Publ. Math. Debrecen
67/1-2 (2005), 79–91

When does an iterate equal a power?

By CHE TAT NG (Waterloo) and WEINIAN ZHANG (Chengdu)

Abstract. Let f be a continuous self-map on the real line, $f^{[m]}$ denote its m-th iterate and f^n its n-th multiplicative power. In this paper we solve the functional equation $f^{[m]} = f^n$ for integers $m \geq 2$, $n \geq 2$. When $m = n$, it reveals functions whose n-th iterate and power agree.

1. Introduction

Let $f : X \to X$ be a map on a set X, $m \geq 0$ an integer. The m-th iterate $f^{[m]}$ of f is defined by

$$f^{[m]}(x) = f(f^{[m-1]}(x)), \quad f^{[0]}(x) \equiv x.$$

When f is bijective, with inverse f^{-1}, iterates with negative exponents are defined by $f^{[-m]} = (f^{-1})^{[m]}$. Sometimes the brackets around m in $f^{[m]}$ are omitted when there is no confusion [1], [4]–[6]. As pointed out in [2], in circumstances where f^m has other natural meaning, such an omission would possibly lead some readers astray. For functions defined on the real line $\mathbb{R}$, let f^m denote its m-th (multiplicative) power.We would like to ask when $f^{[m]}$ and f^m actually agree, i.e.

$$f^{[m]} = f^m. \tag{1}$$

Mathematics Subject Classification: 39B12, 37E05.
Key words and phrases: iteration, periodic orbit, range, iterative root.
Supported by NSERC of Canada Grant OGP 0008182, NNSFC(China) Grant 10171071 and China Education Ministry Research Grants.
Corresponding author: Weinian Zhang (matzwn@sohu.com)

On various domains, this functional equation was discussed ([2], [3]) for $m = -1$. In particular, in [2] the problem leads to a discussion on the 4-th iterative roots of the identity as [4] does. In this paper we seek continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f^{[m]} = f^n \tag{2}$$

for given integers $m \geq 2$, $n \geq 1$.

2. Fundamental results

The following theorem, as a fundamental result, links our problem to iterative roots of special functions.

Theorem 1. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (2) *if and only if there exists an interval I in $\mathbb{R}$ such that*

(i) *I is non-empty, closed relative to $\mathbb{R}$, and is invariant under the power function $F(x) = x^n$,*

(ii) *I is also f-invariant, $f^{[m-1]} = F$ on I, and*

(iii) $\operatorname{ran}(f) \subset I$, *where* $\operatorname{ran}(f)$ *denotes the range of f.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous solution of equation (2):

$$f^{[m]}(x) = (f(x))^n, \quad \forall x \in \mathbb{R}. \tag{3}$$

Putting $y = f(x)$ in it we get

$$f^{[m-1]}(y) = y^n, \quad \forall y \in \operatorname{ran}(f). \tag{4}$$

Since f is continuous, $\operatorname{ran}(f)$ is a non-empty interval. Let I be its closure in $\mathbb{R}$. Then I is also an interval and (iii) holds. Clearly, (iii) implies that I is f-invariant. By continuity, equation (4) can be extended to I and we have (ii):

$$f^{[m-1]}(y) = y^n, \quad \forall y \in I. \tag{5}$$

Being f-invariant, I is also invariant under $f^{[m-1]}$. In view of (5), we get (i) – that I is invariant under the power function F. The converse is easy to check. With (iii), $\operatorname{ran}(f) \subset I$, (ii) implies (4) which is equivalent to (3). $\square$

While there is no need for I to be a closed interval in order to get the converse in the above theorem, its imposition eliminates the need to discuss a broader class of I. Its closure assures that continuous self-maps on I have continuous extension to the larger domain $\mathbb{R}$ without an expansion on its codomain. Although we could impose the condition that $\operatorname{ran}(f)$ is dense in I, we do not do this because it would be inconvenient and unnecessary for the converse.

Referring to condition (i), let

$$a := \inf I, \quad b := \sup I.$$

Thus a is possibly $-\infty$, b is possibly ∞, and $I = [a, b] \cap \mathbb{R}$. The determination of F-invariant I is straight forward. In the next proposition we state the result.

Proposition 1. *For* even n, $n \geq 2$, *I is invariant under $F(x) = x^n$ if and only if its ends a, b are in one of the following four combinations: (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$. For* odd n, $n \geq 3$, *I is invariant under $F(x) = x^n$ if and only if its ends a, b are in one of the following five combinations: (O_1) $a = -\infty$ and $b \in]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or ($O_4$) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$. For $n = 1$, a and b can be chosen arbitrarily with $a \leq b$.*

We now turn our attention to condition (ii). Letting $g = f|_I : I \to I$, we shall solve the equation

$$g^{[m-1]} = F|_I \tag{6}$$

in the next two sections. Together with the above fundamental results, (2) is solved fully. Solving (6) for $m = 2$ is a trivial mission, so in the coming sections we will assume $m > 2$.

