g_X(Y)=g(X,Y)$, for all vector fields Y on M. For a 1-form θ on M, we shall denote by $\theta^\sharp=g^{-1}_\theta$ the vector field on M defined by the usual musical isomorphism, i.e. $g(\theta^\sharp,Y)=\theta(Y)$, for all vector fields Y on M. Remark that, for $p\in T^*M$, we can consider the vector $p^\sharp$, tangent to M in $\pi(p)$. Consider the real valued smooth functions a_1, a_2, b_1, b_2 defined on $[0,\infty)\subset\mathbb{R}$ and define a diagonal natural almost complex structure J on T^*M, by using these coefficients and the Riemannian metric g

$$\begin{cases} JX^H_p=a_1(t)(g_X)^V_p+b_1(t)p(X)p^V_p, \\ J\theta^V_p=-a_2(t)(\theta^\sharp)^H_p-b_2(t)g^{-1}_{\pi(p)}(p,\theta)(p^\sharp)^H_p. \end{cases} \tag{3}$$

We should remark that the vector p^V_p defines the Liouville vector field on T^*M and $(p^\sharp)^H_p$ defines a similar HT^*M-valued vector field.

The expression of J in adapted local frames is given by

$$J\delta_i=a_1(t)g_{ij}\partial^j+b_1(t)p_ip_j\partial^j,$$

$$J\partial^i=-a_2(t)g^{ij}\delta_j-b_2(t)g^{0i}g^{0j}\delta_j.$$

Remark that one can consider the case of the general natural tensor fields J on T^*M. In this case we have another four coefficients a_3, b_3, a_4, b_4 and the computations involved in the study of the corresponding almost complex structure J on T^*M become really complicate (see [9], [10]). In fact, the tensor fields of this type define the most general natural lift of type $(1,1)$ of the metric g.

Proposition 2. *The operator J defines an almost complex structure on T^*M if and only if*

$$a_1a_2=1, \quad (a_1+2tb_1)(a_2+2tb_2)=1. \tag{4}$$

Proof. The relations are obtained easily from the property $J^2 = -I$ of J and Lemma 1.

From the relations obtained in Lemma 1 we can get the explicit expressions of the parameters a_2, b_2

$$a_2 = \frac{1}{a_1}, \quad b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}. \tag{5}$$

The obtained almost complex structures defined by the tensor field J on T^*M are called *natural almost complex structures of diagonal type*, defined by the Riemannian metric g, by using the essential parameters a_1, b_1. We use the word diagonal for these almost complex structures, since the $2n \times 2n$-matrix associated to J, with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ has two $n \times n$-blocks on the second diagonal

$$J = \begin{pmatrix} 0 & -a_2 g^{ij} - b_2 g^{0i} g^{0j} \\ a_1 g_{ij} + b_1 p_i p_j & 0 \end{pmatrix}. \qquad \square$$

Remark. From the conditions (4) we have that the coefficients a_1, a_2, $a_1 + 2tb_1$, $a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0$, $a_2 > 0$, $a_1 + 2tb_1 > 0$, $a_2 + 2tb_2 > 0$ for all $t \geq 0$.

Now we shall study the integrability of the almost complex structure defined by J on T^*M. To do this we need the following well known formulas for the brackets of the vector fields $\partial^i = \frac{\partial}{\partial p_i}, \delta_i = \frac{\delta}{\delta q^i}$, $i = 1, \dots, n$

$$[\partial^i, \partial^j] = 0; \quad [\partial^i, \delta_j] = \Gamma^i_{jk} \partial^k; \quad [\delta_i, \delta_j] = R^0_{kij} \partial^k, \tag{6}$$

where Γ^i_{jk} are the Christoffel symbols defined by the Levi Civita connection $\dot\nabla$, $R^0_{kij} = p_h R^h_{kij}$ and R^h_{kij} are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M.

Theorem 3. *The almost complex structure J on T^*M is integrable if and only if (M, g) has constant sectional curvature c and the function b_1 is given by*

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}. \tag{7}$$

Of course we have to study the conditions under which a_1, b_1 fulfill the conditions $a_1 > 0$, $a_1 + 2tb_1 = \frac{a_1^2 - 2ct}{a_1 - 2ta_1'} > 0$, $\forall t \geq 0$.

Proof. We shall study the vanishing of the Nijenhuis tensor field $N = N_J$ of J, defined by

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \forall X, Y \in \Gamma(T^*M).$$

We have $\delta_k t = 0$, $\partial^k t = g^{0k}$ and, after a straightforward but quite long computation, we get

$$N(\delta_i, \delta_j) = \big\{(a_1 a_1' + 2ta_1' b_1 - a_1 b_1)(p_i g_{jk} - p_j g_{ik}) - R^0_{kij}\big\}\partial^k.$$

Remark that the coefficient of δ_k in the expression of $N(\delta_i, \delta_j)$ becomes 0, due to the usual properties of the Levi Civita connection $\dot{\nabla}$.

