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Ricci curvature along rays

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Abstract. The connection between the behavior of the Ricci curvature and the existence of conjugate points on a complete manifold is studied.

0. Introduction

This note follows in the footsteps of the papers of Ambrose [1] and Wraith [4] examining the connection between the behavior of the Ricci curvature along geodesics and the existence of the conjugate points. The basic setup is as follows.

Let M^n be a complete Riemannian manifold and $\gamma : (0, \infty) \to M$ be a geodesic. Denote by $r(t) = \mathrm{Ricci}(\gamma'(t))$ the Ricci curvature in the direction of the geodesic at $\gamma(t)$. Then Ambrose's result [1] states:

Theorem (Ambrose). *If* $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$, *then there is a point conjugate to* $\gamma(0)$ *along* γ.

The condition $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$ cannot be replaced with the weaker assumption $\limsup_{t\to\infty} \int_0^t r(s)ds = \infty$. At the end of the paper we sketch the construction of a two-dimensional complete manifold with a point O such that $\limsup_{t\to\infty} \int_0^t r(s)ds = \infty$ along every geodesic emanating from O and there are no points conjugate to O on this manifold. So the only way to relax the condition in the above theorem is to put

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some restriction on the Ricci curvature along γ. To this end we have the following.

Theorem 1. *If γ has no conjugate point to $\gamma(0)$ and $r(t) < K$ for some $K > 0$, then $\int_0^t r(s)ds < nK + n$ for all $t \in (0, \infty)$.*

It is well known that if a complete and connected manifold M^n has a point $O \in M^n$ such that every geodesic ray emanating from O has a point conjugate to O along that ray, then M^n is compact (see Lemma 1 of [1]). Combining this with the above we obtain a theorem like that of Myers.

Corollary. *Let M^n be a complete connected Riemannian manifold with Ricci curvature* Ricci $< K$ *for some $K > 0$. Suppose there is a point $O \in M^n$ such that along every geodesic ray $\gamma : (0, \infty) \to M$ emanating from O we have $\int_0^t r(s)ds > nK + n$ for some $t \in (0, \infty)$. Then M^n is compact.*

A similar statement holds if we have some kind of lower bound on the Ricci curvature. In the following theorem we give three versions of this. The different conditions in the theorem are meant to ensure that $\int_0^t r(s)ds$, as a function of t, cannot decrease too fast.

Theorem 2. *Suppose that γ has no conjugate point to $\gamma(0)$. Then $\int_0^t r(s)ds$ is bounded from above as a function of t if any of the following three conditions is satisfied.*

Condition 1: $r(t) > -K$ *for some* $K > 0$ *on* $t \in (0, \infty)$.

Condition 2: $\int_t^{t+1} r(s)ds > -K$ *for some* $K > 0$.

Condition 3: $r'(t) > -K$ *for some* $K > 0$.

Let us denote by $f : M^n \to \mathbb{R}$ the distance function from $\gamma(0)$ and let $m(t) = \Delta f(\gamma(t))$ the trace of the shape operator of distance spheres from $\gamma(0)$ at $\gamma(t)$. It is well known that if $m(t)$ develops a singularity at $t = t_0$, that is, $\lim_{t \to t_0^-} = -\infty$, then $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. It is also well known that $m(t)$ satisfies the Riccati inequality along γ.

$$m'(t) \leq -r(t) - \frac{m^2(t)}{n-1}. \tag{1}$$

Let $y(t)$ be the solution of the Riccati equation

$$y'(t) = -r(t) - \frac{y^2(t)}{n-1}. \tag{2}$$

with the initial condition $y(0) = m(0)$. Then it is clear that if $\gamma(0)$ has no conjugate point on γ, then $m(t) \leq y(t)$ for all $t \geq 0$. Therefore the question of conjugate points can be reduced to the existence of an all time solution of the Riccati equation (2).

1. Riccati equation

In this section we collect some basic results concerning the Riccati equation (2). Let us start with the well known comparison principle [2].

Comparison Principle. *Let $r_1(t) \geq r_2(t)$ be two continuous functions on $[0, T]$ and $y_1(t)$ and $y_2(t)$ be corresponding solutions of (1) and (2) respectively with initial conditions $y_1(0) \leq y_2(0)$. Then for every $t \in [0, T]$ we have $y_1(t) \leq y_2(t)$.*

The next two propositions are crucial to our proof. They are somewhat similar to the comparison principle in that they compare a solution of (1) to a solution of (2) with a specially chosen r.

