Publ. Math. Debrecen
67/1-2 (2005), 9–26

A *G*-version of Smale's theorem

By IMRE MAJOR (Dunaújváros)

Abstract. We will prove the equivariant version of Smale's transversality theorem (see SMALE [5]): suppose that the compact Lie-group G acts on the compact differentiable manifold M on which an invariant Morse-function f and an invariant vector field X are given so that X is gradient-like with respect to f (i.e. $X(f) < 0$ away from critical orbits and X is the gradient of f (w.r.t. a fixed invariant Riemannian metric) on some invariant open subsets about critical orbits of f. Given a bound $\varepsilon > 0$ we will prove the existence of an invariant vector field Y of class C^1 for which vector field $X + Y$ is also gradient-like such that:

(a) $\|Y\|_1 < \varepsilon$ ($\|.\|_1$ is the C^1 norm).

(b) The intersection of the stable and unstable sets of vector field $X + Y$ taken at a pair of critical orbits of f is transverse when restricted to an orbit type of the action.

1. Introduction, basic concepts

Suppose, that a compact Lie group G acts on the compact orientable smooth manifold M^m and let $f : M \to \mathbf{R}$ be an invariant function (i.e. $f(gx) = f(x)$ $(g \in G)$). If an orbit contains a critical point of f then, by invariance, all points of this orbit are critical points and the orbit itself is called a *critical orbit* of f. Choose an invariant Riemannian metric $\mathbf{g}$ and at a point $p \in M$ let

$$\perp_p := [T_pG(p)]^\perp$$

Mathematics Subject Classification: 57R10.
Key words and phrases: group actions, transversality.

be the perpendicular complement of the tangent space to the orbit through point p and let

$$U_p := \exp_p(\perp_p(\epsilon))$$

where $\perp_p(\epsilon)$ is the ϵ-disk about the origin in $\perp_p$ on which the exponential map of the Levi–Civita connection of metric $\mathbf{g}$ is injective. Then a critical point x of f is also a critical point of $f|_{U_x}$. We say that the G-invariant function f is a *G-Morse function* if the Hessian of $f|_{U_x}$ at x is non-degenerate for each critical point x. This property does not depend on the choice of metric $\mathbf{g}$ (see e.g. WASSERMAN [7]). Non-degeneracy of the Hessian also ensures that each critical orbit has an invariant neighborhood (called *tube* about the orbit) that does not contain any other critical orbits. We can suppose, that GU_x is such an invariant neighborhood.

The induced action of the *isotropy subgroup* G_p at a point $p \in M$:

$$G_p \times T_pM/T_pG(p) \to T_pM/T_pG(p)$$

on the normal space is called the *normal action* (see e.g. BREDON [1]).

$$\mathrm{pr} : M \to M/G, \qquad p \longrightarrow G(p)$$

is the *canonical projection.* For a set $N \subset M$ we use notation $\mathcal{N} := \mathrm{pr}(N)$.

Observe that the relation

$$x \sim y \Leftrightarrow \exists\,(g \in G) \ni G_x = gG_yg^{-1} \quad (x, y \in M)$$

is an equivalence relation on M. The equivalence classes provide a partition

$$M = \bigcup_{\alpha \in \mathcal{A}} M_\alpha \tag{1}$$

of M. The index set $\mathcal{A}$ is the set of conjugacy classes of isotropy subgroups of G. It is partially ordered by relation

$$\alpha \prec \beta \;\Leftrightarrow\; \forall\,(x \in M_\alpha)\;\exists\, y \in M_\beta \;\ni\; G_y \subset G_x$$

(the property on the RHS does not depend on the choice of representative $x \in M_\alpha$).

An index $\alpha \in \mathcal{A}$ (or submanifold M_α) is called an *orbit-type* and we say that a point p is *of type* α if $p \in M_\alpha$ (in notation, $[p] = \alpha$). Note, that

$M_\alpha \subset M$ is a G-invariant subset. Theorem 4 of Chapter 3 in [1] implies that M_α and $\mathcal{M}_\alpha$ are smooth manifolds and the partition in formula (1) is locally finite, thus $\mathcal{A}$ is finite when M is compact (cf. 4.3 Theorem, p. 187 and 10.4 Theorem, p. 220 in [1]).

As the boundary of an orbit type M_α is the union of some lower dimensional orbit types that preceed α with respect to $\prec$, a *level* can be associated to each orbit type: the closed ones are at level 0 and inductively, for $i > 0$ an orbit type is at level i if its boundary contains orbit types of level at most $i-1$ and contains at least one orbit type at level $i-1$. (The level is also called the *depth* in the literature.)

Let $m_\alpha := \dim(\mathcal{M}_\alpha)$, $m_\alpha^\perp := \operatorname{codim}(M_\alpha)$ and $o_\alpha := \dim(G/G_p)$ ($p \in M_\alpha$). Notice that then $\dim(M) := m = m_\alpha + m_\alpha^\perp + o_\alpha$. An *$\alpha$-slice* is an m_α-dimensional disk $D \subset M_\alpha$ which intersects an orbit at most once and along which the isotropy subgroup is constant, moreover the union of orbits GD is open in M_α.

For an arbitrary subset $Q \subset M$

$$Q_\alpha := Q \cap M_\alpha$$

is the *α-part of* Q.

