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Hausdorff quasi-uniformities inducing the same hypertopologies

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Abstract. The question is investigated when two quasi-uniformities on a set X give rise to Hausdorff quasi-uniformities inducing the same topologies on the set $\mathcal{P}_0(X)$ of nonempty subsets of X. Some conditions are also given under which such hypertopologies are induced by a unique Hausdorff quasi-uniformity. Our results should be compared to investigations on H-equivalence of uniformities due to Smith, Ward and others.

1. Introduction

Let X be a (nonempty) set, and let $\mathcal{U}$ and $\mathcal{V}$ be two uniformities on X. Ward and Smith have obtained conditions on $\mathcal{U}$ and $\mathcal{V}$ under which the corresponding Hausdorff uniformities induce the same topologies on the set $\mathcal{P}_0(X)$ of nonempty subsets of X. Such uniformities on X are now called H-equivalent according to [24]. Various authors constructed pairs of distinct H-equivalent uniformities, see [6], [7] and [23]. As Hitchcock

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[5] points out ALBRECHT [1] seems to be the first to study the question of H-equivalence of uniformities, but his results similar to those of Ward remained unnoticed. It is known that two uniformities on a set X that are H-equivalent induce the same proximity. Hence, for instance, two distinct metric uniformities on a set X cannot be H-equivalent, since a metric uniformity is always the finest member of its proximity class [21]. SMITH also noted in [21] that a totally bounded uniformity cannot be H-equivalent to any other uniformity.

Similarly, in this article, given two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X we investigate when their corresponding Hausdorff quasi-uniformities induce the same topologies on the set $\mathcal{P}_0(X)$. While it is relatively difficult to construct distinct H-equivalent uniformities, it turns out to be fairly easy to give examples of two distinct quasi-uniformities whose Hausdorff quasi-uniformities induce the same hyperspace topology. Accordingly, in the quasi-uniform setting, it becomes more interesting to determine those quasi-uniformities whose Hausdorff quasi-uniformity induces a hyperspace topology that cannot be induced by Hausdorff quasi-uniformities originating from other quasi-uniformities. Let us note that special instances of the stated problem have already been studied. For instance, the authors in [20] characterized those compatible Hausdorff quasi-uniformities on a topological space X that induce the Vietoris topology on $\mathcal{P}_0(X)$. It follows from their characterization that for any topological space the Pervin quasi-uniformity and the well-monotone quasi-uniformity each induce the Vietoris topology. Obvious variants of our problem deal with appropriate subspaces of $\mathcal{P}_0(X)$ like the set $\mathcal{K}_0(X)$ of nonempty compact subsets of a quasi-uniform space $(X, \mathcal{U})$. We recall in this context that contrary to the situation in the realm of uniform spaces, a Hausdorff quasi-uniformity on $\mathcal{K}_0(X)$ need not induce the Vietoris topology of X [3]. In fact, according to [20], the Hausdorff quasi-uniformity of a quasi-uniform space $(X, \mathcal{U})$ is compatible with the Vietoris topology on the family $\mathcal{K}_0(X)$ of nonempty compact subsets of X if and only if for each $K \in \mathcal{K}_0(X)$, $\mathcal{U}^{-1} \restriction K$ is precompact.

The following definitions are discussed and studied in some detail in [2], [15] and [17]. Let $(X, \mathcal{U})$ be a quasi-uniform space. For any $U \in \mathcal{U}$, let

$$U_+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}, \quad \text{and}$$

$$U_- = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}.$$

Furthermore, set $U_* = U_- \cap U_+$ whenever $U \in \mathcal{U}$. Then $\{U_- : U \in \mathcal{U}\}$ is a base for the *lower quasi-uniformity* on $\mathcal{P}_0(X)$ and $\{U_+ : U \in \mathcal{U}\}$ is a base for the *upper quasi-uniformity* on $\mathcal{P}_0(X)$. Moreover, $\mathcal{U}_* = \mathcal{U}_+ \vee \mathcal{U}_-$ is the so-called *Hausdorff–Bourbaki quasi-uniformity* on $\mathcal{P}_0(X)$. It is obvious that the following equations hold for the conjugate quasi-uniformity $\mathcal{U}^{-1}$ of $\mathcal{U}$: (i) $(\mathcal{U}^{-1})_- = (\mathcal{U}_+)^{-1}$, (ii) $(\mathcal{U}^{-1})_+ = (\mathcal{U}_-)^{-1}$, and (iii) $(\mathcal{U}^{-1})_* = (\mathcal{U}_*)^{-1}$. Observe also that trivially if $\mathcal{U}$ and $\mathcal{V}$ are two quasi-uniformities on a set X, then $\mathcal{U} \subseteq \mathcal{V}$ implies that $\mathcal{T}(\mathcal{U}_-) \subseteq \mathcal{T}(\mathcal{V}_-)$, $\mathcal{T}(\mathcal{U}_+) \subseteq \mathcal{T}(\mathcal{V}_+)$, and $\mathcal{T}(\mathcal{U}_*) \subseteq \mathcal{T}(\mathcal{V}_*)$. Furthermore, $\mathcal{U}_* \vee (\mathcal{U}_*)^{-1} \subseteq (\mathcal{U} \vee \mathcal{U}^{-1})_*$.

