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On approximately t-convex functions

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Dedicated to the 75th birthday of Professor Heinz König

Abstract. A real valued function f defined on an open convex set D is called (ε, p, t)-convex if it satisfies

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \sum_{i=0}^{k} \varepsilon_i |x-y|^{p_i} \quad \text{for } x, y \in D,$$

where $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in [0, 1[^{k+1}$ and $t \in]0, 1[$ are fixed parameters. The main result of the paper states that if f is locally bounded from above at a point of D and (ε, p, t)-convex then it satisfies the convexity-type inequality

$$f(sx + (1-s)y) \le sf(x) + (1-s)f(y) + \sum_{i=0}^{k} \varepsilon_i \phi_{p_i,t}(s)|x-y|^{p_i}$$

for $x, y \in D$, $s \in [0,1]$, where $\phi_{p_i,t} : [0,1] \to \mathbb{R}$ is defined by

$$\phi_{p_i,t}(s) = \max\left\{\frac{1}{(1-t)^{p_i} - (1-t)}; \frac{1}{t^{p_i} - t}\right\}(s(1-s))^{p_i}.$$

The particular case $k = 0$, $p = 0$ of this result is due to PÁLES [Pál00], the case $k = 0$, $p = 0$ and $t = 1/2$ was investigated by NG and NIKODEM [NN93]. The specialization $k = 0$, $\varepsilon_0 = 0$ yields the celebrated theorem of BERNSTEIN and DOETSCH [BD15]. The case $k = 1$, $\varepsilon = (\varepsilon_0, \varepsilon_1)$, $p = (1, 0)$ and $t = 1/2$ was investigated in HÁZY and PÁLES [HP04].

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1. Introduction

Let $(X, |\cdot|)$ be a normed space and $D \subset X$ be a nonempty open convex set throughout this paper. Given a nonnegative constant ε and $t \in]0,1[$, a function $f : D \to \mathbb{R}$ is said to be (ε, t)-convex if

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) + \varepsilon$$

for all $x, y \in D$. If this inequality holds for all $x, y \in D$ and for all $t \in [0,1]$, then f is simply called ε-convex. The following result establishes a connection between these convexity properties (cf. [Pál00]):

Theorem A. *If $f : D \to \mathbb{R}$ is locally bounded from above at a point of D and is (ε, t)-convex on D, then f is $\big(\max\big\{\frac{1}{t}, \frac{1}{1-t}\big\}\varepsilon\big)$-convex.*

If $\varepsilon = 0$, then this result specializes to the celebrated theorem of Bernstein and Doetsch [BD15] (see also [Kuc85] for further references). The particular case $t = 1/2$ was investigated by Ng and Nikodem [NN93].

Motivated by these results, we introduce new approximate convexity concepts and investigate the connection of "t-convexity" and "convexity" in this approximate sense.

Given two parameter vectors $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$ and $p = (p_0, \dots, p_k) \in [0, \infty[^{k+1}$, and a fixed $t \in]0,1[$, a function $f : D \to \mathbb{R}$ is called (ε, p, t)-convex if

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \sum_{i=0}^{k} \varepsilon_i |x-y|^{p_i}$$

for all $x, y \in D$. In the particular case $t = 1/2$ we refer to it as (ε, p)-midconvexity.

Observe that Theorem A concerns the case $k = 0$, $p = 0$. In the case $k = 1$, $\varepsilon = (\varepsilon_0, \varepsilon_1)$, $p = (0,1)$ and $t = 1/2$ the following theorem was proved in [HP04]:

Theorem B. *Let $\varepsilon_0, \varepsilon_1$ be nonnegative constants. If $f : D \to \mathbb{R}$ is locally bounded above at a point of D and $\big((\varepsilon_0, \varepsilon_1), (0,1)\big)$-midconvex on D, then*