The set of differentials of left translations $\{\mathrm{d}L_g \mid g \in G\}$ act on the tangent bundle TM. A vector field X is invariant under this action (called an *invariant vector field*) **iff** its flow is an eqivariant flow (i.e. for trajectory λ_p of vector field X through point p

$$L_g(\lambda_p) = \lambda_{gp}$$

holds.) This implies that the isotropy subgroup is constant along trajectories, in particular, an invariant vector field is tangent to the orbit types.

Definition 1. An invariant vector field X is *gradient-like* for the G-Morse function f if:

(i) Each critical orbit has an invariant neighborhood U such that

$$X|_U = -\operatorname{grad}_{\mathbf{g}}(f)|_U.$$

(ii) $X(f) < 0$ away from critical orbits.

Definition 2. At a critical orbit O:

$$W_O^- = \{p \in M \mid \lim_{t \to -\infty} \lambda_p(t) \in O\}$$

is called the *unstable set* and

$$W_O^+ = \{p \in M \mid \lim_{t \to +\infty} \lambda_p(t) \in O\}$$

is the *stable set* of the flow (of invariant vector field X). (Notations $W_O^s = W_O^+$, $W_O^u = W_O^-$ are also used in the literature.)

Definition 3. A *Morse chart* about critical orbit O is given by:

(i) A splitting of the normal bundle $\perp_O$ of O into two invariant orthogonal subbundles $\perp_O = \perp_O^- \oplus \perp_O^+$.

(ii) An equivariant diffeomorphism $\eta_O : \perp_O(\epsilon) \to U_O$ from the ϵ-disc bundle of the normal bundle of O (with constant ϵ) onto an invariant open neighborhood U_O of O (η_O is the identity on the zero section O of $\perp_O$) such that

$$f \circ \eta_O = -\|P^-\|^2 + \|P^+\|^2 + f(O)$$

where $(P^-, P^+) : \perp_O \to (\perp_O^-, \perp_O^+)$ are the projections that belong to the decomposition in (i). The open set U_O is called a *Morse-tube.*

Observe that $\perp_O^- \longrightarrow O$ and $\perp_O^+ \longrightarrow O$ are G-vector bundles induced from the orthogonal representations on the Eucledian spaces $\perp_x^- := \perp_x \cap \perp_O^-$, $\perp_x^+ := \perp_x \cap \perp_O^+$. The restriction $\mathbf{g}|_{\perp_O}$ is a scalar product on vector bundle $\perp_O$, thus it defines a Reimannian metric $\langle\, ,\rangle$ on $\perp_O$ in the canonical way. The push-forward $\eta_{O*}\langle\, ,\,\rangle$ of this Riemannian metric along map η_O can be patched together with original metric $\mathbf{g}$ by using an invariant cutoff function. Thus we can presume that for the restrictions we have:

$$\mathbf{g}|_{U_O} = \eta_{O*}\langle\, ,\,\rangle|_{U_O}.$$

Lemma (Equivariant Morse Lemma)**.** *Let $f : M \to \mathbf{R}$ be a G-Morse function on a Riemannian G-manifold. Then there is a Morse chart about each critical orbit (see* Wasserman [7]*).*

Note that by the above lemma the stable and unstable sets are, in fact, invariant submanifolds of M. Let $O_1, \dots, O_K$ be the set of critical orbits of f, and abbreviate $W_j^+ := W_{O_j}^+$, etc.

Definition 4. The gradient-like vector field X (or its flow $\Lambda : M \times \mathbf{R} \to M$) is *$G$-Morse–Smale* if it is of class C^1 moreover $W_j^+ \cap M_\alpha$ and $W_k^- \cap M_\alpha$ intersect transversely as submanifolds of M_α for each choice of $\alpha \in \mathcal{A}$, $1 \leq j, k \leq K$. (We refer to this property as *relative transversality* of stable and unstable submanifolds, or α-transversality, when the orbit type α is fixed.)

As we wish to perturb a given gradient-like vector field by an invariant vector field (the flow of which thus keeps orbit types), the above definition seems to be the only plausible generalization for the G-version of Morse–Smale property (see SMALE [5]).

Theorem. *For any given $\varepsilon > 0$ an invariant gradient-like vector field X can be approximated by a G-Morse–Smale vector field X', which is also gradient-like for f such that $\|X - X'\|_1 < \varepsilon$.*

2. Proof of the Theorem

In order to ensure that the perturbed vector field is C^1-close to the original one, we need a family of special coordinate charts. Invariant charts would serve our purpose the best, however, we might not be able to arrange such charts with invariant domains (by compactness of group G, the domain of such a chart cannot be an m-ball).

For each orbit type fix a representative isotropy subgroup $G_\alpha = G_{p_\alpha}$ (where $p_\alpha \in M_\alpha$), a coordinate chart $\big(E_\alpha, \big(\tilde{x}_\alpha^{m_\alpha + m_\alpha^\perp + 1}, \ldots, \tilde{x}_\alpha^m\big)\big)$ about the unit element G_α of quotient group G/G_α, an open neighborhood $E'_\alpha \subset G/G_\alpha$ of G_α such that $\overline{E'_\alpha} \subset E_\alpha$ and elements $g_\alpha^1, \ldots, g_\alpha^{n_\alpha} \in G$ such that the translates

$$g_\alpha^1 E'_\alpha, \ldots, g_\alpha^{n_\alpha} E'_\alpha$$

cover G/G_α. Note that each orbit of type α contains a point with isotropy subgroup G_α. Let $\mathbf{N}_\alpha \xrightarrow{\Pi_\alpha} M_\alpha$ be the normal bundle of orbit type M_α.