3. Cases of odd n

In the special case $n = 1$, (6) is known as a Babbage equation, i.e., $F = \mathrm{id}$ on I, where id denotes the identity map. By Theorem 11.7.1 in [4], either $g = \mathrm{id}$ on I or m has to be odd and g is a strictly decreasing involution. As described in [4], decreasing involutions on an interval have simple geometric interpretation: their graph has to be symmetric with respect to the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$. In the sequel, we need only discuss the cases of an odd $n \geq 3$.

Lemma 1. *Let $n \geq 3$ be odd. Let g be a continuous $(m-1)$-th order iterative root of $F(x) = x^n$ on I. Then* (i) *g is strictly monotonic,* (ii) *g has no periodic points other than $0, 1$ and -1, and they are in fact periodic points of g whenever they are in I,* (iii) *for strictly increasing g, its periodic points can only be of order 1 (a fixed point),* (iv) *for strictly decreasing g , its periodic points are of order 1 and 2, while at most one is of order 1, and $(-1, 1)$ is its only possible 2-cycle in I.*

Proof. (i) With odd n, F is injective. If $g(x_1) = g(x_2)$, then $g^{[m-1]}(x_1) = g^{[m-1]}(x_2)$. Thus $F(x_1) = F(x_2)$, implying $x_1 = x_2$. This shows that g is also injective. Being continuous on the interval, g must be strictly monotonic. (ii) If x_0 is a periodic point of g, then x_0 must be a fixed point of an iterate of F. However, for any integer $k \geq 1$ the function $F^{[k]}(x) = x^{n^k}$ has exactly three fixed points at $0, 1$ and -1. This proves that x_0 can only be $0, 1$ or -1. Conversely, every fixed point x_0 of F is a periodic point of g, as $g^{[m-1]}(x_0) = x_0$ where $m - 1 \geq 1$.

(iii) This is a general observation that the only order preserving (finite) cycles on a linearly ordered set are the trivial 1-cycles. (iv) An order reversing cycle on a linearly orderly set must be a 1-cycle or a 2-cycle. A map with more than one fixed point cannot be order reversing. The 2-cycles $(-1, 0)$ and $(0, 1)$ cannot occur for continuous g, because no fixed points are present in the open intervals $]-1, 0[$ and $]0, 1[$. □

According to the list in Proposition 1, we shall seek roots on intervals $I = [a, b] \cap \mathbb{R}$ where a, b are in one of the following five combinations (O_1) $a = -\infty$ and $b \in\ ]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or (O_4) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$.

In light of Lemma 1, (i), all roots are strictly monotonic. Within the following two subsections, the shorter form "increasing g", for instance, will have the same effect as "strictly increasing g".

3.1. For increasing g on I. The solving of (6) for increasing g on intervals of type (O_3) can be further simplified by solving it "componentwise", as stated more accurately in the following:

Proposition 2. *Let $n \geq 3$ be odd. Then g is a strictly increasing continuous $(m-1)$-th order iterative root of $F(x) = x^n$ on I if and only if it is the union of strictly increasing roots on each of the closed connected sub-intervals of I separated by the fixed points $\{0, \pm 1\} \cap I$.*

Theorem 11.2.2 in [4] gives the results on increasing iterative roots on I. In particular it shows that F possesses increasing iterative roots g of all orders, and they can be constructed by *piecewise defining*.

Moreover, it is easy to show that $g(x) \leq x$ (resp. $\geq x$) for $x \in]-\infty, -1] \cup [0, 1]$ (resp. $x \in [-1, 0] \cup [1, +\infty[$) if g is defined at the point x. For example, if g is an increasing second order root and $g(t_0) > t_0$ for some $t_0 \in [0, 1]$ where g is defined, then $F(t_0) = g(g(t_0)) > g(t_0) > t_0$ since g is strictly increasing. This will contradict $F(t_0) = t_0^n \leq t_0$. This property of g is observed when we apply Theorem 11.2.5 in [4] during a subsequent discussion on the decreasing roots.

In what follows we select a typical interval under the case $-1 < a < 0$ and $b = 0$ to illustrate the general construction of an increasing root.