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + 2\varepsilon_0 + 2\varepsilon_1 \varphi(t)|x-y|$$

for all $x, y \in D$ and $t \in [0,1]$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is the Takagi function defined by

$$\varphi(t) := \sum_{n=0}^{\infty} \frac{\operatorname{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R}).$$

In [HP04] we also showed that

$$\phi(t) \le \varphi(t) \le 1.4\phi(t) \qquad (t \in [0,1]), \tag{1}$$

where the function $\phi : [0,1] \to \mathbb{R}$ is defined by the formula

$$\phi(t) := \max\big(-t\log_2 t,\ -(1-t)\log_2(1-t)\big)$$

$$= \begin{cases} -t\log_2 t & \text{if } \ 0 \le t \le \dfrac{1}{2}, \\ -(1-t)\log_2(1-t) & \text{if } \ \dfrac{1}{2} \le t \le 1. \end{cases} \tag{2}$$

It follows from inequality (1) that the Takagi function cannot be compared with the function $t(1-t)$, that is, there is no constant $c > 0$ so that $\varphi(t) \le ct(1-t)$ is valid for all $t \in [0,1]$. Therefore, it does not follow from Theorem B that locally upper bounded $((\varepsilon_0, \varepsilon_1), (0,1))$-midconvex functions satisfy

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) + 2\varepsilon_0 + c\varepsilon_1 t(1-t)|x-y| \tag{3}$$

for all $x, y \in D$, $t \in [0,1]$, and for some $c > 0$. Moreover, the function $f = \phi$ is $((0,1/2),(0,1))$-midconvex, but there is no constant c such that (3) is valid for some $c > 0$.

In this paper, we search for convexity properties that follow from the (ε, p, t)-convexity and the local upper boundedness property of the function f. More precisely, we intend to find functions $\phi_{p_i,t} : [0,1] \to \mathbb{R}$ $(i = 0, 1, \dots, k)$ so that

$$f\big(sx + (1-s)y\big) \le sf(x) + (1-s)f(y) + \sum_{i=0}^{k} \varepsilon_i \phi_{p_i,t}(s)|x-y|^{p_i} \tag{4}$$

hold for all $x, y \in D$ and all $s \in [0,1]$.

In our main result we prove that if $0 \le p_i < 1$, then (4) holds with the choice

$$\phi_{p_i,t}(s) := \max\left\{\frac{1}{(1-t)^{p_i} - (1-t)}; \frac{1}{t^{p_i} - t}\right\}(s(1-s))^{p_i}.$$

Due to the symmetry, we may assume that $0 < t \leq \frac{1}{2}$ in the rest of the paper.

2. A functional equation and related functional inequalities

Denote by $\mathcal{B}(I)$ the space of bounded real-valued functions defined on $I := [0,1]$ equipped with the usual supremum norm. For fixed $p \geq 0$ and $t \in]0, 1/2]$, introduce the operator $T_{p,t} : \mathcal{B}(I) \to \mathcal{B}(I)$ by

$$(T_{p,t}f)(s) = \begin{cases} (1-t)f\left(\dfrac{s}{1-t}\right) + \left(\dfrac{s}{1-t}\right)^p & 0 \leq s \leq \dfrac{1}{2}, \\ (1-t)f\left(\dfrac{1-s}{1-t}\right) + \left(\dfrac{1-s}{1-t}\right)^p & \dfrac{1}{2} \leq s \leq 1. \end{cases}$$

Our first result concerns the solution of the functional equation

$$\varphi(s) = (T_{p,t}\varphi)(s) \qquad (s \in [0,1]) \tag{5}$$

and the corresponding functional inequalities

$$\Phi(s) \leq (T_{p,t}\Phi)(s) \qquad (s \in [0,1]), \tag{6}$$

$$\Psi(s) \geq (T_{p,t}\Psi)(s) \qquad (s \in [0,1]). \tag{7}$$

Theorem 1. *There exists a unique bounded function $\varphi : [0,1] \to \mathbb{R}$ such that (5) holds. Furthermore, φ is continuous, nonnegative and is symmetric with respect to $s = 1/2$, i.e., $\varphi(s) = \varphi(1-s)$ for all $s \in [0,1]$. In addition, if $\Phi : [0,1] \to \mathbb{R}$ and $\Psi : [0,1] \to \mathbb{R}$ are bounded solutions of (6) and (7), respectively, then $\Phi \leq \varphi \leq \Psi$ holds.*

