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On Douglas metrics

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Abstract. In this paper, we discuss Douglas metrics with relatively isotropic Landsberg curvature or isotropic mean Berwald curvature. Then we introduce the Finsler metrics of isotropic Berwald curvaure. We prove an equivalence among the above metrics.

1. Introduction

In Finsler geometry, there are several important classes of Finsler metrics. The Berwald metrics were first investigated by L. Berwald. Every Finsler metric F on a manifold M induced a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics. By definition, a Finsler metric F is a Berwald metric if the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_xM$ at every point x, i.e.,

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k.$$

Riemannian metrics are special Berwald metrics. In fact, Berwald metrics are "almost Riemannian" in the sense that every Berwald metric is affinely equivalent to a Riemannian metric, i.e., the geodesics of any Berwald metric are the geodesics of some Riemannian metric [12]. The Douglas metrics are more generalized ones than Berwald metrics. A Finsler metric is called

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a Douglas metric if the spray coefficients $G^i = G^i(x, y)$ are in the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x, y)y^i. \tag{1}$$

Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

There are two imoportant non-Riemannian quantities: the mean Berwald curvature (E-curvature) and the Landsberg curvature (L-curvature). If $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is the spray of a Finsler metric F, then the mean Berwald tensor $\mathcal{E} = E_{ij} dx^i \otimes dx^j$ and the Landsberg tensor $\mathcal{L} = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ are respectively defined by

$$E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right), \quad L_{ijk} := -\frac{1}{2} y^s g_{ms} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}.$$

Clearly, for Berwad metrics, $\mathcal{E} = 0$ and $\mathcal{L} = 0$. Finsler metrics with $\mathcal{E} = 0$ are called weakly Berwald metrics and those with $\mathcal{L} = 0$ are called Landsberg metrics. There are many weakly Berwald metrics which are non-Berwaldian, but so far, we do not know if every Landsberg metric is Berwaldian. This is a long existing open problem in Finsler geometry. The class of Douglas metrics is much larger than that of Berwald metrics. In this paper, we are prove the following theorem for a general Douglas metrics.

Theorem 1.1. *Let (M, F) be a non-Riemannian Douglas manifold of dimension $n \geq 3$. Then the following are equivalent:*

(a) *F has isotropic mean Berwald curvature;*

(b) *F has relatively isotropic Landsberg curvature.*

We shall introduce Finsler metrics with isotropic Berwald curvature. We shall prove that, if a Finsler metric is a Douglas metric satisfying condition (a) or (b) in Theorem 1.1, then it must have isotropic Berwald curvature, and vice versa. This particularly implies that any Douglas metric with vanishing Landsberg curvature must be a Berwald metric.

2. Preliminaries

Let M be a manifold of dimension n and let π^*TM denote the pull-back tangent bundle over the slit tangent bundle $TM_0 := TM \setminus \{0\}$ by the natural projection $\pi : TM_0 \to M$. Denote by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ be the local natural frame and $\{dx^i, dy^i\}$ the local natural coframe for $T(TM_0)$ in a standard local coordinate system (x^i, y^i) in TM_0. Denote by $\{\partial_i\}$ the local natural frame for π^*TM corresponding to the local natural frame $\{\frac{\partial}{\partial x^i}\}$ for the tangent bundle TM.

Let F be a Finsler metric on manifold M of dimension n. We have two tensors

$$g := g_{ij} dx^i \otimes dx^j, \qquad h := h_{ij} dx^i \otimes dx^j,$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and $h_{ij} := F F_{y^i y^j} = g_{ij} - F_{y^i} F_{y^j}$. The first important quantity is the Cartan tensor $\mathcal{C} = C_{ijk} dx^i \otimes dx^j \otimes dx^l$, where $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^k y^l}$. Let G be the induced spray. $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a special vector field on TM_0, which is defined by

$$G^i = \frac{1}{4} g^{il} \big\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \big\},$$

where $(g^{ij}) := (g_{ij})^{-1}$. The spray determines the geodesics of F by

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0.$$

More precisely, the geodesics of F are the projections of the integral curves of G.