From the condition $N(\delta_i, \delta_j) = 0$ we get

$$R^0_{kij} = (a_1 a_1' + 2ta_1' b_1 - a_1 b_1)(p_i g_{jk} - p_j g_{ik}).$$

Differentiating with respect to p_h, taking $p = 0$ and using Schur theorem, it follows that the curvature tensor field of $\dot{\nabla}$ (in the case where M is connected and $\dim M > 2$) must have the expression

$$R^h_{kij} = c\big(\delta^h_i g_{kj} - \delta^h_j g_{ki}\big),$$

where c is a constant. Then we obtain the expression (7) of b_1.

Next it follows by a straightforward computation that $N(\partial^i, \delta_j) = 0$, $N(\partial^i, \partial^j) = 0$, whenever $N(\delta_i, \delta_j) = 0$.

Hence the condition $N = 0$ implies that (M, g) must have constant sectional curvature c, and b_1 must be given by (7). Conversely, if (M, g) has constant curvature c and b_1 is given by (7), it follows in a straightforward way that $N = 0$. $\square$

Remark. In the case where $a_1^2 - 2ct = 0$, we have $a_1 a_1' - c = 0$, $a_1 - 2ta_1' = 0$ too. So, this case must be thought of as a singular case and should be considered separately.

2. Natural almost Hermitian structures on T^*M

Consider the following symmetric $M-$tensor fields on T^*M, defined by the components

$$G^{(1)}_{ij} = c_1 g_{ij} + d_1 p_i p_j, \quad G^{ij}_{(2)} = c_2 g^{ij} + d_2 g^{0i} g^{0j}, \tag{8}$$

where c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in [0, \infty)$.

Obviously, $G^{(1)}$ is of type $(0,2)$ and $G_{(2)}$ is of type $(2,0)$. We shall assume that the matrices defined by $G^{(1)}$ and $G_{(2)}$ are positive definite. This happens iff $c_1 > 0$, $c_2 > 0$, $c_1 + 2td_1 > 0$, $c_2 + 2td_2 > 0$. Then the following Riemannian metric may be considered on T^*M

$$G = G^{(1)}_{ij} dq^i dq^j + G^{ij}_{(2)} Dp_i Dp_j, \tag{9}$$

where $Dp_i = dp_i - \Gamma^0_{ij} dq^j$ is the absolute (covariant) differential of p_i with respect to the Levi Civita connection $\dot{\nabla}$ of g (recall that $\Gamma^0_{ij} = p_h \Gamma^h_{ij}$). Equivalently, we have

$$G(\delta_i, \delta_j) = G^{(1)}_{ij}, \quad G(\partial^i, \partial^j) = G^{ij}_{(2)}, \quad G(\partial^i, \delta_j) = G(\delta_j, \partial^i) = 0.$$

Remark that HT^*M, VT^*M are orthogonal to each other with respect to G, but the Riemannian metrics induced from G on HT^*M, VT^*M are not the same, so the considered metric G on T^*M is not a metric of Sasaki type. The $2n \times 2n$-matrix associated to G, with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ has two $n \times n$-blocks on the first diagonal

$$G = \begin{pmatrix} G^{(1)}_{ij} & 0 \\ 0 & G^{ij}_{(2)} \end{pmatrix}.$$

The Riemannian metric G is called a *natural lift of diagonal type* of g. Remark also that the system of 1-forms $(dq^1, \dots, dq^n, Dp_1, \dots, Dp_n)$ defines a local frame on T^*T^*M, dual to the local frame $(\delta_1, \dots, \delta_n, \partial^1, \dots$ $\dots, \partial^n)$ adapted to the direct sum decomposition (1).

We shall consider another two M-tensor fields $H_{(1)}$, $H^{(2)}$ on T^*M, defined by the components

$$H^{jk}_{(1)} = \frac{1}{c_1} g^{jk} - \frac{d_1}{c_1(c_1 + 2td_1)} g^{0j} g^{0k},$$

$$H^{(2)}_{jk} = \frac{1}{c_2} g_{jk} - \frac{d_2}{c_2(c_2 + 2td_2)} p_j p_k.$$

The components $H^{jk}_{(1)}$ define an M-tensor field of type $(2,0)$ and the components $H^{(2)}_{jk}$ define an M-tensor field of type $(0,2)$. Moreover, the matrices associated to $H_{(1)}$, $H^{(2)}$ are the inverses of the matrices associated to $G^{(1)}$ and $G_{(2)}$, respectively, i.e. we have

$$G^{(1)}_{ij} H^{jk}_{(1)} = \delta^k_i, \quad G^{ij}_{(2)} H^{(2)}_{jk} = \delta^i_k.$$

Now, we shall be interested in the conditions under which the metric G is almost Hermitian with respect to the almost complex structure J, considered in the previous section, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on T^*M.