Proof. It is immediate to see that $T_{p,t}$ is a contraction with contraction factor $1-t$ on $\mathcal{B}(I)$. Hence, by the Banach fixed point theorem, there exists a unique function $\varphi \in \mathcal{B}(I)$ such that $T_{p,t}\varphi = \varphi$, i.e., (5) is satisfied.

Let the sequence $\varphi_n : [0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned} \varphi_1 &:= 0, \\ \varphi_{n+1}(s) &:= (T_{p,t}\varphi_n)(s). \end{aligned} \tag{8}$$

By induction, one can see that φ_n is a continuous, nonnegative, and symmetric (with respect to $1/2$) function on $[0,1]$ for all $n \in \mathbb{N}$. By the Banach fixed point theorem, this sequence uniformly tends to φ (i.e., to the fixed point of $T_{p,t}$). Therefore φ is also continuous, nonnegative, and symmetric.

Finally, let $\Phi : [0,1] \to \mathbb{R}$ and $\Psi : [0,1] \to \mathbb{R}$ be bounded solutions of (6) and (7). Observe that $T_{p,t}$ is monotone with respect to the pointwise ordering in $\mathcal{B}(I)$. Thus, applying the operator $T_{p,t}^n$ to the inequalities (6) and (7), we get that

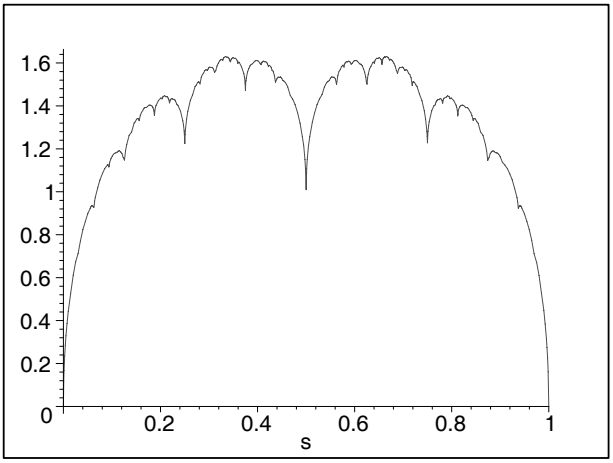
$$T_{p,t}^n \Phi \le T_{p,t}^{n+1} \Phi \quad \text{and} \quad T_{p,t}^n \Psi \ge T_{p,t}^{n+1} \Psi \quad \text{for all } n \in \mathbb{N}.$$

It follows from these inequalities that

$$\Phi \le T_{p,t}^n \Phi \quad \text{and} \quad \Psi \ge T_{p,t}^n \Psi \quad \text{for all } n \in \mathbb{N}.$$

Taking the limit $n \to \infty$, we obtain $\Phi \le \varphi \le \Psi$. □

In the sequel, the unique solution φ of (5) will be denoted by $\varphi_{p,t}$. The picture of $\varphi_{p,t}$ in the case of $p = 1/2$, $t = 1/2$ is as follows:



In order to compare $\varphi_{p,t}$ with a function that is defined in more computable terms, introduce the function $\phi_p : [0,1] \to \mathbb{R}$ by the following

formula:

$$\phi_p(s) := \big(s(1-s)\big)^p. \tag{9}$$

To obtain our result on the comparison of $\varphi_{p,t}$ and ϕ_p, we need the following lemma.