Let r be a continuous function on $[a, b]$. Denote by $r^+ = \max(0, r)$ and by $r^- = -\min(0, r)$. Then $r = r^+ - r^-$. Let $M = \int_a^b r^+$ and $N = \int_a^b r^-$. Then we have.

Proposition 1. *Let r be a continuous function on $[a, b]$ and m be a solution of (1) with initial condition $m(a) \leq T$. Let r_0 be the distribution on $[a, b]$ defined by $r_0 = M\delta_a - N\delta_b$, where δ_x denotes the Dirac delta distribution at x. Suppose m_0 is a positive solution of (2) with initial condition $m_0(a) = T$. Then $m(b) \leq m_0(b)$. The above statement remains true for $T = \infty$ as well.*

We also need the following.

Proposition 2. *Let r be a continuous function on $[a, a+n-1]$ with the property that $\int_a^t r(s)ds \geq 0$ for all $t \in [a, a+n-1]$. Suppose m is* *a solution of* (1) *with initial condition* $m(a) < -1$. *Then* m *develops a singularity in* $[a, a+n-1]$, *that is, there is a* $t_0 \in (a, a+n-1]$ *such that* $\lim_{t \to t_0^-} m(t) = -\infty$.

The rest of the section is devoted to the proof of these propositions.

Proof of Proposition 1. Assume, on the contrary, that $m(b) > m_0(b)$. Let t_0 be the last point where $m(t_0) = m_0(t_0)$, that is $m(t) > m_0(t) > 0$ for all $t \in (t_0, b]$. Then we have

$$\begin{aligned} m_0(b) - m_0(t_0) &= \int_{t_0}^{b} m_0'(s)ds \\ &= \int_{t_0}^{b} -r_0(s) - \frac{m_0^2(s)}{n-1} \\ &\geq \int_{t_0}^{b} -r(s) - \frac{m^2(s)}{n-1} \\ &\geq \int_{t_0}^{b} m'(s)ds = m(b) - m(t_0). \end{aligned}$$

Since $m(t_0) = m_0(t_0)$ we obtain $m_0(b) \geq m(b)$ that contradicts our assumption, therefore the proof is complete.

In the case when $T = \infty$ set $r_{0,\epsilon} = M\delta_{a+\epsilon} - N\delta_b$ and denote by $m_{0,\epsilon}$ the positive solution of (2) on the interval $[a+\epsilon, b]$ with initial condition $m_{0,\epsilon}(\epsilon) = m(\epsilon) + \frac{1}{\epsilon}$. Then by Proposition 1 we have $m(b) \leq m_{0,\epsilon}(b)$. Since $\lim_{\epsilon \to 0} m_{0,\epsilon}(b) = m_0(b)$ the conclusion of the proposition holds. □

Proof of Proposition 2. Let $r_0 \equiv 0$ on $[a, a+n-1]$ and denote by m_0 the solution of (2) with initial condition $m_0(a) = -1$. Then we know from the explicit solution of (2), $\left(m_0(t) = \frac{-1}{1-\frac{t-a}{n-1}}\right)$, that m_0 develops a singularity at $t = a+n-1$. The conclusion of the Proposition will follow if we can show that $m(t) < m_0(t)$ for every $t \in [a, a+n-1]$ as long as $m(t)$ is defined.

Suppose this is not true. Then, since $m(a) < m_0(a) = -1$, there is a $t_0 \in [a, a+n-1]$ such that $m(t_0) = m_0(t_0)$ and $m(t) < m_0(t)$ for all

$t \in [a, t_0]$. Then

$$m_0(a) - m_0(t_0) = \int_a^{t_0} r_0(s) + \frac{m_0^2(s)}{n-1} ds$$
$$< \int_a^{t_0} r(s) + \frac{m^2(s)}{n-1} ds = m(a) - m(t_0),$$

that contradicts the assumption on the initial condition. □

2. Proof of the theorems

PROOF OF THEOREM 1. The proof proceeds by contradiction. Assume, on the contrary, that there is a point t_3 such that $\int_0^{t_3} r(s)ds \geq nK + n$. Then $t_3 > 1$ since $r(s) < K$.