Considering the coefficients of g_{ij}, g^{ij} in the conditions

$$\begin{cases} G(J\delta_i, J\delta_j) = G(\delta_i, \delta_j), \\ G(J\partial^i, J\partial^j) = G(\partial^i, \partial^j), \end{cases} \tag{10}$$

we can express the parameters c_1, c_2 with the help of the parameters a_1, a_2 and a proportionality factor $\lambda = \lambda(t)$

$$c_1 = \lambda a_1, \quad c_2 = \lambda a_2, \tag{11}$$

where the coefficients a_1, a_2 are related by (4). Since we made the assumption $a_1 > 0$, $a_2 > 0$, it follows $\lambda > 0$.

Next, considering the coefficients of $p_i p_j$, $g^{0i} g^{0j}$ in the relations (10), we can express the parameters $c_1 + 2td_1, c_2 + 2td_2$ with the help of the parameters $a_1 + 2tb_1$, $a_2 + 2tb_2$ and a new parameter $\lambda + 2t\mu$

$$\begin{cases} c_1 + 2td_1 = (\lambda + 2t\mu)(a_1 + 2tb_1), \\ c_2 + 2td_2 = (\lambda + 2t\mu)(a_2 + 2tb_2). \end{cases} \tag{12}$$

Remark that $\lambda + 2t\mu = \lambda(t) + 2t\mu(t)$ is a positive smooth function of $t \in [0, \infty)$. It was much more convenient to consider the proportionality factor in such a form in the expression of the parameters c_1+2td_1, c_2+2td_2. Of course, we can obtain easily from (12) the explicit expressions of the coefficients d_1, d_2

$$\begin{cases} d_1 = \lambda b_1 + \mu(a_1 + 2tb_1), \\ d_2 = \lambda b_2 + \mu(a_2 + 2tb_2). \end{cases} \tag{13}$$

Hence we may state

Theorem 4. *Let J be the natural, almost complex structure of diagonal type on T^*M, given by (3), where the coefficients a_1, a_2, b_1, b_2 are related by (4). The family of the natural Riemannian metrics G on T^*M, of diagonal type, such that (T^*M, G, J) is an almost Hermitian manifold, is given by (9) where the coefficients c_1, c_2 are related to a_1, a_2 by (11), and $c_1 + 2td_1$, $c_2 + 2td_2$ are related to $a_1 + 2tb_1$, $a_2 + 2tb_2$ by (12), the proportionality coefficients being $\lambda > 0$ and $\lambda + 2t\mu > 0$.*

Remark. A result of the same kind can be obtained in the case of the natural almost Hermitian structures of general type on T^*M (see [9]).

Consider now the two-form ϕ defined by the almost Hermitian structure (G, J) on T^*M

$$\phi(X,Y) = G(X, JY),$$

for all vector fields X, Y on T^*M.

Proposition 5. *The expression of the 2-form ϕ in a local adapted frame $(\partial^1, \dots, \partial^n, \delta_1, \dots, \delta_n)$ on T^*M, is given by*

$$\phi(\partial^i, \partial^j) = 0, \quad \phi(\delta_i, \delta_j) = 0, \quad \phi(\partial^i, \delta_j) = \lambda\delta^i_j + \mu g^{0i} p_j,$$

or, equivalently

$$\phi = (\lambda\delta^i_j + \mu g^{0i} p_j) Dp_i \wedge dq^j, \tag{14}$$

where $Dp_i = dp_i - \Gamma^0_{ih} dq^h$ is the absolute differential of p_i.

The proof is obtained by using the definition of ϕ and computing the values $\phi(\partial^i, \partial^j)$, $\phi(\delta_i, \delta_j)$, $\phi(\partial^i, \delta_j)$.

Theorem 6. *The almost Hermitian structure (G, J) on T^*M is almost Kählerian if and only if*

$$\mu = \lambda'.$$

Proof. We shall study the vanishing of the exterior differential $d\phi$. The expressions of $d\lambda$, $d\mu$, dg^{0i} and dDp_i are obtained in a straightforward way, by using the property $\dot{\nabla}_k g_{ij} = 0$ (hence $\dot{\nabla}_k g^{ij} = 0$)

$$d\lambda = \lambda' g^{0i} Dp_i, \quad d\mu = \mu' g^{0i} Dp_i, \quad dg^{0i} = g^{ik} Dp_k - g^{0h} \Gamma^i_{hk} dq^k,$$

$$dDp_i = -\frac{1}{2} R^0_{ikl} dq^k \wedge dq^l + \Gamma^l_{ik} dq^k \wedge Dp_l.$$

Then we have

$$\begin{aligned} d\phi &= (d\lambda \delta^i_j + d\mu g^{0i} p_j + \mu dg^{0i} p_j + \mu g^{0i} dp_j) \wedge Dp_i \wedge dq^j \\ &\quad + (\lambda \delta^i_j + \mu g^{0i} p_j) dDp_i \wedge dq^j. \end{aligned}$$