Definition 1.1. Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. Then

(i) $\mathcal{U}$ and $\mathcal{V}$ are called *QH-equivalent* if $\mathcal{T}(\mathcal{U}_*) = \mathcal{T}(\mathcal{V}_*)$ on $\mathcal{P}_0(X)$ (similarly we shall use the self-explanatory term *QH-finer*);

(ii) $\mathcal{U}$ and $\mathcal{V}$ are called *doubly QH-equivalent* if both $\mathcal{T}(\mathcal{U}_*) = \mathcal{T}(\mathcal{V}_*)$ and $\mathcal{T}((\mathcal{U}^{-1})_*) = \mathcal{T}((\mathcal{V}^{-1})_*)$ on $\mathcal{P}_0(X)$.

Given a quasi-uniformity $\mathcal{U}$ on a set X, we shall denote by $Q(\mathcal{U})$ the collection of all quasi-uniformities which are QH-equivalent to $\mathcal{U}$. A straightforward application of Zorn's lemma shows that $Q(\mathcal{U})$ contains maximal elements (with respect to set inclusion). Of course, two uniformities $\mathcal{U}$ and $\mathcal{V}$ are H-equivalent if and only if they are QH-equivalent. Note that $\mathcal{U}$ and $\mathcal{V}$ are H-equivalent if and only if the restrictions of $\mathcal{U}_*$ and $\mathcal{V}_*$ induce the same topology on the set 2^X of nonempty closed subsets of X. Two H-equivalent uniformities $\mathcal{U}$ and $\mathcal{V}$ are trivially doubly QH-equivalent. Hence the examples of distinct H-equivalent uniformities show that doubly QH-equivalent quasi-uniformities may differ.

For a quasi-uniform space $(X, \mathcal{U})$, as usual, we shall denote by $\mathcal{U}_\omega$ (resp. $\delta_\mathcal{U}$) the finest totally bounded quasi-uniformity coarser than $\mathcal{U}$ (resp. the quasi-proximity induced by $\mathcal{U}$ on X). If $\mathcal{V}$ is another quasi-uniformity on X and $\delta_\mathcal{U} = \delta_\mathcal{V}$, then we say that $\mathcal{U}$ and $\mathcal{V}$ are *qp-equivalent*. Let $\pi(\mathcal{U})$ denote the collection of all quasi-uniformities on X which are *qp*-equivalent to $\mathcal{U}$. We refer the reader to [4] for undefined notation and basic facts about quasi-uniformities.

2. Necessary conditions for QH-equivalence of two quasi-uniformities

In this section, we shall provide some necessary conditions for the QH-equivalence of two quasi-uniformities on the same set.

Lemma 2.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. Then the following statements are equivalent.*

(i) $\mathcal{V}_\omega \subseteq \mathcal{U}_\omega$.

(ii) $\mathcal{T}(\mathcal{V}_+) \subseteq \mathcal{T}(\mathcal{U}_+)$ *on* $\mathcal{P}_0(X)$.

(iii) $\mathcal{T}(\mathcal{V}_+) \subseteq \mathcal{T}(\mathcal{U}_*)$ *on* $\mathcal{P}_0(X)$.