By definition, F is a Berwald metric if the spray coefficients $G^i = G^i(x,y)$ are quadratic in $y \in T_xM$ at every point $x \in M$. For a general Finsler metrics it is natural to consider the following term

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

It is easy to verify that $\mathcal{B} := B_j{}^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_0. We call $\mathcal{B}$ the *Berwald tensor.* The mean Berwald tensor $\mathcal{E} = E_{ij} dx^i \otimes dx^j$ is defined by

$$E_{ij} := \frac{1}{2} B_m{}^m{}_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right). \tag{2}$$

The Landsberg tensor $\mathcal{L} = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := -\frac{1}{2}y^s g_{sm} B_i{}^m{}_{jk} = -\frac{1}{2}y^s g_{sm} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}. \tag{3}$$

We have the following important Bianchi identity:

$$\frac{\partial L_{jkl}}{\partial y^m} - \frac{\partial L_{jkm}}{\partial y^l} = \frac{1}{2} g_{il} B_m{}^i{}_{kj} - \frac{1}{2} g_{im} B_l{}^i{}_{kj}. \tag{4}$$

See (10.12) in [11] for a proof.

Definition 2.1. Let F be a Finsler metric on an n-dimensional manifold M.

(a) F has isotropic mean Berwald curvature if

$$E_{ij} = \frac{n+1}{2} c F_{y^i y^j}. \tag{5}$$

(b) F has relatively isotropic Landsberg curvature if

$$L_{ijk} + cFC_{ijk} = 0. \tag{6}$$

(c) F has isotropic Berwald curvature if

$$B_j{}^i{}_{kl} = c\{F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^j y^k y^l} y^i\}. \tag{7}$$

Here $c = c(x)$ in (5)–(7) is a scalar function on M.

Since $h_{ij} = FF_{y^i y^j}$ and $h^i_j := g^{ik}h_{jk} = \delta^i_j - F^{-2} g_{js} y^s y^i$, (7) can expressed as

$$B_j{}^i{}_{kl} = cF^{-1}\{h_{jk}h^i_l + h_{jl}h^i_k + h_{kl}h^i_j + 2C_{jkl}y^i\}. \tag{8}$$

Example 2.2. The Funk metric $\Theta = \Theta(x, y)$ on a strongly convex domain in $\mathbb{R}^n$ has isotropic Berwald curvature with $c = \frac{1}{2}$.

Let

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right),$$

It is easy to verify that $\mathcal{D} := D_j{}^i{}_{kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_0. We call $\mathcal{D}$ the *Douglas tensor*. By a direct computation, one can express $D_j{}^i{}_{kl}$ as follows.

$$D_j{}^i{}_{kl} := B_j{}^i{}_{kl} - \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\, y^i\right\}. \tag{9}$$

It is easy to verify that (1) holds if and only if

$$G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k, \tag{10}$$

and (10) holds if and only if $\mathcal{D}=0$. Thus Douglas metrics are also characterized by the curvature equation $\mathcal{D}=0$. By (9), one can easily see that every Berwald metric is a Douglas metric. There are many non-Berwaldian Douglas metrics. For example, a Randers metric $F=\alpha+\beta$ is a Douglas metric if and only if β is closed [1].

In [6], we prove that a Randers metric $F=\alpha+\beta$ with β closed has isotropic mean Berwald curvature if and only if it has relatively isotropic Landsberg curvature [6]. In this paper, we generalize this result.

Proposition 2.3. *Let F be a non-Riemannian Finsler metric on a manifold of dimension $n\geq 3$. The following are equivalent.*

(a) *F is of isotropic Berwald curvature;*

(b) *F is a Douglas metric with isotropic mean Berwald curvature;*

(c) *F is a Douglas metric with relatively isotropic Landsberg curvature.*

From Proposition 2.3, we see that every Finsler metric of isotropic Berwald curvature is a Douglas metric.

Besides Randers metrics, there are many interesting Douglas metrics.