By replacing the expressions of $d\lambda$, $d\mu$, dg^{0i} and dDp_i then using, again, the property $\dot{\nabla}_k g_{ij} = 0$, doing some algebraic computations with the exterior products, then using the well known symmetry properties of g_{ij}, Γ^h_{ij}, and of the Riemann–Christoffel tensor field, as well as the Bianchi identities, it follows that

$$d\phi = \frac{1}{2}(\lambda' - \mu) g^{0h} Dp_h \wedge Dp_i \wedge dq^i.$$

Therefore we have $d\phi = 0$ if and only if $\mu = \lambda'$. □

Theorem 7. *The almost Hermitian structure (G,J) on T^*M is Kählerian if and only if the base manifold M has constant sectional curvature, the parameter b_1 is given by (7) and $\mu = \lambda'$.*

Proof. The family of natural almost Hermitian structures (G, J) of diagonal type on T^*M depends on four essential coefficients a_1, b_1, λ, μ. According to the result of Theorem 3, the integrability of J is equivalent to the property of M to have constant sectional curvature c and the condition for b_1 to be given by (7). Then, from Theorem 6, we get the (G, J) is almost Kählerian if and only if $\mu = \lambda'$. Combining these two results one obtains the result of our theorem. □

Remark. A natural Kählerian structure (G, J) of diagonal type on T^*M is defined by two essential coefficients a_1, λ. Using (7), these coefficients must satisfy the supplementary conditions $a_1 > 0$, $a_1 + 2tb_1 = \frac{a_1^2 - 2ct}{a_1 - 2ta_1'} > 0$, $\lambda > 0$, $\lambda + 2t\lambda' > 0$.

3. The Levi Civita connection of the metric G and its curvature tensor field

Recall that the Levi Civita connection $\dot\nabla$ on a Riemannian manifold (M,g) is determined by the conditions

$$\dot\nabla g = 0, \quad \dot T = 0,$$

where $\dot T$ is its torsion tenor field. The explicit expression of this connection is obtained from the formula

$$\begin{aligned}2g(\dot\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z)\\ - g([X,Z],Y) - g([Y,Z],X), \quad \forall X,Y,Z{\in}\Gamma(M).\end{aligned}$$

We shall use this formula in order to obtain the expression of the Levi Civita connection ∇ of G on T^*M. The final result can be stated as follows

Theorem 8. *The Levi Civita connection ∇ of G has the following expression in the local adapted frame $(\partial^1,\dots,\partial^n,\ \delta_1,\dots,\delta_n)$*

$$\nabla_{\partial^i}\partial^j = Q^{ij}_h\partial^h, \qquad \nabla_{\delta_i}\partial^j = -\Gamma^j_{ih}\partial^h + P^{hj}_i\delta_h,$$

$$\nabla_{\partial^i}\delta_j = P^{hi}_j\delta_h, \qquad \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S_{hij}\partial^h,$$

where Q^{ij}_h, P^{hi}_j, S_{hij} are M-tensor fields on T^*M, defined by

$$Q^{ij}_h = \frac{1}{2}H^{(2)}_{hk}(\partial^i G^{jk}_{(2)} + \partial^j G^{ik}_{(2)} - \partial^k G^{ij}_{(2)}),$$

$$P^{hi}_j = \frac{1}{2}H^{hk}_{(1)}(\partial^i G^{(1)}_{jk} - G^{il}_{(2)}R^0_{ljk}),$$

$$S_{hij} = -\frac{1}{2}H^{(2)}_{hk}\partial^k G^{(1)}_{ij} + \frac{1}{2}R^0_{hij}.$$

Replacing the expressions of the involved M-tensor fields and assuming that the base manifold (M,g) has constant sectional curvature, one obtains

$$Q^{ij}_h = -\frac{c_2' - 2d_2}{2(c_2+2d_2t)}g^{ij}p_h + \frac{c_2'}{2c_2}(\delta^j_h g^{0i} + \delta^i_h g^{0j}) + \frac{-2c_2'd_2 + c_2d_2'}{2c_2(c_2+2d_2t)}p_h g^{0i}g^{0j},$$

$$P_j^{hi} = -\frac{cc_2 - d_1}{2c_1} g^{hi} p_j + \frac{cc_2 + d_1}{2(c_1 + 2d_1 t)} \delta_j^i g^{0h} + \frac{c_1'}{2c_1} \delta_j^h g^{0i}$$
$$+ \frac{-c_1' d_1 + cc_2 d_1 - d_1^2 + c_1 d_1'}{2c_1(c_1 + 2d_1 t)} p_j g^{0h} g^{0i},$$

$$S_{hij} = \frac{-c_1'}{2(c_2 + 2d_2 t)} g_{ij} p_h + \frac{cc_2 - d_1}{2c_2} g_{hj} p_i - \frac{cc_2 + d_1}{2c_2} g_{hi} p_j$$
$$- \frac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2d_2 t)} p_h p_i p_j.$$

In the case of a Kähler structure on T^*M, the final expressions of these M-tensor fields can be obtained by doing the necessary replacements. However, the final expressions are quite complicate but they may be obtained quite automatically by using the Mathematica package RICCI for doing tensor computations (see [4]).