Lemma. *Let $0 \le p < 1$ be an arbitrary constant and $\gamma_{p,t} : [0, 1-t] \to \mathbb{R}$ be defined by*

$$\gamma_{p,t}(s) := (1-s)^p(1-t)^p - (1-t)^{1-p}(1-t-s)^p.$$

Then $\gamma_{p,t}$ is a positive and increasing function.

Proof. Since $0 < 1-t < 1$ and $0 \le p < 1$, therefore

$$\gamma_{p,t}(0) = (1-t)^p - (1-t) > 0.$$

The function $\gamma_{p,t}$ is differentiable and

$$\gamma'_{p,t}(s) = p\big(-(1-s)^{p-1}(1-t)^p + (1-t-s)^{p-1}(1-t)^{1-p}\big).$$

Since $p \ge 0$, therefore it is enough to prove that, for $s \in [0, 1-t]$,

$$-(1-s)^{p-1}(1-t)^p + (1-t-s)^{p-1}(1-t)^{1-p} \geqq 0,$$

which is equivalent to the inequality

$$1 - \frac{t}{1-s} \leqq (1-t)^{\frac{2p-1}{p-1}}.$$

One can easily see, that the left-hand side of this inequality is a monotone function of s, and the inequality holds at the endpoints $s = 0$ and $s = 1-t$. Therefore, it is also valid for all $s \in [0, 1-t]$. Thus, $\gamma_{p,t}$ is increasing and it is also positive on $[0, 1-t]$. □

The functions ϕ_p and $\varphi_{p,t}$ have the following property:

Theorem 2. *If $0 \le p < 1$, then*

$$\frac{\phi_p(s)}{\gamma_{p,t}(1/2)} \le \varphi_{p,t}(s) \le \frac{\phi_p(s)}{\gamma_{p,t}(0)} \quad \text{for} \quad s \in [0,1]. \tag{10}$$

PROOF. In the first step we prove that the function $\Phi = \dfrac{\phi_p}{\gamma_{p,t}(1/2)}$ is a solution of the functional inequality (6).

We consider first the case $0 \le s \le 1/2$. The function $\gamma_{p,t}$ is monotone increasing, therefore $\gamma_{p,t}(s) \le \gamma_{p,t}(1/2)$ for all $0 \le s \le 1/2$, i.e.,

$$\frac{\gamma_{p,t}(s)}{\gamma_{p,t}(1/2)} = \frac{(1-s)^p(1-t)^p - (1-t)^{1-p}(1-s-t)^p}{\gamma_{p,t}(1/2)} \le 1,$$

which implies

$$\frac{(1-s)^p(1-t)^p}{\gamma_{p,t}(1/2)} \le \frac{(1-t)^{1-p}(1-s-t)^p}{\gamma_{p,t}(1/2)} + 1.$$

Multiplying by $(\frac{s}{1-t})^p$, we get

$$\frac{s^p(1-s)^p}{\gamma_{p,t}(1/2)} \le \frac{(1-t)}{\gamma_{p,t}(1/2)} \left(\frac{s}{1-t}\right)^p \left(1 - \frac{s}{1-t}\right)^p + \left(\frac{s}{1-t}\right)^p,$$

i.e.,

$$\frac{\phi_p(s)}{\gamma_{p,t}(1/2)} \le \frac{(1-t)}{\gamma_{p,t}(1/2)} \phi_p\left(\frac{s}{1-t}\right) + \left(\frac{s}{1-t}\right)^p.$$

Similarly, if $1/2 \le s \le 1$, then from $\gamma_{p,t}(1-s) \leqq \gamma_{p,t}(1/2)$, we get that

$$\frac{\phi_p(s)}{\gamma_{p,t}(1/2)} \le \frac{(1-t)}{\gamma_{p,t}(1/2)} \phi_p\left(\frac{1-s}{1-t}\right) + \left(\frac{1-s}{1-t}\right)^p.$$

Thus, we have proved that $\Phi = \dfrac{\phi_p}{\gamma_{p,t}(1/2)}$ indeed satisfies the functional inequality (6). Due to Theorem 1, the left hand side inequality in (10) holds.