Proof. (i) $\Rightarrow$ (ii): Let $A \in \mathcal{P}_0(X)$ and $V \in \mathcal{V}$. Then $(A, X \setminus V(A)) \notin \delta_\mathcal{V}$. Since $\mathcal{V}_\omega \subseteq \mathcal{U}_\omega$, we have $(A, X \setminus V(A)) \notin \delta_\mathcal{U}$. Thus, there exists a $U \in \mathcal{U}$ such that $U(A) \cap (X \setminus V(A)) = \emptyset$. We conclude that $U_+(A) \subseteq V_+(A)$. Hence $\mathcal{T}(\mathcal{V}_+) \subseteq \mathcal{T}(\mathcal{U}_+)$.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): Suppose the contrary, that is, (iii) holds but $\mathcal{V}_\omega \not\subseteq \mathcal{U}_\omega$. Then there are $A, B \subseteq X$ such that $V_0(A) \cap B = \emptyset$ for some $V_0 \in \mathcal{V}$, but $U(A) \cap B \neq \emptyset$ for every $U \in \mathcal{U}$. Let $\mathcal{B} = \{F \in \mathcal{P}_0(X) : F \cap B \neq \emptyset\}$. For each $U \in \mathcal{U}$, pick a point $b_U \in U(A) \cap B$, and define $A_U = A \cup \{b_U\}$. Then $A_U \in U_*(A) \cap \mathcal{B}$ whenever $U \in \mathcal{U}$. Therefore $A \in \mathrm{cl}_{\mathcal{T}(\mathcal{U}_*)} \mathcal{B}$. On the other hand, $(V_0)_+(A) \cap \mathcal{B} = \emptyset$, thus $A \notin \mathrm{cl}_{\mathcal{T}(\mathcal{V}_+)} \mathcal{B}$. We have reached a contradiction which implies that the assertion holds. □

Corollary 2.2. *Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on the same set X are qp-equivalent if and only if $\mathcal{T}(\mathcal{U}_+) = \mathcal{T}(\mathcal{V}_+)$ on $\mathcal{P}_0(X)$.* □

Let κ be an infinite cardinal. A quasi-uniform space $(X, \mathcal{U})$ is called *κ-precompact* if for every $U \in \mathcal{U}$, there exists a subset F of X such that $|F| < \kappa$ and $X = U(F)$. As usual, we shall call ω-precompact (resp. ω_1-precompact) quasi-uniform spaces *precompact* [4] (resp. *preLindelöf* [14]). Let $P(\kappa, \mathcal{U})$ denote the collection of all κ-precompact subspaces of $(X, \mathcal{U})$.

Theorem 2.3. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. If $\mathcal{U}$ and $\mathcal{V}$ are QH-equivalent, then*

(i) $\mathcal{U}_\omega = \mathcal{V}_\omega$, *i.e., $\mathcal{U}$ and $\mathcal{V}$ are qp-equivalent; and*

(ii) $P(\kappa, \mathcal{U}^{-1}) = P(\kappa, \mathcal{V}^{-1})$ *for any cardinal $\kappa \geq \omega$.*

Proof. (i). This follows directly from Lemma 2.1.

(ii). Suppose the contrary, that is, $P(\kappa, \mathcal{U}^{-1}) \neq P(\kappa, \mathcal{V}^{-1})$ for some infinite cardinal κ. Without loss of generality, we may assume that there exists some $A \in \mathcal{P}_0(X)$ such that $A \in P(\kappa, \mathcal{U}^{-1}) \setminus P(\kappa, \mathcal{V}^{-1})$. For each $U \in \mathcal{U}$, choose an $F_U \subseteq A$ such that $|F_U| < \kappa$ and $A \subseteq U^{-1}(F_U)$. Then the net $(F_U)_{U \in \mathcal{U}}$ converges to A in $\mathcal{T}(\mathcal{U}_*)$, but there exists $V_0 \in \mathcal{V}$ such that $A \nsubseteq V_0^{-1}(F_U)$ whenever $U \in \mathcal{U}$, since otherwise A would be κ-precompact in $(X, \mathcal{V}^{-1})$. Thus $F_U \notin (V_0)_-(A)$ whenever $U \in \mathcal{U}$. It follows that the net $(F_U)_{U \in \mathcal{U}}$ does not converge to A with respect to $\mathcal{T}(\mathcal{V}_-)$, so certainly not with respect to $\mathcal{T}(\mathcal{V}_*)$ either. This is a contradiction. □

Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X, and $A \subseteq X$. We say that $\mathcal{U}$ is *quasi-uniformly finer*, abbreviated as *qu-finer*, than $\mathcal{V}$ on A if for any $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that $U(x) \subseteq V(x)$ whenever $x \in A$. For any $V \in \mathcal{V}$, A is called *V-discrete* if $(x, y) \in (A \times A) \cap V$ implies $x = y$. Moreover, A is said to be *$\mathcal{V}$-discrete* if it is V-discrete for some $V \in \mathcal{V}$.