Now we shall indicate the obtaining of the components of the curvature tensor field of the connection ∇.

The curvature tensor K field of the connection ∇ is obtained from the well known formula

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components of K with respect to the adapted local frame $(\partial^1, \dots, \partial^n, \delta_1, \dots, \delta_n)$ can be expressed easily

$$K(\partial^i, \partial^j)\partial^k = PPP_h^{ijk} \partial^h = (\partial^i Q_h^{jk} - \partial^j Q_h^{ik} + Q_l^{jk} Q_h^{il} - Q_l^{ik} Q_h^{jl}) \partial^h,$$

$$K(\partial^i, \partial^j)\delta_k = PPQ_k^{ijh} \delta_h = (\partial^i P_k^{hj} - \partial^j P_k^{hi} + P_k^{lj} P_l^{hi} - P_k^{li} P_l^{hj}) \delta_h,$$

$$K(\delta_i, \delta_j)\partial^k = QQP_{ijh}^k \partial^h = (-R_{hij}^k - R_{lij}^0 Q_h^{lk} + S_{hil} P_j^{lk} - S_{hjl} P_i^{lk}) \partial^h,$$

$$K(\delta_i, \delta_j)\delta_k = QQQ_{ijk}^h \delta_h = (R_{kij}^h + S_{ljk} P_i^{hl} - S_{lik} P_j^{hl} - R_{lij}^0 P_k^{hl}) \delta_h,$$

$$K(\partial^i, \delta_j)\delta_k = PQQ_{jkh}^i \partial^h = (\partial^i S_{hjk} + S_{ljk} Q_h^{il} - S_{hjl} P_k^{li}) \partial^h,$$

$$K(\partial^i, \delta_j)\partial^k = PQP_j^{ikh} \delta_h = (\partial^i P_j^{hk} + P_l^{hi} P_j^{lk} - Q_l^{ik} P_j^{hl}) \delta_h.$$

The explicit expressions of these components are obtained after some quite long and hard computations, made by using the package RICCI.

Next, the components of the Ricci tensor field are obtained as traces of K

$$\operatorname{Ric} PP^{jk} = \operatorname{Ric}(\partial^j, \partial^k) = PPP_h^{hjk} - PQP_h^{jkh},$$

$$\operatorname{Ric} QQ_{jk} = \operatorname{Ric}(\delta_j, \delta_k) = QQQ^h_{hjk} + PQQ^h_{jkh},$$

$$\operatorname{Ric}(\partial^j, \delta_k) = \operatorname{Ric}(\delta_k, \partial^j) = 0.$$

4. The cotangent bundle T^*M as a Kähler Einstein manifold

From the explicit expressions of the components of the Ricci tensor field on T^*M one obtains the common Einstein factor Ef, appearing in the condition for the Kählerian manifold (T^*M, G, J) to be an Einstein space

$$\mathrm{Ef} = -n\frac{a_1^2 a_1' \lambda - 2a_1 c\lambda + a_1^3 \lambda' + 2a_1' c\lambda t - 2a_1 c\lambda' t}{2a_1\lambda^2(a_1 - 2a_1' t)} - (a_1^2 - 2ct)(a_1 a_1' \lambda^2$$

$$+ a_1^2 \lambda\lambda' - a_1'^2 \lambda^2 t + a_1 a_1'' \lambda^2 t - a_1^2 \lambda'^2 t + a_1^2 \lambda\lambda'' t - 2a_1'^2 \lambda\lambda' t^2$$

$$+ 2a_1 a_1'' \lambda\lambda' t^2 + 2a_1 a_1' \lambda'^2 t^2 - 2a_1 a_1' \lambda\lambda'' t^2)/(a_1\lambda^2(a_1 - 2a_1' t)^2(\lambda + 2\lambda' t)).$$

Next we consider the differences

$$\operatorname{Diff} QQ_{jk} = \operatorname{Ric} QQ_{jk} - \mathrm{Ef}\ \ G^{(1)}_{jk} = \frac{a_1^2 - 2ct}{2a_1^2\lambda^2(a_1 - 2a_1' t)^4(\lambda + 2\lambda' t)^2}\gamma p_j p_k,$$

$$\operatorname{Diff} PP^{jk} = \operatorname{Ric} PP^{jk} - \mathrm{Ef}\ \ G^{jk}_{(2)}$$

$$= \frac{1}{2a_1^2\lambda^2(a_1 - 2a_1' t)^2(a_1^2 - 2ct)(\lambda + 2\lambda' t)^2}\gamma g^{0j} g^{0k},$$