It is a well known fact that G is a principal bundle over quotient group G/G_α by the canonical projection $G \longrightarrow G/G_\alpha$ with structure group G_α, so we can fix a section $\sigma_\alpha : E_\alpha \to G$, $\sigma_\alpha(G_\alpha) = e$ of this bundle over contractible neighborhood E_α.

Definition 5. A coordinate chart $(U, (x^1, \dots, x^m))$ is *adapted to orbit type* α if there exists a relatively compact α-slice $U^*_\alpha \subset M_\alpha$ with isotropy subgroup G_α, a coordinate chart $(\mathcal{U}^*_\alpha, x^1_\alpha, \dots, x^{m_\alpha}_\alpha))$ (with respect to the smoothness structure on quotient manifold $\mathcal{M}_\alpha$), an index $1 \le i \le n_\alpha$ and $\epsilon > 0$ such that:

(i) $U = \exp(\mathbf{N}_\alpha(\epsilon)|_{U^i_\alpha})$ with $U^i_\alpha = g^i_\alpha E_\alpha U^*_\alpha$, i.e. U is the exp-image of the restriction of the ϵ-disc bundle of $\mathbf{N}_\alpha$ to subset U^i_α.

(ii) $x^i(q) = x^i_\alpha \circ pr \circ \Pi_\alpha \circ \exp^{-1}(q)$ $(i = 1, \dots, m_\alpha,\ q \in U)$.

(iii) By clause (i), $\forall\ r \in U\ \exists!$ pair $(gG_\alpha, q) \in E_\alpha \times U^*_\alpha$ such that:

$$\Pi_\alpha \circ \exp^{-1}(r) = g^i_\alpha g q$$

holds. Define

$$x^i(r) = \tilde{x}^i_\alpha(gG_\alpha), \quad (i = m_\alpha + m^\perp_\alpha + 1, \dots, m). \tag{2}$$

(iv) Fix an orthonormal frame bundle $(\mathbf{v}_1, \dots, \mathbf{v}_{m^\perp_\alpha})$ of trivial bundle $\mathbf{N}_\alpha|_{U^*_\alpha}$ and extend it to U^i_α by

$$\mathbf{v}_j(g^i_\alpha g q) = \mathrm{d}L_{g^i_\alpha \sigma_\alpha(gG_\alpha)}(\mathbf{v}_j(q)) \quad (j = 1, \dots, m^\perp_\alpha,\ q \in U^*_\alpha)$$

Define:

$$\exp^{-1}(r) = \sum_{j=1}^{m^\perp_\alpha} x^{m_\alpha + j}(r)\mathbf{v}_j \quad (r \in U). \tag{3}$$

Note that the first m_α coordinates do not depend on the choice of the rest of the coordinates. For the definition of the C^1-norm of a vector field we need to fix a compact subset within each chart, which we can get as follows:

Fix $K^*_\alpha \subset U^*_\alpha$ compact set, $0 < \epsilon' < \epsilon$ real number, let $K^i_\alpha = g^i_\alpha \overline{E'_\alpha} K^*_\alpha$ and define

$$K = \exp\left(\overline{\mathbf{N}}_\alpha(\epsilon')|_{K^i_\alpha}\right) \subset U.$$

We will call $\mathrm{int}(K)$ the *strong interior* of chart $(U, (x^1, \dots, x^m))$. In the sequel we will presume that each adapted chart has a fixed compact set in its interior (even if we don't state this explicitely).

To cover M choose adapted charts

$$\left(U_j, (x^1_j, \dots, x^m_j)\right) \quad (j = 1, \dots, j_0)$$

so that their strong interior cover the level-0 orbit types. Then the complement of the union of strong interiors of these charts in any of the level-1 strata is compact, so it can be covered by the strong interiors of charts

$$\big(U_j, (x_j^1, \dots, x_j^m)\big) \quad (j = j_0 + 1, \dots, j_1)$$

so that each chart is adapted to the stratum at issue, a.s.o. Finally we get a finite family of adapted coordinate charts so that their strong interiors $\mathrm{int}(K_1), \dots, \mathrm{int}(K_{j_L})$ cover M (here L is the highest level).