Theorem 2.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. If $\mathcal{U}$ and $\mathcal{V}$ are QH-equivalent, then*

(i) *$\mathcal{U}$ is qu-finer than $\mathcal{V}$ on each $\mathcal{V}$-discrete set; and*

(ii) *$\mathcal{V}$ is qu-finer than $\mathcal{U}$ on each $\mathcal{U}$-discrete set.*

Proof. Since (i) and (ii) are similar, we shall prove (i) only. Let A be a V-discrete subset of X, where $V \in \mathcal{V}$. By our assumption, there exists some $U \in \mathcal{U}$ such that $U_*(A) \subseteq V_-(A)$. Next, we shall show that $U(a) \subseteq V(a)$ for every $a \in A$. To this end, let $a \in A$ and $y \in U(a)$. Let $B = \{y\} \cup (A \setminus \{a\})$. It can be checked easily that $B \in U_*(A) \subseteq V_-(A)$. Thus, $a \in A \subseteq V^{-1}(B)$. Since A is V-discrete, then we have $a \in V^{-1}(y)$. It follows that $y \in V(a)$. Thus, $\mathcal{U}$ is *qu*-finer than $\mathcal{V}$ on each $\mathcal{V}$-discrete set. □

Remark 2.5. In fact, to show Theorem 2.4 (i), we only need the condition "$\mathcal{T}(\mathcal{V}_-) \subseteq \mathcal{T}(\mathcal{U}_*)$". □

3. Sufficient conditions for QH-equivalence of two quasi-uniformities

In this section, we shall provide some sufficient conditions that make two quasi-uniformities on the same set QH-equivalent. First, we introduce a notion which is slightly weaker than that of $\mathcal{V}$-discreteness of a subset in a quasi-uniform space $(X, \mathcal{V})$. Let V be an entourage of a quasi-uniform space $(X, \mathcal{V})$. An indexed subset $A = \{x_\alpha : \alpha < \gamma\}$ of X is said to be *V-separated* provided that $(x_\alpha, x_\beta) \in V$ and $\alpha \leq \beta < \gamma$ implies $x_\alpha = x_\beta$, and is called *$\mathcal{V}$-separated* if it is V-separated for some $V \in \mathcal{V}$.

Proposition 3.1. *A subset A indexed by some ordinal of a quasi-uniform space $(X, \mathcal{V})$ is $\mathcal{V}$-discrete if and only if it is both $\mathcal{V}$-separated and $\mathcal{V}^{-1}$-separated.*

PROOF. The proof is straightforward, so it is omitted. □

The next simple example shows that $\mathcal{V}$-separatedness and $\mathcal{V}$-discreteness of a subset in a quasi-uniform space $(X, \mathcal{V})$ are different.

Example 3.2. Let ω be the set of nonnegative integers equipped with the usual order $\leq$. Let $\mathcal{V}$ be the quasi-uniformity on ω generated by the base $\{D\}$ where $D^{-1} = \leq$, that is D is the order dual to $\leq$. Clearly ω with its usual order $\leq$ is a $\mathcal{V}$-separated set that is not $\mathcal{V}$-discrete. □

Theorem 3.3. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. If*

(i) $\mathcal{V}_\omega \subseteq \mathcal{U}_\omega$, *and*

(ii) $\mathcal{U}$ *is quasi-uniformly finer than* $\mathcal{V}$ *on each* $\mathcal{V}^{-1}$*-separated set,*

then $\mathcal{T}(\mathcal{V}_*) \subseteq \mathcal{T}(\mathcal{U}_*)$.

PROOF. By (i) and Lemma 2.1, we have $\mathcal{T}(\mathcal{V}_+) \subseteq \mathcal{T}(\mathcal{U}_+)$. Hence, it suffices to show $\mathcal{T}(\mathcal{V}_-) \subseteq \mathcal{T}(\mathcal{U}_-)$. To this end, let $A \in \mathcal{P}_0(X)$ and $V \in \mathcal{V}$. Choose some $W \in \mathcal{V}$ such that $W^2 \subseteq V$. Starting with any point $x_0 \in A$, by transfinite induction, we construct a subset $S_A = \{x_\alpha : \alpha < \gamma\}$ of A such that

(iii) $x_\alpha \in A \setminus W^{-1}(\{x_\beta : \beta < \alpha\})$ for every $\alpha < \gamma$;

(iv) $A \subseteq W^{-1}(S_A)$.

It is clear from (iii) that S_A is W^{-1}-separated, and thus $\mathcal{V}^{-1}$-separated. By (ii), there exists some $U \in \mathcal{U}$ such that $U(x) \subseteq W(x)$ whenever $x \in S_A$.