where $\gamma = nC_n + \beta$ and C_n, β are expressions involving a_1, λ and their derivatives up to third order. The condition for (T^*M, G, J) to be Kähler Einstein is given by $\operatorname{Diff} QQ_{jk} = 0$, $\operatorname{Diff} PP^{jk} = 0$ or, equivalently, $\gamma = 0$. If we ask for Kähler Einstein structures on T^*M to be independent of the dimension n of M, then we must have $C_n = 0$, $\beta = 0$. The coefficient of n in the expression of γ is

$$C_n = -(a_1 - 2a_1' t)(a_1^2 - 2ct)(\lambda + 2\lambda' t)^2(2a_1 a_1'^2 \lambda^2 + a_1^2 a_1'' \lambda^2 + 2a_1^2 a_1' \lambda\lambda'$$

$$- 2a_1^3\lambda'^2 + a_1^3\lambda\lambda'' - 2a_1'^3\lambda^2 t - 2a_1 a_1'^2\lambda\lambda' t + 2a_1^2 a_1''\lambda\lambda' t$$
$$+ 4a_1^2 a_1'\lambda'^2 t - 2a_1^2 a_1'\lambda\lambda'' t).$$

Excluding the cases for which we have singularities, we can obtain from the condition $C_n = 0$ the expression

$$\left\{\begin{aligned} a_1'' = & -(2a_1 a_1'^2\lambda^2 + 2a_1^2 a_1'\lambda\lambda' - 2a_1^3\lambda'^2 + a_1^3\lambda\lambda'' - 2a_1'^3\lambda^2 t - \\ & - 2a_1 a_1'^2\lambda\lambda' t + 4a_1^2 a_1'\lambda'^2 t - 2a_1^2 a_1'\lambda\lambda'' t)/(a_1^2\lambda^2 + 2a_1^2\lambda\lambda' t). \end{aligned}\right. \tag{15}$$

Differentiating the expression of a_1'' with respect to t, then replacing a_1'' from (15), we get a quite complicate expression for the derivative of third order $a_1^{(3)}$

$$\begin{aligned} a_1^{(3)} = & (12a_1^2 a_1'^3\lambda^4 - 24ta_1 a_1'^4\lambda^4 + 12t^2 a_1'^5\lambda^4 + 18a_1^3 a_1'^2\lambda^3\lambda' - 30ta_1^2 a_1'^3\lambda^3\lambda' \\ & + 12t^2 a_1 a_1'^4\lambda^3\lambda' + 18ta_1^3 a_1'^2\lambda^2\lambda'^2 - 24t^2 a_1^2 a_1'^3\lambda^2\lambda'^2 - 12a_1^5\lambda\lambda'^3 \\ & + 36ta_1^4 a_1'\lambda\lambda'^3 - 24t^2 a_1^3 a_1'^2\lambda\lambda'^3 - 12ta_1^5\lambda'^4 + 24t^2 a_1^4 a_1'\lambda'^4 + 3a_1^4 a_1'\lambda^3\lambda'' \\ & - 12ta_1^3 a_1'^2\lambda^3\lambda'' + 12t^2 a_1^2 a_1'^3\lambda^3\lambda'' + 9a_1^5\lambda^2\lambda'\lambda'' - 24ta_1^4 a_1'\lambda^2\lambda'\lambda'' \\ & + 12t^2 a_1^3 a_1'^2\lambda^2\lambda'\lambda'' + 12ta_1^5\lambda\lambda'^2\lambda'' - 24t^2 a_1^4 a_1'\lambda\lambda'^2\lambda'' - a_1^5\lambda^3\lambda^{(3)} \\ & + 2ta_1^4 a_1'\lambda^3\lambda^{(3)} - 2ta_1^5\lambda^2\lambda'\lambda^{(3)} + 4t^2 a_1^4 a_1'\lambda^2\lambda'\lambda^{(3)})/(a_1^4(\lambda^2 + 2t\lambda\lambda')^2). \end{aligned}$$

Next, replacing these expressions of a_1'', $a_1^{(3)}$ in the condition $\beta = 0$, we get the following interesting relation

$$\left\{\begin{aligned} & \lambda(a_1 - 2ta_1')^3(a_1^2 - 2ct)(a_1\lambda - ta_1'\lambda + ta_1\lambda')(a_1'\lambda \\ & \quad + a_1\lambda')(a_1^2 a_1'\lambda + 2ca_1\lambda + a_1^3\lambda' - 2cta_1'\lambda + 2cta_1\lambda') = 0. \end{aligned}\right. \tag{16}$$

The vanishing of the factors λ,$a_1 - 2ta_1'$ and $a_1^2 - 2ct$ will be not considered since the corresponding situations lead to singularities. Thus we have the following three essential cases

1) The first and most interesting situation which will be studied is that when the last factor in (16) vanishes. From the corresponding relation one gets

$$a_1' = -\frac{2ca_1\lambda + a_1^3\lambda' + 2cta_1\lambda'}{\lambda(a_1^2 - 2ct)}. \tag{17}$$