To obtain the right hand side inequality in (10), it suffices to prove that the function $\Psi = \dfrac{\phi_p}{\gamma_{p,t}(0)}$ is a bounded solution of the functional inequality (7) and use Theorem 1 again. □

Remark 1. An inequality analogous to (10) in the case $p = 1$, $t = 1/2$ was derived in [HP04]; in this case $\phi_{p,t}$ can be compared to the function ϕ defined by (2). It is not clear, however, what the asymptotic magnitude of $\phi_{p,t}$ is for $p > 1$. This problem is left open in this paper.

3. Regularity properties of (ε, p, t)-convex functions

In our next results, we deal with boundedness and continuity properties of (ε, p, t)-convex functions.

Theorem 3. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in [0, \infty[^{k+1}$, and $t \in]0, 1/2]$. If $f : D \to \mathbb{R}$ is (ε, p, t)-convex and locally bounded from above at a point $w \in D$, then f is locally bounded on D.*

Proof. First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0 := \{w\}, \qquad D_{n+1} := tD_n + (1-t)D.$$

Then, it follows by induction that

$$D_n = t^n w + (1 - t^n)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption f is locally upper bounded at $w \in D_0$. Assume that f is locally upper bounded at each point of D_n. For $x \in D_{n+1}$, there exists $x_0 \in D_n$ and $y_0 \in D$ such that $x = tx_0 + (1-t)y_0$. By the inductive assumption, there exists $r > 0$ and a constant $M_0 \geq 0$ such that $f(x') \leq M_0$ for $|x_0 - x'| < r$. Then, by the (ε, p, t)-convexity of f, for $x' \in U_0 := U(x_0, r)$, we have

$$\begin{aligned} f\left(tx' + (1-t)y_0\right) &\leq tf(x') + (1-t)f(y_0) + \sum_{i=0}^{k} \varepsilon_i |x' - y_0|^{p_i} \\ &\leq tM_0 + (1-t)f(y_0) + \sum_{i=0}^{k} \varepsilon_i (|x' - x_0| + |x_0 - y_0|)^{p_i} \\ &\leq tM_0 + (1-t)f(y_0) + \sum_{i=0}^{k} \varepsilon_i (|x_0 - y_0| + r)^{p_i} =: M. \end{aligned}$$

Therefore, for $x \in U := tU_0 + (1-t)y_0 = U(tx_0 + (1-t)y_0, tr)$, we get that $f(x) \leq M$. Thus f is locally bounded from above on D_{n+1}.

On the other hand, one can easily see that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Indeed, for fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{(1/t)^n x - w}{(1/t)^n - 1}.$$

Then $x_n \to x$ if $n \to \infty$. The set being open, $x_n \in D$ for some n. Therefore

$$x = \frac{w + ((1/t)^n - 1)x_n}{(1/t)^n} = t^n w + (1 - t^n)x_n \in t^n w + (1 - t^n)D = D_n.$$

Thus f is locally bounded from above on D.

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, hence there exists $\varrho > 0$ and $M > 0$ such that

$$\sup_{U(q,\varrho)} f \le M.$$

Let $x \in U(q, \varrho)$ and $y := \frac{1}{1-t}q - \frac{t}{1-t}x$. Then, by (ε, p, t)-convexity,

$$f(q) = f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \sum_{i=0}^{k} \varepsilon_i \frac{1}{(1-t)^{p_i}} |x - q|^{p_i},$$

which implies

$$\begin{aligned} f(x) &\ge \frac{1}{t}f(q) - \frac{1-t}{t}f(y) - \frac{1}{t}\sum_{i=0}^{k} \varepsilon_i \frac{1}{(1-t)^{p_i}} |x - q|^{p_i} \\ &\ge \frac{1}{t}f(q) - \frac{1-t}{t}M - \frac{1}{t}\sum_{i=0}^{k} \varepsilon_i \frac{1}{(1-t)^{p_i}} \varrho^{p_i} =: M^*. \end{aligned}$$

Therefore f is locally bounded from below at any point of D. □

The next theorem essentially weakens the local boundedness assumption if the underlying space is of finite dimension. It can be derived from Theorem 3 adopting the argument followed in [HP04] (that is based on Steinhaus' and Piccard's theorems (cf. [Ste20], [Pic42])).