Next, we shall show $U_-(A) \subseteq V_-(A)$. Suppose $B \in U_-(A)$, that is, $A \subseteq U^{-1}(B)$. For any point $a \in A$, by (iv), there exists an $x_\alpha \in S_A$ such that $x_\alpha \in W(a)$. It follows that $W(x_\alpha) \subseteq W^2(a) \subseteq V(a)$. On the other hand, since $x_\alpha \in A$, we have $U(x_\alpha) \cap B \neq \emptyset$. Thus, because $U(x_\alpha) \subseteq W(x_\alpha)$, also $V(a) \cap B \neq \emptyset$. Therefore, $A \subseteq V^{-1}(B)$ and $B \in V_-(A)$. We conclude that $\mathcal{T}(\mathcal{V}_-) \subseteq \mathcal{T}(\mathcal{U}_-)$. Therefore, $\mathcal{T}(\mathcal{V}_*) \subseteq \mathcal{T}(\mathcal{U}_*)$. □

Corollary 3.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. If*

(i) $\mathcal{U}_\omega = \mathcal{V}_\omega$,

(ii) *$\mathcal{U}$ is quasi-uniformly finer than $\mathcal{V}$ on each $\mathcal{V}^{-1}$-separated set, and*

(iii) *$\mathcal{V}$ is quasi-uniformly finer than $\mathcal{U}$ on each $\mathcal{U}^{-1}$-separated set,*

then $\mathcal{U}$ and $\mathcal{V}$ are QH-equivalent. □

Corollary 3.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-uniformities on a set X. If*

(i) $\mathcal{U}_\omega = \mathcal{V}_\omega$, *and*

(ii) *both $\mathcal{U}^{-1}$ and $\mathcal{V}^{-1}$ are hereditarily precompact,*

then $\mathcal{U}$ and $\mathcal{V}$ are QH-equivalent.

Proof. If $\mathcal{V}^{-1}$ is hereditarily precompact, then each $\mathcal{V}^{-1}$-separated set must be finite. Since $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\mathcal{V})$, then $\mathcal{U}$ is quasi-uniformly finer than $\mathcal{V}$ on each finite subset of X. It follows from Theorem 3.3 that $\mathcal{U}$ is QH-finer than $\mathcal{V}$. In a similar way, $\mathcal{V}$ is QH-finer than $\mathcal{U}$. Therefore, $\mathcal{U}$ and $\mathcal{V}$ are QH-equivalent. □

4. *QH*-singularity

According to [25], a uniformity $\mathcal{U}$ on a set X is called *H-singular* if there exists no distinct uniformity on X which is H-equivalent to $\mathcal{U}$. Similarly, we say that a quasi-uniformity $\mathcal{U}$ on a set X is *QH-singular* (*bi-QH-singular*) if there is no other quasi-uniformity on X which is QH-equivalent (doubly QH-equivalent) to it. We shall also say that a quasi-uniformity $\mathcal{U}$ is *doubly QH-singular* provided that both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are QH-singular. Of course, each doubly QH-singular quasi-uniformity is QH-singular and each QH-singular quasi-uniformity is bi-QH-singular. Observe also that each QH-singular uniformity is doubly QH-singular. In [21], Smith noted that each totally bounded uniformity is H-singular. Similarly we have the following result.

Theorem 4.1. *Each totally bounded quasi-uniformity is bi-QH-singular.*

Proof. Let $\mathcal{U}$ be a totally bounded quasi-uniformity on a set X and suppose that $\mathcal{V}$ is a quasi-uniformity on X such that $\mathcal{U}$ and $\mathcal{V}$ are doubly QH-equivalent. It follows from Theorem 2.3 that $\mathcal{V}$ belongs to the quasi-uniformity class of $\mathcal{U}$ and that both $\mathcal{V}$ and $\mathcal{V}^{-1}$ are hereditarily precompact, since both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact. We conclude that $\mathcal{V}$ is totally bounded [10, Lemma 1.1] and thus equal to $\mathcal{U}$. □

In the following, we consider when a given quasi-uniformity on a set is (doubly) QH-singular.