Example 2.4. Let $F=(\alpha^2+\beta^2)/\alpha$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta=b_iy^i$ be a 1-form on a manifold M with $b:=\sqrt{a_{ij}b^ib^j}<1$ at any point $x\in M$. Then F is a Finsler metric. M. Matsumto [10] has proved that when $n=\dim M\geq 3$, F is a Douglas metric if and only if

$$b_{i|j} = c\big\{(1+2b^2)a_{ij} - 3b_ib_j\big\}.$$

where $c=c(x)$ is a scalar function on M. In this case, we can show that

$$G^i = \bar{G}^i + c\big\{Py^i + \alpha^2 b^i\big\},$$

where $P=-2\beta^3/(\alpha^2+\beta^2)$ and $\bar{G}^i=\bar{G}^i(x,y)$ are the spray coefficients of α. One has

$$\frac{\partial G^m}{\partial y^m} = \frac{\partial \bar{G}^m}{\partial y^m} + c\{(n+1)P + 2\beta\}.$$

Thus

$$E_{ij} = \frac{n+1}{2} c P_{y^i y^j}.$$

Clearly, F does not have isotropic mean Berwald curvature.

The Douglas tensor is a projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent,

$$G^i = \bar{G}^i + P y^i,$$

where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$. Thus if a Finsler metric is projectively equivalent to a Berwald metric, then it is a Douglas metric. However, it is still an open problem whether or not every Douglas metric is (locally) projectively equivalent to a Berwald metric.

3. Proof of Theorem 1.1

Lemma 3.1. *Let $F = F(x, y)$ be a Douglas metric on an n-dimensional manifold M. If $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$, then $\mathcal{L} + cF\mathcal{C} = 0$.*

Proof. By (9), $\mathcal{D} = 0$ if and only if

$$B_j{}^i{}_{kl} = \frac{2}{n+1} \left\{ E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\, y^i \right\}. \tag{11}$$

Plugging $E_{jk} = \frac{1}{2}(n+1)cF_{y^i y^j}$ into (11), one obtains

$$\begin{aligned} B_j{}^i{}_{kl} &= c\big\{ F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^j y^k y^l} y^i \big\} \\ &= cF^{-1}\big\{ h_{jk}h^i_l + h_{jl}h^i_k + h_{kl}h^i_j + 2C_{jkl}y^i \big\}. \end{aligned} \tag{12}$$

Contracting with (12) with $-\frac{1}{2}y^s g_{si}$, one immediately obtains

$$L_{jkl} = -\frac{1}{2} y^s g_{si} B_j{}^i{}_{kl} = -cFC_{jkl}. \qquad \Box$$

The converse is almost true.

Lemma 3.2. *Let $F = F(x, y)$ be a non-Riemannian Douglas metric on an n-dimensional manifold M $(n \geq 3)$. If $\mathcal{L} + cF\mathcal{C} = 0$, then $\mathcal{E} = \frac{1}{2}(n+1)\lambda F^{-1} h$ for some scalar function $\lambda = \lambda(x)$. Moreover, $\lambda(x) = c(x)$ at any point x where F_x is not Euclidean.*

Proof. By assumption, (11) holds and

$$L_{jkl} = -cFC_{jkl}. \tag{13}$$

Contracting ${B_j}^i{}_{kl}$ with $h_i^m := \delta_i^m - F^{-2} g_{is} y^s y^m$ and using (3) and (13), one obtains

$$h_i^m {B_j}^i{}_{kl} = {B_j}^m{}_{kl} + 2F^{-2} L_{jkl} y^m = {B_j}^m{}_{kl} - 2cF^{-1} C_{jkl} y^m. \tag{14}$$

Contracting (11) with h_i^m and using (14), one obtains

$${B_j}^m{}_{kl} = \frac{2}{n+1}\big\{E_{jk} h_l^m + E_{jl} h_k^m + E_{kl} h_j^m\big\} + 2cF^{-1} C_{jkl} y^m. \tag{15}$$