Theorem 4. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in [0, \infty[^{k+1}$, and $t \in]0, 1/2]$. Let D be an open convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ be an (ε, p, t)-convex function. Assume that there exist a*

Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S \subset D$ and a Lebesgue-measurable (resp. a Baire-measurable) function $g : S \to \mathbb{R}$ such that $f \leq g$ on S. Then f is locally bounded on D.

The next result states that if all the components of p are positive then the local upper boundedness of an (ε, p, t)-convex function yields its continuity as well. The proof is analogous to what was followed for $(\varepsilon, 0)$-midconvexity in [HP04].

Theorem 5. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in]0, \infty[^{k+1}$, and $t \in]0, 1/2]$. If $f : D \to \mathbb{R}$ is (ε, p, t)-convex and locally bounded from above at a point of D, then it is continuous.*

4. Main results

The following result offers a generalization of the theorems of Bernstein and Doetsch [BD15], Ng and Nikodem [NN93] and the results of Páles [Pál00] and Házy and Páles [HP04].

Theorem 6. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in [0, \infty[^{k+1}$, and $t \in]0, 1/2]$. If $f : D \to \mathbb{R}$ is (ε, p, t)-convex and locally bounded from above at a point of D, then*

$$f(sx + (1-s)y) \leq sf(x) + (1-s)f(y) + \sum_{i=0}^{k} \varepsilon_i \varphi_{p_i,t}(s)|x-y|^{p_i} \qquad (11)$$

for all $x, y \in D$ and $s \in [0, 1]$, (where $\varphi_{p_i,t}$ is the fixed point of the operator $T_{p_i,t}$ defined in Section 2).

Proof. Due to Theorem 3, f is locally bounded at each point of D. Thus f is bounded on each compact subset of D.

Let $x, y \in D$ be fixed and denote by $K_{x,y}$ the upper bound of the function

$$s \mapsto f(sx + (1-s)y) - sf(x) - (1-s)f(y) \qquad (s \in [0, 1]).$$

We are going to show, by induction on n, that

$$f\big(sx+(1-s)y\big) \le sf(x)+(1-s)f(y)+(1-t)^n K_{x,y} + \sum_{i=0}^{k} \varepsilon_i (T^n_{p_i,t}\mathbf{0})(s)|x-y|^{p_i} \tag{12}$$

for all $x, y \in D$ and $s \in [0,1]$, where $\mathbf{0}$ denotes the identically zero function on $[0,1]$.

For $n=0$, the statement follows from the definition of $K_{x,y}$.

Assume that (12) is true for some $n \in \mathbb{N}$. Suppose that $s \in [1/2,1]$. Then, due to the (ε,p,t)-convexity of f, we get

$$f(sx+(1-s)y) = f\left(tx+(1-t)\left(\frac{s-t}{1-t}x+\frac{1-s}{1-t}y\right)\right)$$

$$\le tf(x)+(1-t)f\left(\frac{s-t}{1-t}x+\frac{1-s}{1-t}y\right)+\sum_{i=0}^{k}\varepsilon_i\left(\frac{1-s}{1-t}\right)^{p_i}|x-y|^{p_i}.$$