Theorem 4.2. *Let X be a nonempty set. Then*

(i) *The discrete uniformity $\mathcal{D}$ on X is doubly QH-singular;*

(ii) *For any quasi-uniformity $\mathcal{U}$ on X, if $\mathcal{U}_\omega$ is QH-singular then every quasi-uniformity in $\pi(\mathcal{U})$ is hereditarily precompact.*

Proof. (i): Let $\mathcal{H}$ be a quasi-uniformity on X that is QH-equivalent to $\mathcal{D}$, but $\mathcal{D} \neq \mathcal{H}$. Then for each $V \in \mathcal{H}$, there is a point $x_V \in X$ such that $V^{-1}(x_V) \neq \{x_V\}$. Set $D_V = \{x_V\} \cup (X \setminus V^{-1}(x_V))$. Since

$$X = V^{-1}(x_V) \cup (X \setminus V^{-1}(x_V)) \subseteq V^{-1}(D_V),$$

we conclude that the net $(D_V)_{V \in \mathcal{H}}$ converges to X with respect to $\mathcal{T}(\mathcal{H}_*)$. However, for the entourage $U = \Delta \in \mathcal{D}$, we have

$$U^{-1}(D_V) = \{x_V\} \cup (X \setminus V^{-1}(x_V)) \neq X.$$

It follows that the net $(D_V)_{V \in \mathcal{H}}$ does not converge to X with respect to $\mathcal{T}(\mathcal{D}_*)$. Therefore, $\mathcal{D}$ and $\mathcal{H}$ are not QH-equivalent, which is a contradiction. Hence, the uniformity $\mathcal{D}$ is doubly QH-singular.

(ii): If there exists a quasi-uniformity $\mathcal{V} \in \pi(\mathcal{U})$ which is not hereditarily precompact, then according to the construction in [11], for every p-filter σ on ω there is a quasi-uniformity $\mathcal{L}_\sigma$ on X such that $\mathcal{U}_\omega \subseteq \mathcal{L}_\sigma \subseteq \mathcal{V}$. Note that $\mathcal{L}_\sigma^{-1}$ is hereditarily precompact, since the conjugates of the subbasic entourages are clearly hereditarily precompact. By Corollary 3.4, each $\mathcal{L}_\sigma$ and $\mathcal{U}_\omega$ are QH-equivalent. Since $\mathcal{L}_\sigma$ is not totally bounded, $\mathcal{L}_\sigma$ and $\mathcal{U}_\omega$ are distinct. So, $\mathcal{U}_\omega$ is not QH-singular. This is a contradiction. □

Corollary 4.3. *Let $\mathcal{U}$ be a totally bounded quasi-uniformity on a set X. Then $\mathcal{U}$ is the only member of its quasi-proximity class if and only if $\mathcal{U}$ is doubly QH-singular.*

Proof. (⇒) If $\mathcal{U}$ is totally bounded and the unique member of its quasi-proximity class, then the same applies to $\mathcal{U}^{-1}$. By Theorem 2.3 (i), both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are QH-singular.

(⇐) Suppose that both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are QH-singular. Let $\mathcal{V} \in \pi(\mathcal{U})$ be any member. Since $\mathcal{U} = \mathcal{U}_\omega$ and $(\mathcal{U}_\omega)^{-1} = (\mathcal{U}^{-1})_\omega$, by Theorem 4.2 (ii), $\mathcal{V}$ is doubly hereditarily precompact. Then, by [10, Lemma 1.1], $\mathcal{V}$ is totally bounded. Thus, $\mathcal{V} = \mathcal{U}$. It follows that $\mathcal{U}$ is the only member of its quasi-proximity class. □

Example 4.4. There is a uniform space $(X, \mathcal{U})$ such that $\mathcal{U}$ is H-singular, but not QH-singular. Let $X = \omega$, and define

$$V_n = \{(a, a) : a \in n\} \cup ((X \setminus n) \times (X \setminus n))$$

for each $n \in \omega$. Then $\{V_n : n \in \omega\}$ generates a transitive metrizable totally bounded uniformity $\mathcal{U}$ on ω. Since such a uniformity is at the same time the finest as well as the coarsest member of its proximity class, it is H-singular. Note that $T = \bigcup_{n\in\omega} \{n\} \times (n+1)$ is a transitive reflexive relation on ω. Let $\mathcal{V}$ be the quasi-uniformity generated on X by $\{V_n : n \in \omega\} \cup \{T\}$. One checks that $\mathcal{V}$ belongs to the quasi-proximity class of $\mathcal{U}$: Let $\mathcal{H}$ be the quasi-uniformity generated by $\{T\}$ on X. Since for any $B \subseteq \omega$, $T(B) = n$ or ω and for each subset n of ω, we have $V_n(n) = n$, we conclude that $\mathcal{H}_\omega \subseteq \mathcal{U}$.