Given a vector field Y with local coordinates:

$$Y|_{U_j} = \sum_{i=1}^{m} y_j^i \frac{\partial}{\partial x_j^i} \quad (j = 1, \dots, j_L)$$

its C^1-norm is defined as:

$$\|Y\|_1 := \sum_{i,j,k} \left(\sup_{K_j} |y_j^i| + \sup_{K_j} \left| \frac{\partial}{\partial x_j^k}\, y_j^i \right| \right) \tag{4}$$

We will perturb vector field X into a G-Morse–Smale vector field in succession of increasing order of the level of strata. Although domains $U_1, \dots, U_{j_L}$ cover M, each stratum has parts covered by some of the U_j's that are adapted to a stratum at a lower level. To attain α-transversality, we should adjust X in charts that are adapted to orbit type α. It seems so that in order to end up with a C^1 vector field, we need to modify vector field X in finite steps. To cover a non-compact stratum, however, we need to use infinitely many adapted charts. To overcome this discrepancy, for each orbit type α we will choose a countable family of adapted coordinate charts with set of domains $\underline{Q}^{(\alpha)}$ which is a finite union

$$\underline{Q}^{(\alpha)} = \underline{Q}^{(\alpha);1} \cup \dots \cup \underline{Q}^{(\alpha);k_\alpha}$$

of sub-families such that:

Property I. For a fixed $i = 1, \dots, k_\alpha$ the closure of domains in

$$\underline{Q}^{(\alpha);i} = \left\{ Q_1^{(\alpha);i}, Q_2^{(\alpha);i}, \dots, Q_j^{(\alpha);i}, \dots \right\}$$

are pairwise disjoint.

Property II. The strong interiors of domains in $\underline{Q}^{(\alpha)}$ cover M_α.

Property III.

$$Q_j^{(\alpha);i} \cap Q_l^{(\beta);k} \neq \emptyset \text{ for some } i,j,k,l \\ \Rightarrow \alpha \text{ and } \beta \text{ can be compared w.r.t. } \prec$$

Remark. Given a family of open subsets about strata, it is standard to impose a condition similar to Property III (see e.g. MATHER [3]). It is also shown there, that arbitrary system of tubes about strata can be trimmed down so that Property III holds.

Definition 6. A cover of each stratum by adapted charts with the above three properties is called a *stratified cover of M*.

Proposition 1. *A compact G-manifold has a stratified cover.*

PROOF. We will use induction on the level of strata. We have already constructed a finite cover of each level-0 strata. For an orbit type M_α at the $(k+1)^{\text{th}}$ level the complement $M_\alpha \setminus \bigcup_{j=1}^{j_k} U_j$ is compact (here the U_j'-s are the domains constructed above) so it can be covered by the strong interiors of finitely many charts adapted to M_α, thus it is enough to choose a stratified cover for each set

$$M_\alpha \cap U_j \quad 1 \leq j \leq j_k$$

separately. For a fixed j domain U_j is adapted to M_β for some $\beta \prec \alpha$.

Let $U_\beta = U_j \cap M_\beta$. Choose $\epsilon' < \epsilon$ and let $S_{\epsilon'} \subset \mathbf{N}_\beta$ denote the ϵ'-sphere bundle of the normal bundle of stratum M_β. The level of orbit type $(\exp|_{U_\beta})^{-1}(M_\alpha) \cap S_{\epsilon'}$ of the normal action is at most k, thus by induction we can choose a stratified cover $\underline{Q}'^{(\alpha)}$ of the subset

$$(S_{\epsilon'})_\alpha := (\exp|_{U_\beta})^{-1}(M_\alpha) \cap S_{\epsilon'}.$$

This means that family $\underline{Q}'^{(\alpha)}$ is a finite union

$$\underline{Q}'^{(\alpha)} = \bigcup_{i=1}^{k} \underline{Q}'^{(\alpha);i}$$

where each subset $\underline{Q}'^{(\alpha);i}$ consists of relatively compact open sets (in the topology of $S_{\epsilon'}$) with pairwise disjoint closure. Consider the subsets

$$\mathbf{R}_1 := \left\{ \left(\frac{1}{2n+1}, \frac{2}{4n-1} \right) \mid n \in \mathbf{Z}^+ \right\}$$

$$\mathbf{R}_2 := \left\{ \left(\frac{1}{2n}, \frac{4}{8n-5} \right) \mid n \in \mathbf{Z}^+ \right\}$$

Let

$$\underline{Q}_1^{(\alpha);i} := \left\{ (a,b) \times Q \mid (a,b) \in \mathbf{R}_1,\ Q \in \underline{Q}'^{(\alpha);i} \right\}$$

$$\underline{Q}_2^{(\alpha);i} := \left\{ (a,b) \times Q \mid (a,b) \in \mathbf{R}_2,\ Q \in \underline{Q}'^{(\alpha);i} \right\}.$$

Then the exp-images of sets $\underline{Q}_1^{(\alpha);i}$, $\underline{Q}_2^{(\alpha);i}$ $(i = 1, \ldots, k)$ provide a stratified cover for $M_\alpha \cap U_j$. (Pairwise disjointness and relative compactness follow trivially; intervals (a,b) can serve as new coordinates.) □

At each step X will denote the invariant gradient-like vector field that has already been adjusted along certain strata (so we will not re-denote X in every single step).

For the proof of our theorem suppose inductively that relative transversality of ascending and descending submanifolds has been attained below critical level $f(O) = c$. Fix critical orbit O and a Morse-chart $\eta_O : \perp_O(\epsilon^*) \to U_O$ about O. We will perturb X by an invariant vector field with support contained in the Morse-tube U_O (in the case when we have more than one critical orbits at level $(f = c)$, we choose disjoint Morse-tubes about them). This way we will not influence relative transverse intersections that have already been established in previous steps.