Differentiating a_1' with respect to t and replacing a_1' from (17) in the obtained result, one gets the same expression for a_1'' as that obtained from (15), after the replacing of a_1' from (17). Next, computing $a_1^{(3)}$ and replacing again a_1' from (17) one gets $\mathrm{Diff}\, QQ_{jk} = 0$ and $\mathrm{Diff}\, PP^{jk} = 0$. Thus if the relation (17) is fulfilled, one obtains that (T^*M, G, J) is Kähler Einstein. Next one obtains the expresion

$$\mathrm{Ef} = 2c(n+1)\frac{a_1}{\lambda(a^2+2ct)},$$

of the Einstein factor which must be a constant. We shall take $\mathrm{Ef} = \frac{k(n+1)}{2}$, where k is a constant. It follows that we can express λ as a function of a_1 (although the above computations could suggest to express a_1 as a function of λ)

$$\lambda = \frac{4c}{k}\frac{a_1}{a_1^2+2ct}. \tag{18}$$

Differentiating (18) with respect to t it follows that (17) is identically fulfilled. Hence the expression (18) of λ is obtained from a prime integral of (17).

Remark. The same result is obtained if we express from the equation $C_n = 0$ the derivative λ'' as a function of λ, λ', a_1, a_1', a_1''.

Recall that the Kähler manifold (T^*M, G, J) has constant holomorphic sectional curvature k if its curvature tensor field K can be expressed by the relation

$$\begin{aligned} K(X,Y)Z = \frac{k}{4}(&G(Z,Y)X - G(Z,X)Y \\ &+ G(Z,JY)JX - G(Z,JX)JY + 2G(X,JY)JZ), \end{aligned}$$

where X, Y, Z are vector fields on T^*M.

We shall use an adapted local frame $(\partial^1, \dots, \partial^n,\ \delta_1, \dots, \delta_n)$ in order to obtain the expressions for the components of K in the case where (T^*M, G, J) has constant holomorphic curvature. Introduce the following M-tensor fields

$$J^{(1)}_{ij} = a_1 g_{ij} + b_1 p_i p_j, \quad J^{kl}_{(2)} = a_2 g^{kl} + b_2 g^{0k} g^{0l}.$$

Remark that, up to a sign, the M-tensor fields $J^{(1)}_{ij}$, $J^{kl}_{(2)}$ are the components of the tensor field J, defining the integrable almost complex structure on T^*M. Next we have

$$K(\delta_i,\delta_j)\delta_k=\frac{k}{4}\big(G^{(1)}_{jk}\delta^h_i-G^{(1)}_{ik}\delta^h_j\big)\delta_h,$$

$$K(\partial^i,\partial^j)\partial^k=\frac{k}{4}\big(G^{kj}_{(2)}\delta^i_h-G^{ki}_{(2)}\delta^j_h\big)\partial^h,$$

$$K(\delta_i,\delta_j)\partial^k=\frac{k}{4}\big(J^{(1)}_{ih}J^{(1)}_{jl}-J^{(1)}_{il}J^{(1)}_{jh}\big)G^{kl}_{(2)}\partial^h,$$

$$K(\partial^i,\partial^j)\delta_k=\frac{k}{4}\big(J^{ih}_{(2)}J^{jl}_{(2)}-J^{il}_{(2)}J^{jh}_{(2)}\big)G^{(1)}_{kl}\delta_h,$$

$$K(\partial^i,\delta_j)\partial^k=\frac{k}{4}\big(-J^{(1)}_{jl}J^{ih}_{(2)}G^{kl}_{(2)}-G^{ki}_{(2)}\delta^h_j-2J^{(1)}_{jl}J^{kh}_{(2)}G^{il}_{(2)}\big)\delta_h,$$

$$K(\partial^i,\delta_j)\delta_k=\frac{k}{4}\big(G^{(1)}_{kj}\delta^i_h+G^{(1)}_{kl}J^{il}_{(2)}J^{(1)}_{jh}+2G^{il}_{(2)}J^{(1)}_{jl}J^{(1)}_{kh}\big)\partial^h.$$

In our case, i.e. when a_1, λ are related by (18), one obtains that the components of K are given just by the above relations, hence the Kähler Einstein manifold (T^*M,G,J) has constant holomorphic curvature k. Hence we may state the following result.