As before set $r^+ = \max(0, r)$ and $r^- = -\min(0, r)$. Let $A = \int_0^1 r^+(s)ds$ and $B = \int_0^1 r^-$. Then again from the bound on $r(s)$ we see that $A < K$.

Let $y(t)$ be the solution of (2) with $r = A\delta_0 - B\delta_1$ and initial condition $y(0) = \infty$, where δ_x denotes the Dirac distribution at x. Then $y(t) = (n-1)/t$ for $t < 1$ and $y(1) = B + n - 1$. Proposition 1 now implies that

$$m(1) \leq B + n - 1. \tag{3}$$

From the indirect assumption it follows that there are points $1 < t_1 < t_2 < t_3$ such that $\int_0^{t_1} r(s)ds = K + n$, $\int_0^{t_2} r(s)ds = nK + n$ and for all $t \in (t_1, t_2)$ we have $\int_0^t r(s)ds > K + n$. From the upper bound on r we conclude that $t_2 - t_1 \geq n - 1$.

First we claim that $m(t_1) < -1$. From the definition of A and B we have

$$K + n = \int_0^{t_1} r(s)ds = A - B + \int_1^{t_1} r(s)ds,$$

hence

$$\int_1^{t_1} r(s)ds = B + n + K - A > B + n.$$

Then from (1) and (3) we get

$$m(t_1) \le m(1) + \int_1^{t_1} -r(s) - \frac{m^2(s)}{n-1} ds$$
$$< B + n - 1 - (B+n) = -1.$$

Next we apply Proposition 2 to the interval $[t_1, t_2]$. Recall that for all $t \in (t_1, t_2)$ we have $\int_0^t r(s)ds > K + n$. This implies that for all $t \in (t_1, t_2)$ we have $\int_{t_1}^t r(s)ds > 0$. Since $t_2 - t_1 \ge n - 1$ Proposition 2 implies that m develops a singularity between t_1 and t_2. This contradicts the assumption that γ has no conjugate points and the proof is complete. □

Proof of Theorem 2. Assume, on the contrary, that $\limsup_{t\to\infty} \int_0^t r(s)ds = \infty$. Then there are numbers $1 < t_1 < t_2$ such that

$$\int_1^{t_1} r(s)ds \ge m(1) + 1.1, \tag{4}$$

and for all $t \in (t_1, t_2)$ we have

$$\int_{t_1}^t r(s)ds > 0 \quad \text{with} \quad \int_{t_1}^{t_2} r(s)ds = (n-1)^2 K \quad \text{and} \quad r(t_2) \ge 0. \tag{5}$$

Then from (1) and (4) we have

$$m(t_1) \le m(1) + \int_1^{t_1} -r(s) - \frac{m^2(s)}{n-1} ds < m(1) - m(1) - 1.1 < -1. \tag{6}$$

On the other hand combining (5) with any of the three conditions of the theorem implies that for all $t \in [t_1, t_2 + n - 1]$ we have $\int_{t_1}^t r(s)ds \ge 0$. Then an application of Proposition 2 to the interval $[t_1, t_1 + n - 1]$ shows that $m(s)$ develops a singularity. This leads to a contradiction that completes the proof of the theorem. □

Remark. The same argument gives a proof of Ambrose's theorem.

Proof of the Theorem of Ambrose. Assume that $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$. Then there is a number $t_1 > 1$ such that $\int_1^{t_1} r(s)ds > m(1) + 1$ and for all $t \in [t_1, \infty)$ we have $\int_{t_1}^t r(s)ds \ge 0$. Then, as in (6), we have $m(t_1) < -1$ and an application of Proposition 2. yields the desired contradiction. □

Remark. The fact that the Ambrose Theorem can be derived from the Riccati inequality for mean curvature was established in [5].

3. An example

The purpose of this section is to sketch the construction of a two-dimensional complete Riemannian manifold M^2 with a point O such that every geodesic γ emanating from O is free of conjugate points and $\limsup_{t\to\infty} \int_0^t r(s)ds = \infty$ along γ.

Our manifold M^2 will be a surface of rotation in $\mathbb{R}^3$ obtained by rotating the graph of a function $f : (0, \infty) \to \mathbb{R}$ with $f(0) = 0$ around the x-axis. The origin of the coordinate system will play the role of the point O. The metric on the surface will be the natural metric inherited from the ambient space $\mathbb{R}^3$.