Notations. At a point $x \in O$ let S_x, S_x^-, S_x^+ denote the spheres with radius $\epsilon < \epsilon^*$ (about the origin) in subspaces $\perp_x$, $\perp_x^-$, $\perp_x^+$ respectively. Let $B = \perp_x (\epsilon^*)$ denote the ϵ^*-ball about the origin of $\perp_x$. As for the rest of this paper we will work in ball B, we can (ab)use notation by denoting the trajectory of vector field

$$X|_B := \mathrm{d}\eta_O^{-1}(X|_{U_O}) = \mathrm{grad}(f \circ \eta_O)$$

through point $p \in B$ by λ_p.

Fix $x \in O$ and drop it from the subscripts. Let $\hat{G} := G_x$ be the isotropy subgroup at x and choose an invariant open neighborhood

$$\nu := \nu_x \subset \eta_O^{-1}\big(f^{-1}(c-\epsilon) \cap U_O\big) \cap \perp_x$$

of outbound shpere $S^- := S_x^-$. Then ν is partitioned by the orbit types of action $\hat{G} \times \nu \to \nu$ as

$$\nu = \bigcup_{[x] \preceq \alpha} \nu_\alpha \,.$$

For a subset $Z \subset \nu$ and an orbit type $[x] \preceq \alpha$ let $Z_\alpha := Z \cap \nu_\alpha$ be the α-part of set Z in orbit type ν_α. Let $\{O_1, \ldots, O_k\}$ denote the set of critical orbits that are connected with O by a trajectory of X and reside below critical level c. Set

$$\Sigma_j := \nu \cap \eta_O^{-1}(W_{O_j}^+ \cap U_O), \quad \Sigma := \bigcup_{j=1}^{k} \Sigma_j \,.$$

Observe that the intersection $W_O^- \cap W_j^+$ is relative transverse (with respect to the partition $M = \bigcup_{\alpha \in \mathcal{A}} M_\alpha$) if and only if for all orbit types α preceeded by $[x]$ the intersection $S_\alpha^- \cap (\Sigma_j)_\alpha$ is transverse in submanifold ν_α. Recall notation $U_x := \eta_O(B)$ and choose a point $p \in U_x \cap W_O^- \cap W_j^+ \cap M_\alpha$. We have

$$T_p(W_O^-)_\alpha + T_p(W_j^+)_\alpha = T_p(W_O^- \cap U_x)_\alpha + T_p(W_j^+ \cap U_x)_\alpha + T_p G(p) \quad (5)$$

thus $T_p(W_O^-)_\alpha + T_p(W_j^+)_\alpha = T_p M_\alpha$ if and only if

$$T_p(W_O^- \cap U_x)_\alpha + T_p(W_j^+ \cap U_x)_\alpha = T_p(U_x)_\alpha$$

This means that it is enough to ensure relative transversality of intersection $S^- \cap \Sigma_j$ with respect to ν.

Strategy of proof: first we will construct the perturbing vector field on ball B, then we extend it along the G-action to a vector field on $\perp_O(\epsilon^*)$, finally we push it forward along Morse coordinate system $\eta_O : \perp_O(\epsilon^*) \to U_O$. We will proceed by induction on the level of orbit types of action $\hat{G} \times S^- \to S^-$. In the sequel “level” will always be meant in this sense. We will use the same method to perturb vector field X for both the base step and the induction step, so by induction suppose that we need to define the perturbing vector field for orbit type α and we are done with all orbit types at lower levels. We can presume $G_\alpha \subset \hat{G}$. Then the fixed point set of orthogonal action $G_\alpha \times \perp_x \to \perp_x$ is a linear subspace

$$V = V^- \oplus V^+$$

where $V^- \subset \perp_x^-$, $V^+ \subset \perp_x^+$ and $\dim(V^-) \neq \emptyset$.

Note, that the normalizer N_α of subgroup $G_\alpha \subset \hat{G}$ acts on linear subspace V as well as on

$$V_\alpha^- := [V^- \text{ less the points that are fixed} \\ \text{by a group strictly larger than } G_\alpha].$$

Let Q^* be an α-slice for action $N_\alpha \times V_\alpha^- \to V_\alpha^-$ such that it is a union of (open) rays in V_α^- (thus it is a cone $\mathbf{C}(Q^* \cap S^-)$ over α-slice $Q^* \cap S^-$ for action $N_\alpha \times S^- \to S^-$). Let $D^+ \subset V^+$ be a small disc about the origin. Then $Q^* \oplus D^+$ is an α-slice of action $N_\alpha \times V \to V$ and also for action $\hat{G} \times \perp_x \to \perp_x$, thus

$$(Q^* \oplus D^+) \cap \nu$$

is an α-slice for action $\hat{G} \times \nu \to \nu$. This leads to the following conclusion, which is crucial for the proof:

Observation 1. For an α-slice Q^{**} of action $\hat{G} \times S^- \to S^-$ there exists an α-slice of action

$$\hat{G} \times \nu \to \nu$$

which is a product of Q^{**} and the fiber. We will call such a slice a *product slice associated* to Q^{**} and its image under G (i.e. the union of G-paths through its points) is the *G-extension* of the associated slice.