On the other hand, using (12), we have that

$$f\left(\frac{s-t}{1-t}x+\frac{1-s}{1-t}y\right) \le \frac{s-t}{1-t}f(x)+\frac{1-s}{1-t}f(y) + (1-t)^n K_{x,y} + \sum_{i=0}^{k}\varepsilon_i (T^n_{p_i,t}\mathbf{0})\left(\frac{1-s}{1-t}\right)|x-y|^{p_i}.$$

Combining these two inequalities, we obtain

$$f\left(sx+(1-s)y\right) \le sf(x)+(1-s)f(y)+(1-t)^{n+1}K_{x,y} + \sum_{i=0}^{k}\varepsilon_i\left((1-t)(T^n_{p_i,t}\mathbf{0})\left(\frac{1-s}{1-t}\right)+\left(\frac{1-s}{1-t}\right)^{p_i}\right)|x-y|^{p_i}$$

$$= sf(x)+(1-s)f(y)+(1-t)^{n+1}K_{x,y}+\sum_{i=0}^{k}\varepsilon_i(T^{n+1}_{p_i,t}\mathbf{0})(s)|x-y|^{p_i}.$$

Thus, we proved (12) for $s \in [1/2,1]$. A completely similar argument shows that (12) is also valid for $s \in [0,1/2]$.

To complete the proof of the theorem, we take the limit $n \to \infty$ in (12) and we get (11). □

Applying the right hand side inequality of Theorem 2, if the parameters p_i are smaller than 1, we immediately get the following result.

Corollary. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \dots, p_k) \in [0, 1[^{k+1}$, and $t \in]0, 1/2]$. If $f : D \to \mathbb{R}$ is (ε, p, t)-convex and locally bounded from above at a point of D, then*

$$f(sx + (1-s)y) \le sf(x) + (1-s)f(y) + \sum_{i=0}^{k} \frac{\varepsilon_i}{(1-t)^{p_i} - (1-t)} \big(s(1-s)|x-y|\big)^{p_i} \quad (13)$$

for all $x, y \in D$ and $s \in [0, 1]$.

Proof. It follows from Theorem 2 that, for $0 \le p_i < 1$,

$$\varphi_{p_i,t} \le \frac{\phi_{p_i}}{\gamma_{p_i,t}(0)} = (1-t)^{p_i} - (1-t)\big(s(1-s)\big)^{p_i},$$

where the function $\gamma_{p,t}$ was defined in Lemma. Thus the statement is an immediate consequence of the previous theorem. □

References

[BD15] F. Bernstein and G. Doetsch, Zur Theorie der konvexen Funktionen, *Math. Annalen* **76** (1915), 514–526.

[HP04] A. Házy and Zs. Páles, On approximately midconvex functions, *Bull. London Math. Soc.* **36**, no. 3 (2004), 339–350.

[Kuc85] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, Państwowe Wydawnictwo Naukowe, *Uniwersytet Śląski*, *Warszawa – Kraków – Katowice*, 1985.

[Mro01] J. Mrowiec, Remark on approximately Jensen-convex functions, *C. R. Math. Rep. Acad. Sci. Canada* **23**, no. 1 (2001), 16–21.

[NN93] C. T. Ng and K. Nikodem, On approximately convex functions, *Proc. Amer. Math. Soc.* **118**, no. 1 (1993), 103–108.

[Pál00] Zs. Páles, Bernstein–Doetsch-type results for general functional inequalities, *Rocznik Nauk.-Dydakt. Prace Mat.* **17** (2000), 197–206, Dedicated to Professor Zenon Moszner on the occasion of his seventieth birthday.

[Pál03] Zs. Páles, On approximately convex functions, *Proc. Amer. Math. Soc.* **131**, no. 1 (2003), 243–252.

[Pic42] S. Piccard, Sur les ensembles parfaits, Mém. Univ. Neuchâtel, vol. 16, *Secrétariat de l'Université Neuchâtel*, 1942.

[Ste20] H. Steinhaus, Sur les distance des points des ensembles de mesure positif, *Fund. Math.* **1** (1920), 99–104.

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