Hence, by [4, Proposition 1.40], we have $\mathcal{U} = \mathcal{U} \vee \mathcal{H}_\omega = \mathcal{U}_\omega \vee \mathcal{H}_\omega = (\mathcal{U} \vee \mathcal{H})_\omega = \mathcal{V}_\omega$. Since the quasi-uniformity $\mathcal{V}^{-1}$ is hereditarily precompact, by Corollary 3.5, we conclude that $\mathcal{V}$ is QH-equivalent to $\mathcal{U}$. Hence, $\mathcal{U}$ is not QH-singular. □

Given a topological space X, let $\mathcal{P}$ be the *Pervin quasi-uniformity* [4]. Recall that a family $\mathcal{L}$ of subsets of X is *well-monotone* if the partial order $\subseteq$ of set inclusion is a well-order on $\mathcal{L}$. The compatible quasi-uniformity

$\mathcal{M}$ on X which has as a subbase the set of all binary relations that are associated with the well-monotone open covers of X under the Fletcher construction is called the *well-monotone quasi-uniformity* of X [8], denoted by $\mathcal{M}$. Recall that X is said to be *hereditarily compact* [22] if every nonempty subspace of X is compact. It is well-known that a space is hereditarily compact if and only if each strictly increasing sequence of open subsets in it is finite.

Theorem 4.5. *Let X be a topological space. Then the following statements* (i)–(iii) *are equivalent:*

(i) *$\mathcal{P}$ is QH-singular;*

(ii) *$\mathcal{M}$ is QH-singular;*

(iii) *X is hereditarily compact.*

Furthermore, the following statements (iv)–(vi) *are equivalent:*

(iv) *$\mathcal{P}$ is doubly QH-singular;*

(v) *$\mathcal{M}$ is doubly QH-singular;*

(vi) *X admits a unique compatible quasi-uniformity.*

If X is Hausdorff, then all the above statements (i)–(vi) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii): By Proposition 6 of [20], both $\mathcal{P}$ and $\mathcal{M}$ induce the Vietoris topology. Thus, if $\mathcal{P}$ is QH-singular, then $\mathcal{P} = \mathcal{M}$. This implies that $\mathcal{M}$ is QH-singular as well.

(ii) $\Rightarrow$ (iii): Suppose that X is not hereditarily compact. Then the well-monotone quasi-uniformity and the Pervin quasi-uniformity of X are distinct, but both induce the Vietoris topology on $\mathcal{P}_0(X)$. Hence, $\mathcal{M}$ is not QH-singular, a contradiction.

(iii) $\Rightarrow$ (i): Let X be hereditarily compact, and let $\mathcal{U}$ be a quasi-uniformity on X that is QH-equivalent to $\mathcal{P}$. Then $\mathcal{U}$ belongs to the Pervin quasi-proximity class by Theorem 2.3 (i). (In fact, all compatible quasi-uniformities of a hereditarily compact space belong to this quasi-proximity class.) Now $\mathcal{P}$ is totally bounded, thus $\mathcal{P}^{-1}$ is hereditarily precompact, and hence according to Theorem 2.3 (i), $\mathcal{U}^{-1}$ is hereditarily precompact. Since X is hereditarily compact, $\mathcal{U}$ is hereditarily precompact. Because doubly hereditarily precompact quasi-uniformities are totally bounded [10, Lemma 1.1], we conclude that $\mathcal{U}$ is totally bounded. It follows that $\mathcal{U} = \mathcal{U}_\omega = \mathcal{P}_\omega = \mathcal{P}$. Hence, $\mathcal{P}$ is QH-singular.

(iv) $\Rightarrow$ (v): If both $\mathcal{P}$ and $\mathcal{P}^{-1}$ are QH-singular, then by Corollary 4.3, $\mathcal{P}$ is the unique quasi-uniformity in its quasi-proximity class. Hence, $\mathcal{P} = \mathcal{M}$, and thus both $\mathcal{M}$ and $\mathcal{M}^{-1}$ are QH-singular.

(v) $\Rightarrow$ (vi): If both $\mathcal{M}$ and $\mathcal{M}^{-1}$ are QH-singular, then by the equivalence of (ii) and (iii) above, we conclude that X is hereditarily compact. Thus, $\mathcal{M} = \mathcal{P}$ [13, Remark 1], which implies that $\mathcal{P}$ is doubly QH-singular. By Corollary 4.3, $\mathcal{P}$ is the unique quasi-uniformity in its quasi-uniformity class. Hence, the fine quasi-uniformity is totally bounded. It follows from a result of [18] that X admits a unique quasi-uniformity.