Level-0 orbit types are compact, so are their projections to orbit space $\mathcal{M}$. In the base step of induction we define the perturbing vector fields along these projections and then we will lift them into invariant vector fields defined on M. This is done in complete analogy with the non-equivariant case (see Smale [5]).

Remark. We could have chosen to work out the whole reasoning in the orbit space, lifting the resulting vector fields (isotopies) to M afterwards.

In spite of the simplicity of some parts of this way of reasoning, other technical difficulties would have arisen (e.g. the quotient space $\mathcal{V}$ is not a linear space, etc).

Construction of the stratified cover. First we choose a family of α-slices

$$Q_j^{(\alpha);i*} \subset S_\alpha^- \cap V^- \quad (i = 1, \ldots, k_\alpha,\ j = 1, \ldots, n, \ldots)$$

for the action

$$\hat{G} \times S^- \to S^-$$

together with associated product slices

$$Q_j^{(\alpha);i*} \oplus D^+$$

so that

$$\overline{\hat{G}Q_j^{(\alpha);i*}} \cap \overline{\hat{G}Q_{j'}^{(\alpha);i*}} = \emptyset \quad (j \neq j').$$

Intersections

$$\nu \cap \mathbf{C}\big(Q_j^{(\alpha);i*} \oplus D^+\big)$$

will be α-slices for action

$$\hat{G} \times \nu \to \nu$$

so if $\mathbf{n}_\alpha$ is the normal bundle of $\nu_\alpha \subset \nu$ then with the aid of a small disc bundle of

$$\mathbf{n}_\alpha\big|_{\hat{G}\mathbf{C}(Q_j^{(\alpha);i*} \oplus D^+) \cap \nu}$$

we can define a stratified cover

$$\underline{Q}^{(\alpha)} = \underline{Q}^{(\alpha);1} \cup \cdots \cup \underline{Q}^{(\alpha);k_\alpha}$$

of the α-part ν_α (the cover is taken in ν). Choose a smaller disk $\overline{D'^+} \subset D^+$ and fix compact subsets

$$K_j^{(\alpha);i*} \subset Q_j^{(\alpha);i*}, \quad K_j^{(\alpha);i} \subset Q_j^{(\alpha);i}$$

With $\epsilon' < \epsilon$ define

$$\tilde{Q}_j^{(\alpha);i} := \Big\{ q \in \lambda_p \mid p \in Q_j^{(\alpha);i} \text{ and } c - \epsilon < f \circ \eta_O(q) < c - \epsilon' \Big\} \tag{6}$$

$$\tilde{K}_j^{(\alpha);i} := \left\{ q \in \lambda_p \mid \ p \in K_j^{(\alpha);i} \text{ and } c - \epsilon < f \circ \eta_O(q) < c - \epsilon' \right\} \qquad (7)$$

The α^{th} step will consist of k_α substeps so that in the i^{th} substep the support of the perturbing vector field is contained in the extension

$$G\left(\bigcup_{j=1}^{\infty} \tilde{Q}_j^{(\alpha);i}\right) \qquad (i = 1, \ldots, k_\alpha).$$

The following observations can be made:

Observation 2. The perturbed vector field remains the same on lower level strata, so does the transversality of intersections $S_\beta^- \cap \Sigma_\beta$ $(\beta \prec \alpha)$ (that have already been established by induction).

Observation 3. By the fact that transverse intersections are stable under small perturbations of class C^1, in each sub-step we can choose the perturbing vector field to be so small that it will not destroy transverse intersections that had already been established in previous sub-steps. Thus it is enough to describe the i^{th} sub-step, or, as the domains

$$\left\{ \tilde{Q}_j^{(\alpha);i} \mid j = 1, \ldots, n, \ldots \right\}$$

are pairwise disjoint, it is enough to describe how to modify X within one such domain.

Observation 4. The bound we imposed in the Theorem divided by the total number of sub-steps (i.e.

$$\varepsilon' := \frac{\varepsilon}{\sum_{\alpha \in \mathcal{A}} k_\alpha} \Big)$$

provides a bound we should use in formula (4) in each sub-step.

Observation 5. For each triple (α, i, j) there is a bound $\varepsilon_j^{(\alpha);i}$ so that if in the i^{th} sub-step we choose the perturbing vector field with C^1-norm measured on domain $\tilde{Q}_j^{(\alpha);i}$ smaller than $\varepsilon_j^{(\alpha);i}$, then its C^1-norm defined by formula (4) is smaller than ε'^* (this is a well-known fact that follows from relative compactness of the domains, see e.g. Sternberg [6]).

Observation 6. The major difficulty is to ensure that after the final step we end up with a C^1 vector field. This will give us additional conditions on the size of perturbation we can make in each sub-step. These conditions are described in the Lemma below.