Theorem 9. *Assume that the Riemannian manifold (M,g) has constant sectional curvature c and consider the natural integrable almost complex structure J defined on its cotangent bundle T^*M by (3), where the coefficients a_1, a_2, b_1, b_2 are related by (4) and (7). There exists a family of Kähler Einstein structures (G,J) defined by (9), on T^*M, where the coefficients c_1, c_2, d_1,d_2 are expressed by (11), (13) and the factors λ, μ are given by $\mu=\lambda'$ and by (18). Moreover, the obtained Kähler Einstein structure has constant holomorphic sectional curvature k.*

Remark. The parameter a_1 is not quite arbitrary. In fact, the following conditions must be fulfilled

$$a_1>0,\quad a_1+2tb_1=\frac{a_1a_1'-c}{a_1-2ta_1'}>0,$$

$$\lambda=\frac{4c}{k}\frac{a_1}{a_1^2+2ct}>0,\quad \lambda+2t\lambda'=\frac{4c}{k}\frac{(a_1-2ta_1')(a_1^2-2ct)}{(a_1^2+2ct)^2}>0.$$

Examples 1. Assume $c > 0$ and consider the function $a_1 = B + \sqrt{B^2 + 2ct}$, where B is a positive constant. We have $a_1' = \frac{c}{\sqrt{B^2+2ct}}$ and one checks easily that all the conditions from the above remark are fulfilled. In the case where $c < 0$ one can consider the function $a_1 = B + \sqrt{B^2 - 2ct}$, where B is a positive constant, and a simple algebraic computation shows that λ is a constant and all the conditions from the above remark are fulfilled. In fact, the case $\lambda = 1$ has been considered, in the case of the tangent bundle, in [8], [11].

2. The next situation is obtained when

$$a_1'\lambda + a_1\lambda' = 0$$

It follows that $a_1\lambda = k$, a constant (this constant is not related to the constant used in the study of the first case). Then we have

$$\lambda = \frac{k}{a_1}, \quad \lambda' = -k\frac{a_1'}{a_1^2}, \quad \lambda'' = k\frac{2a_1'^2 - a_1 a_1''}{a_1^3},$$

$$\lambda^{(3)} = k\frac{6a_1 a_1' a_1'' - 6a_1'^3 - a_1^2 a_1^{(3)}}{a_1^4}$$

With these values of λ, λ', λ'', $\lambda^{(3)}$ one gets that the conditions $\operatorname{Diff} QQ_{jk} = 0$ and $\operatorname{Diff} PP^{jk} = 0$ are fulfilled identically.

If we study the property of (T^*M, G, J) to have constant holomorphic sectional curvature, we get that the components $K(\delta_i, \delta_j)\delta_k$, $K(\delta_i, \delta_j)\partial^k$, $K(\partial^i, \partial^j)\delta_k$, $K(\partial^i, \partial^j)\partial^k$ can be expressed just like in the case 1. However, the last two components $K(\partial^i, \delta_j)\partial^k$, $K(\partial^i, \delta_j)\delta_k$ are quite different from the expression obtained in the case 1. Hence (T^*M, G, J) cannot have constant holomorphic sectional curvature. Then, we may state the following result

Theorem 10. *Consider the Kählerian structure (G, J) on T^*M obtained in Theorem 7, depending on the essential parameters a_1, λ. If $\lambda = \frac{k}{a_1}$, then the manifold (T^*M, G, J) is Kähler Einstein. It cannot have constant holomorphic sectional curvature.*

Remark. In this case, the following conditions must be fulfilled

$$a_1 > 0, \quad a_1 + 2tb_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'} > 0,$$

$$k>0, \quad \frac{a_1^2}{k}(\lambda+2t\lambda')=a_1-2ta_1'>0,$$

hence the functions a_1^2-2ct and $\frac{t}{a_1^2}$ must be increasing.

Remark. The case $\lambda=1$ (and $k=1$) has been considered, in the case of the tangent bundle, in [7].

3. The last case is obtained when $a_1\lambda-ta_1'\lambda+ta_1\lambda'=0$. One sees easily that, in this case, one has $a_1=kt\lambda$, where k is a constant (this constant is not related to the constant used in the previous cases). One sees easily that $a_1(0)=0$, thus this situation should be excluded. However, we can study the properties of the Kählerian structure (G,J) on the manifold T_0^*M obtained from T^*M by excluding the zero section. Next one gets that the conditions Diff $QQ_{jk}=0$ and Diff $PP^{jk}=0$ are fulfilled identically, so the Kähler manifold (T_0^*M,G,J) is Einstein.

Theorem 11. *it Consider the Kählerian structure (G,J) on T^*M, obtained in Theorem 7, depending on the essential parameters a_1, λ. If $a_1=kt\lambda$ then the manifold (T_0^*M,G,J) is Kähler Einstein.*

Remark. The function λ must fulfill the conditions obtained from $a_1>0$, $a_1+2tb_1>0$, $\lambda>0$, $\lambda+2t\lambda'>0$.

Remark. The case $\lambda=1$ has been considered, in the case of the tangent bundle, in [14].

Remark. The authors agree with the referee who claimed more details about some involved computations necessary to get our results. Remark that some results are quite simple if we think of the huge formulas used in order to get them. E.g. the expanded expression of the expression from the formula (16) contains about 180 terms. However, after some quite obvious factorizations, one sees that the essential expression contains only 30 terms. The authors are ready to reveal details of the computations (symbolic computations made by using the Ricci program) used in the paper to reader who claim this.

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