Plugging (13) and (15) into (4), one obtains

$$E_{km} h_{jl} + E_{jm} h_{kl} - E_{kl} h_{jm} - E_{jl} h_{km} = 0. \tag{16}$$

Contracting (16) with g^{jm} yields

$$E_{kl} = \frac{1}{2}(n+1)\lambda F^{-1} h_{kl}, \tag{17}$$

where

$$\lambda := \frac{2}{n^2-1} F g^{jm} E_{jm}.$$

Next we are going to show that $\lambda = \lambda(x, y)$ is independent of $y \in T_xM$ at any point $x \in M$. Plugging (17) into (11) and (15) respectively, one obtains

$$\begin{aligned} B^i_{jkl} &= \frac{\lambda}{F}\big\{h_{jk}\delta_l^i + h_{jl}\delta_k^i + h_{kl}\delta_j^i\big\} + \big[\lambda F^{-1} h_{jk}\big]_{y^l} y^i \\ &= \frac{\lambda}{F}\big\{h_{jk} h_l^i + h_{jl} h_k^i + h_{kl} h_j^i\big\} + 2cF^{-1} C_{jkl} y^i. \end{aligned}$$

Comparing the above two identities yields

$$\lambda_{y^l} h_{jk} = 2(c-\lambda) C_{jkl}. \tag{18}$$

Contracting (18) with g^{jk} yields

$$\lambda_{y^l} = \frac{2}{n-1}(c-\lambda)I_l. \tag{19}$$

Plugging (19) into (18), one obtains

$$(c-\lambda)\big\{(n-1)C_{jkl} - I_l h_{jk}\big\} = 0. \tag{20}$$

Contracting the above identity with g^{jk} yields

$$(n-2)(c-\lambda)I_l = 0.$$

Since $n \geq 3$, the above equation becomes

$$(c-\lambda)I_l = 0.$$

Then it follows from (19) that $\lambda_{y^l} = 0$. Thus $\lambda = \lambda(x)$ is independent of $y \in T_xM$.

Now it follows from (18) that

$$(c-\lambda)C_{jkl} = 0.$$

At any point $x \in M$ where $F_x = F|_{T_xM}$ is not Euclidean, $C_{jkl}(x,y) \neq 0$ for some $y \in T_xM \setminus \{0\}$. Then $c(x) = \lambda(x)$. This completes the proof. □

Lemma 3.2 might be true too in dimension two. But we do not find a short proof for this conjecture.

4. Isotropic Berwald curvature

In this section, we are going to prove Proposition 2.3.

Lemma 4.1. *Let F be a Finsler metric on an n-dimensional manifold M. The following are equivalent.*

(a) *F is of isotropic Berwald curvature satisfying (8) for a scalar function $c = c(x)$ on M;*

(b) *$\mathcal{D} = 0$ and $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$ for a scalar function $c = c(x)$ on M.*

Proof. Assume that (8) holds. Then by (2), one obtains

$$E_{jk} = \frac{1}{2} B_j{}^m{}_{km} = \frac{1}{2}(n+1)cF^{-1}h_{jk} = \frac{1}{2}(n+1)cF_{y^i y^j}.$$

Plugging it into (18) yields (11). Thus $D_j{}^i{}_{kl} = 0$. Conversely, suppose that $F = F(x,y)$ is a Douglas metric with $\mathcal{E} = \frac{1}{2}(n+1)cF^{-1}h$. Plugging $E_{jk} = \frac{1}{2}(n+1)cF_{y^j y^k}$ into (11) yields (12). Thus F has isotropic Berwald curvature. This is already proved in Lemma 3.1. □

By Lemmas 3.2 and 4.1, one can easily show the following

Lemma 4.2. *Let F be a Finsler metric on an n-dimensional manifold M. The following hold*

(a) *If F is of isotropic Berwald curvature satisfying* (8) *for a scalar function $c = c(x)$ on M, then $\mathcal{D} = 0$ and $\mathcal{L} + cF\mathcal{C} = 0$;*

(b) *($n \geq 3$) If $\mathcal{D} = 0$ and $\mathcal{L} + cF\mathcal{C} = 0$ for a scalar function $c = c(x)$ on M, then F is of isometric Berwald curvature satisfying* (8) *for a scalar function $\lambda = \lambda(x)$. In this case, $\lambda(x) = c(x)$ at point x where F_x is not Euclidean.*

One immediately obtains the following corollary.

Corollary 4.3 ([2])**.** *Let F be a Douglas metric on a manifold M of dimension $n \geq 3$. Suppose that $\mathcal{L} = 0$, then F is a Berwald metric.*

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