By the above observations it is enough to show that for arbitrary indices (α, i, l) (that we fix and drop, using notation $\varepsilon'' := \varepsilon_l^{(\alpha);i}$) the following holds:

Proposition 2. *Given a domain $\tilde{Q} = \tilde{Q}_l^{(\alpha);i}$ with compact subset $\tilde{K} = \tilde{K}_l^{(\alpha);i}$ and $\varepsilon'' > 0$, there exists an invariant vector field Y with support in $G\tilde{Q}$ such that:*

(i) *Vector field $X + \mathrm{d}\eta_O(Y)$ is gradient like for G-Morse function f.*

(ii) $\|Y\|_{1;\tilde{Q}} < \varepsilon''$ *(i.e. the C^1-norm on $G\tilde{Q}$ is smaller than ε''.)*

(iii) *The intersection*

$$\left[S_\alpha^- \cap \hat{G}K_l^{(\alpha);i}\right] \cap \left[\Sigma_\alpha \cap \hat{G}K_l^{(\alpha);i}\right]$$

(notations stand for objects of the flow of vector field $X|_B + Y$) is transverse in orbit type ν_α, thus by formula (5) intersection

$$\left[W_O^- \cap G\tilde{K} \cap M_\alpha\right] \cap \left[W_{O'}^+ \cap G\tilde{K} \cap M_\alpha\right] \tag{8}$$

is transverse in orbit type M_α for any other critical orbit O'.

Proof. Using notations as above, $\dim(V^+) = 0$ implies $\nu_\alpha = S_\alpha^-$ thus transversality of intersections in formula (8) follow trivially.

Otherwise note that $V \cap \Sigma_j$ is the fixed point set of subgroup $G_\alpha \subset \hat{G}$ for the action $\hat{G} \times \Sigma_j \to \Sigma_j$, thus it is a submanifold of Σ_j. Let Q_p^* be an α-slice at point $p \in \Sigma_j \cap V_\alpha^- (= \Sigma_j \cap S_\alpha^-)$ for action $N_\alpha \times (V \cap \Sigma_j) \to (V \cap \Sigma_j)$ and let $V_p^* := T_p Q_p^*$ (it is an affine subspace of V). Intersection $(\Sigma_j)_\alpha \cap S_\alpha^-$ is α-transverse at p iff $\dim(V_p^{*+}) = \dim(V^+)$, or in other words if the origin is a regular value of the projection to the second factor

$$P^+|_{V \cap \Sigma_j} : V \cap \Sigma_j \to V^+$$

(see also formula (5)).

By Sard's theorem the set of critical values of the above projection is of measure-0 for $j = 1, \dots, k$ thus the same holds for their union. This means that we can choose an arbitrarily small vector $\mathbf{v} \in D^+$ so that the constant section $\mathbf{C}(K_l^{(\alpha);i*} \oplus \mathbf{v}) \cap \nu_\alpha$ intersects $(\Sigma_j)_\alpha$ α-transversely (i.e., relative to ν_α) for $j = 1, \ldots, k$.

Choose cutoff function

$$\phi : Q_l^{(\alpha);i*} \to [0,1], \quad \phi\big(K_l^{(\alpha);i*}\big) = 1, \quad \mathrm{supp}(\phi) \subset Q_l^{(\alpha);i*}$$

and let H_D be an isotopy of disk D^+ which moves the origin into $\mathbf{v}$ and fixes a neighborhood of the boundary. Define isotopy

$$H : \left(Q_l^{(\alpha);i*} \oplus D^+\right) \times [0,1] \to Q_l^{(\alpha);i*} \oplus D^+, \; H((p,\mathbf{w}),t) = (p, H_D(\mathbf{w}, \phi(p)t).$$

Using rays this defines an isotopy of set $\mathbf{C}(Q_l^{(\alpha);i*} \oplus D^+) \cap \nu$. An application of the argument in Milnor ([4], pp. 42–43) to the restriction $X_B|_F$ where

$$F := \left\{ q \in \lambda_p \mid p \in \mathbf{C}\big(Q_l^{(\alpha);i*} \oplus D^+\big) \cap \nu \text{ and } c - \epsilon < f \circ \eta_O(q) < c - \epsilon' \right\}$$

produces a vector field Y^* which is tangent to F so that the difference between moving along the flow-lines of vector fields $X_B|_F$ and $X_B|_F + Y^*$ shows up in the application of map H_1 on level set $(f = c - \epsilon)$. Let $\hat{Y}$ be the extension of Y^* along the action onto set $\hat{G}F$. This set is an open subset of the α-part B_α of disk $B = \perp_x(\epsilon^*)$.

For a $\delta > 0$ choose cutoff function

$$\psi : [0,\delta] \to [0,1], \quad \psi(0) = 1, \; \psi(\delta) = (0), \; \frac{\mathrm{d}\psi}{\mathrm{d}t} \leq 0.$$

Let $\mathbf{N}_\alpha$ be the normal bundle of the α-part $B_\alpha \subset B$ (taken in ball B). Consider the Levi–Civita connection of the Euclidean metric on vector bundle $\mathbf{N}_\alpha$ and let Y' be the horizontal lift of $\hat{Y}$ (i.e. a vector field on $\mathbf{N}_\alpha$ with vertical components $= 0$ at each point so that $\mathrm{d}\pi(Y'|_p) = \hat{Y}|_{\pi(p)}$). Let

$$Y = \exp_*(\psi(\|v\|)Y'|_v) \quad (v \in \mathbf{N}_\alpha)$$

and extend Y via the action onto tube U_O. Let Y be the 0 vector field outside of set $G\exp(\mathbf{N}_\alpha(\delta)|_{\hat{G}F})$. Then clauses (i) and (iii) hold for vector field $X_B + Y$ (its restriction to a smaller neighborhood of critical orbit O is gradient-like for f).