(vi) $\Rightarrow$ (iv): This is obvious.

Finally, if X is Hausdorff, then by Theorem 2.36 in [4], (iii) and (vi) are equivalent; indeed X is finite. Hence, all the statements of (i)–(vi) are equivalent. □

Example 4.6. (a) If a quasi-uniformity is unique in its quasi-proximity class, then it is doubly QH-singular according to Corollary 4.3. So, for instance, the coarsest quasi-uniformity of a locally compact T_2-space X is doubly QH-singular provided that X is compact or non-Lindelöf (see [19]).

(b) The coarsest compatible quasi-uniformity of a topological space need not be QH-singular: Just consider the Pervin quasi-uniformity of a topological space X (see e.g. [16]) that admits a unique quasi-proximity, but is not hereditarily compact. The assertion follows from Theorem 4.5. □

Remark 4.7. It is shown in [21] that any two metrizable uniformities on the same set cannot be H-equivalent, and any two uniformities on the same set, at least one of which is totally bounded, cannot be H-equivalent. However, there are no quasi-uniform analogues for these facts. First, note that both $\mathcal{U}$ and $\mathcal{V}$ defined in Example 4.4 have a countable base, and hence are quasi-metrizable. Second, $\mathcal{P}$ is always totally bounded for any space X, and by Theorem 4.5, it is not QH-singular if X is not hereditarily compact (see however Theorem 4.1). □

Question 4.8. Is there a quasi-uniformity that is (doubly) QH-singular, but not totally bounded or discrete?

Each quasi-uniformity $\mathcal{V}$ such that $\mathcal{T}(\mathcal{V}^*)$ is pseudocompact is unique in its quasi-proximity class and thus doubly QH-singular, since $\mathcal{T}(\mathcal{W}^*) = \mathcal{T}(\mathcal{V}^*)$ is pseudocompact and thus $\mathcal{W}^*$ totally bounded for any quasi-uniformity $\mathcal{W}$ which is QH-equivalent to $\mathcal{V}$.

Question 4.9. Characterize those totally bounded quasi-uniformities which are QH-singular.

5. QH-equivalence classes

Example 5.1. There exists a quasi-uniform space $(X, \mathcal{U})$ *such that* $Q(\mathcal{U})$ *contains simultaneously transitive and nontransitive quasi-uniformities as well as bicomplete and nonbicomplete quasi-uniformities.* Let $X = \omega$ be equipped with the lower topology $\mathcal{T} = \{\emptyset, \omega\} \cup \{[0, n[\,: n \in \omega\}$. It is known that this space has a unique compatible quasi-proximity [9, Example 1], since its topology is the unique base that is closed under finite unions and finite intersections. Clearly, the well-monotone quasi-uniformity on X is the fine quasi-uniformity and is bicomplete. Hence all quasi-uniformities compatible with the given topology are QH-equivalent, since they induce the Vietoris topology [20, Proposition 6]. Observe that the Pervin quasi-proximity class contains nontransitive quasi-uniformities [12], while the (nonbicomplete) Pervin quasi-uniformity is transitive. □

Question 5.2. Let $\mathcal{V}$ be a quasi-uniformity and let κ be the number of QH-equivalence classes into which the quasi-proximity class $\pi(\mathcal{V})$ of $\mathcal{V}$ splits. Which cardinalities κ can occur? Is $|Q(\mathcal{V})| = 1$ or $|Q(\mathcal{V})| \geq 2^{2^{\aleph_0}}$?

Question 5.3. Can the fine transitive quasi-uniformity and the fine quasi-uniformity of a topological space be distinct, but QH-equivalent?

6. Quasi-uniformities of algebraic structures

It is known that if the left and right uniformities of a topological group are distinct, then they are not H-equivalent [21].

Question 6.1. If the left and right quasi-uniformities of a paratopological group are distinct, can they be QH-equivalent?

Question 6.2. Ward [24] states that the left uniformity of a locally compact topological group is H-singular. When is it QH-singular?

Remark 6.3. The (left) uniformity of a compact topological group is QH-singular, because it is the coarsest compatible quasi-uniformity [4, Proposition 1.47] and thus its quasi-proximity class is a singleton according to Example 4.6(a). □

Question 6.4. Is it possible that for a paratopological group, $\mathcal{U}_L$ and $\mathcal{U}_R$ are QH-equivalent, but $\mathcal{U}_L \vee \mathcal{U}_R$ and $\mathcal{U}_R$ are not QH-equivalent?

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