First we construct $f(t)$ to be piecewise smooth. In fact $f(t)$ will be piecewise linear and decreasing on $[1, \infty)$. Although the manifold obtained this way will not be smooth it has the advantage of maintaining simplicity and clarity. Smoothing off the corners of $f(t)$ will give a smooth function, which in turn will give rise to a smooth manifold.

Before we start off with the construction of f we have to make a remark on the curvature of the manifold arising this way. Since M^2 is rotationally symmetric we may assume that γ is the intersection of the 1st quadrant of the xy-plane and M^2. Therefore we need to compute the curvature at the points $(t, f(t))$ only. When f is smooth this is given by the formula (Gray [3], Chapter 18)

$$K = \frac{-f''}{f(1 + f'^2)^2}. \tag{7}$$

Since f is piecewise linear on $[1, \infty)$ the curvature of the surface will be zero at $(t, f(t))$ for those points where f is smooth and $t > 1$.

The only problem is when two linear pieces join together. At these points the manifold is not smooth so it has no curvature in the classical sense. Even if we round off the corner and take limits the sectional curvature will approach $\pm\infty$. But if we integrate the curvature of the rounded off manifold near the point and take the limit the result will be a finite value depending on the left and right derivatives and the value of f. We will need an estimate of this value.

Let us assume that f is piecewise linear and t_0 is one of the points where two linear pieces meet. Denote the left and right derivatives by $s_- = D_- f(t_0)$ and $s_+ = D_+ f(t_0)$.

If we smooth off the corner of the graph of f at t_0 then for the integral of the sectional curvature (which is the same as the Ricci curvature in this case) we have from (7)

$$C_1 \frac{s_- - s_+}{f(t_0)} < \int_{t_0-\delta}^{t_0+\delta} r(s)ds < C_2 \frac{s_- - s_+}{f(t_0)} \quad \text{for} \quad 0 > s_-,\ s_+ \geq -1, \tag{8}$$

where $C_1, C_2 > 0$ are some constants.

We are now ready to construct f. First set $f(t) = \sqrt{1-(1-t)^2}$ for $t \in [0,1]$. The graph of this function is the quarter unit circle centered around the point $x = 1$.

Let f be linear on $[1, a_1]$ with slope $s_1 = -1/2$ and on $[a_1, a_1 + \epsilon_1/10]$ with slope -1, where $\epsilon_1 = f(a_1)$ and $a_1 = 3/2$.

Suppose that f had been constructed on $[0, a_i + \epsilon_i/10]$, where $\epsilon_i = f(a_i)$ and $\frac{-1}{2} \leq s_i < 0$ denotes the slope of f on $[a_{i-1} + \epsilon_{i-1}/10, a_i]$.

Choose s_{i+1} such that $s_i, -\frac{\epsilon_i}{3} < s_{i+1} < 0$. Let f be linear on $[a_i + \epsilon_i/10, a_{i+1}]$ with slope s_{i+1} and on $[a_{i+1}, a_{i+1} + \frac{\epsilon_{i+1}}{10}]$ with slope -1, where $\epsilon_{i+1} = f(a_{i+1})$. Choose a_{i+1} such that $a_{i+1} > 2 + a_i$ (this is why we required $-\frac{\epsilon_i}{3} < s_{i+1} < 0$) and

$$\frac{C_1}{2\epsilon_{i+1}} > i + 1 + \left| \int_0^{a_{i+1}-1} r(s)ds + 1 \right|.$$

This will imply that the integral of the curvature near the point a_{i+1} will dominate the integral of the curvature on $[0, a_{i+1} - 1]$. To be more precise from (8) we obtain

$$\int_{a_{i+1}-\delta}^{a_{i+1}+\delta} r(s)ds > C_1 \frac{s_{i+1} - (-1)}{\epsilon_{i+1}} > \frac{C_1}{2\epsilon_{i+1}},$$

for some sufficiently small $\delta > 0$.

From this we have

$$\int_0^{a_{i+1}+(\epsilon_{i+1}/20)} r(s)ds > i+1.$$

Therefore $\limsup_{t\to\infty}\int_0^t r(s)ds = \infty$, while the condition $a_{i+1} > 2+a_i$ implies that f is defined on $\mathbb{R}$. Rounding off the corners of f will produce a smooth function which in turn will give rise to a smooth surface with the required properties.

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