As in Definition 5, choose a coordinate system $(x^2, \dots, x^{m_\alpha})$ on α-slice $Q_l^{(\alpha);i*} \oplus D^+$. Use G-Morse function $f \circ \eta_O$ as the first coordinate x^1. As in clause (iv) of Definition 5, fixing an orthonormal frame bundle on $\mathbf{N}_\alpha|_F$ introduces further coordinates $x^{m_\alpha+1}, \dots, x^{m_\alpha+m_\alpha^\perp}$. Finally, supplementing these with the coordinates in formula (2) provides a finite family of coordinate charts on invariant open set GF. Note, that only the first $m_\alpha + m_\alpha^\perp$ coordinates of vector field Y are non-zero and the coordinates themselves are bounded by $\|\mathbf{v}\|$. By relative compactness of set $Q_l^{(\alpha);i*}$ we can choose an upper bound N for the absolute value of the derivatives of cutoff functions ϕ and ψ. Then

$$2\left(N\|\mathbf{v}\| + \frac{2\|\mathbf{v}\|}{\epsilon - \epsilon'}\right)$$

will serve as an upper bound for the absolute values of the first derivatives of coordinates of vector field Y. This shows that by choosing $\|\mathbf{v}\|$ small enough, one can arrange that clause (ii) of our proposition holds as well. □

Differentiability of the final vector field can be ensured by choosing $\|\mathbf{v}\|$ according to how close the domain $\tilde{Q}_j^{(\alpha);i}$ is from the boundary of B_α. Let

$$\mathrm{Fr}(B_\alpha) := \overline{B}_\alpha \setminus B_\alpha$$

be the *frontier* of orbit type B_α and let d stand for the Euclidean distance on ball B. Then, by relative compactness of set $Q_j^{(\alpha);i*}$ in S_α, the distance

$$d_{ij} := d\big(\hat{G}\tilde{Q}_j^{(\alpha);i},\ \mathrm{Fr}(B_\alpha)\big) > 0.$$

In the i^{th} substep the components of $\mathrm{supp}(Y)$ are contained in subset

$$\bigcup_{j=1}^{\infty} G\tilde{Q}_j^{(\alpha);i}.$$

By invariance it is enough to prove differentiability of restriction $Y|_B$. By construction, Y is a smooth vector field in a tube about α-part B_α (i.e. on a set

$$\exp(\mathbf{N}_\alpha(\zeta))$$

where $\mathbf{N}_\alpha(\zeta)$ is an appropriate disk bundle of the normal bundle of $B_\alpha \subset B$) and Y is the zero vector field outside of this tube. This shows that we can have problems with differentiability of vector field Y only at the points of the frontier $\mathrm{Fr}(B_\alpha)$. We will prove that imposing an additional condition ensures that vector field Y is of class C^1 at points of $\mathrm{Fr}(B_\alpha)$. We will formulate this condition by utilizing the fact that B is a Euclidean disk, thus its tangent bundle is trivial by natural identification

$$TB = B \times \mathbf{R}^n$$

($n = \dim(B)$.) This way we can look at vector field $Y|_B$ as a map

$$Y_B : B \to \mathbf{R}^n.$$

Lemma. *Vector field Y is of class C^1 whenever*

$$\|Y(q)\| < d_{ij}^3 \quad \big(q \in \tilde{Q}_j^{(\alpha);i},\ i = 1, \ldots, k_\alpha,\ j \in \mathbf{N}\big)$$

Proof. A point $q \in \mathrm{Fr}(B_\alpha)$ belongs to $q \in B_\beta$ for some orbit type $\beta \prec \alpha$. By definition Y is differentiable at point q if

$$\lim_{p \to q} \frac{\|Y(p) - Y(q)\|}{\|p - q\|} = 0. \tag{9}$$

Value $Y(q) = \mathbf{0}$ for points of the frontier, thus $Y(p) \neq \mathbf{0}$ implies that $p \in \hat{G}\tilde{Q}_j^{(\alpha);i}$ for some indices $i \in \{1, \ldots, k_\alpha\}$, $j \in \mathbf{N}$. But then

$$\|Y(p)\| < d_{ij}^3 \leq \|p - q\|^3$$

thus the limit in formula (9) is indeed 0, together with the limit of the first partial derivatives of Y. □

Remark. The proof of the lemma has been built heavily on the Eucledian structure on disk B. In a general setting a somewhat more complicated proof would work: one considers the isotopy induced by time-dependent vector field

$$Y_t := \xi(t)Y \quad (t \in [0, 1])$$

where $\xi : [0, 1] \to [0, 1]$, $\xi[0, \epsilon) = 0$ is some cutoff function. One puts the analoguos condition on the displacement of this isotopy (measured in the metric distance on B). Then a similar argument proves that the isotopy is of class C^1, consequently vector field Y is also of class C^1.

This paper is an improved version of Chapter 3 of my Ph.D. thesis [2]. I wish to express my gratitude to my advisor, Professor Dan Burghelea for his help.

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IMRE MAJOR
POLYTECHNIC OF DUNAÚJVÁROS
H-2401 DUNAÚJVÁROS, P.O. BOX 152
HUNGARY

E-mail: major.imre@freemail.hu

(Received March 17, 2003; revised October 28, 2004)