Let g be a strictly increasing continuous self-map on $[a, 0]$ and

$$g^{[m-1]}(x) = F(x) = x^n \quad (x \in [a, 0]). \tag{7}$$

According to Lemma 1, g has 0 as its unique fixed point. Letting

$$c_j = g^{[j]}(a), \quad j = 0, \ldots, m-1 \tag{8}$$

we first get $-1 < a = c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m-2} \leq c_{m-1} = F(a) = a^n < 0$ from the range condition $\operatorname{ran}(F|_I) = \operatorname{ran}(g^{[m-1]}) \subset \operatorname{ran}(g^{[m-2]}) \subset \cdots \subset \operatorname{ran}(g) \subset [a, 0]$ while using $g(0) = 0$. Because g has 0 as its unique fixed point, the above inequalities must be strict:

$$-1 < a = c_0 < c_1 < c_2 < \cdots < c_{m-2} < c_{m-1} = a^n < 0.$$

Recall that $m > 2$ has been assumed. A *fundamental region* for g is $[c_0, c_{m-2}]$, in the sense that the restriction

$$g_0 := g|_{[c_0, c_{m-2}]}$$

can be initiated reasonably freely, and it determines g on the full $[a, 0]$. The *initial* g_0 is an order preserving homeomorphism, mapping $[c_0, c_{m-2}]$ onto $[c_1, F(a)]$, and $g_0(c_j) = c_{j+1}$ for each $j = 0, \ldots, m-2$. It implies

$$g_0^{[m-2]}([c_0, c_1]) = [c_{m-2}, c_{m-1}] \tag{9}$$

in particular, and g on $[a, 0]$ is uniquely determined by g_0 via:

Step 1. For each $\ell \geq 1$ and $x \in [F^{[\ell]}(c_0), F^{[\ell]}(c_{m-2})]$, there exists a unique $y \in [c_0, c_{m-2}]$ such that $F^{[\ell]}(y) = x$. We have $g(x) = g(F^{[\ell]}(y)) = F^{[\ell]}(g(y))$. Thus $g(x) = F^{[\ell]}(g_0(y))$.

It corresponds to the observation that for each $\ell \geq 0$, g maps $[F^{[\ell]}(c_{m-3}), F^{[\ell]}(c_{m-2})]$ homeomorphically onto $[F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$, order preserving.

Step 2. For each $\ell \geq 0$ and $x \in [F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$, there exists a unique $y \in [c_{m-2}, F(a)]$ such that $F^{[\ell]}(y) = x$. Further, by (9), there exists a unique $z \in [c_0, c_1]$ such that $g_0^{[m-2]}(z) = y$. We have $g(x) = g(F^{[\ell]}(y)) = F^{[\ell]}(g(g_0^{[m-2]}(z))) = F^{[\ell+1]}(z)$.

It reflects that g maps $[F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$ onto $[F^{[\ell+1]}(a), F^{[\ell+1]}(c_1)]$ homeomorphically, order preserving.

Step 3. $g(0) = 0$.

Conversely, let $(c_j)_{j=0}^{m-1}$ be a strictly increasing sequence with $c_0 = a$ and $c_{m-1} = F(a)$, and let g_0 be an order preserving homeomorphism from $[c_0, c_{m-2}]$ onto $[c_1, c_{m-1}]$ satisfying (8), we can verify that the above three steps well define an extension of g_0 to a continuous increasing g satisfying (7).

3.2. For decreasing g on I.

Proposition 3. *Let $n \geq 3$ be odd. Let g be a strictly decreasing continuous $(m-1)$-th order iterative root of $F(x) = x^n$ on I. Then the five interval types listed in Proposition 1 are confined further to (C_1) I is one of the degenerated singletons $\{-1\}$, $\{0\}$ and $\{1\}$, (C_2) $I = \mathbb{R}$ or $[-1, 1]$, or (C_3) $I = [a, b]$ where $-1 < a < 0$, $0 < b < 1$.*

PROOF. Amongst the five interval types (O_1) $a = -\infty$ and $b \in]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or (O_4) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$, we shall rule some out quickly. When $a = -\infty$, b cannot be finite because the condition $\mathrm{ran}(F|_I) \subset \mathrm{ran}(g)$ cannot be met by finite b. For the same reason, a finite a cannot be paired with $b = \infty$. By Lemma 1, I cannot contain two points in $\{-1, 0, 1\}$ without having the third. This rules out both $a = -1$ and $0 \leq b < 1$, and $-1 < a \leq 0$ and $b = 1$. The case $a = 0$ and $0 < b < 1$ is not admissible because the order reversing g cannot map $[0, b]$ into $[0, b]$ while keeping 0 fixed. For the same reason the case $-1 < a < 0$ and $b = 0$ is not admissible. □

In the following we give the construction of g on intervals I listed in Proposition 3. On a degenerated singleton I, the answer for g is trivial. For the rest of this subsection, we shall assume that I is not a singleton. Because F is strictly increasing, $m - 1$ must be even, say $m - 1 = 2k$.

Let
$$\phi := g^{[2]}.$$

Then ϕ is continuous, strictly increasing and satisfies
$$\phi^{[k]}(x) = x^n. \tag{10}$$

As illustrated in the previous subsection we can solve for all increasing ϕ from (10). By Lemma 1, ϕ has no periodic points other than $0, 1$ and -1 and they are in fact fixed points of ϕ whenever they are in I.

For each solved ϕ, which is perhaps not a power function, we continue to solve for continuous and strictly decreasing g from
$$g^{[2]} = \phi. \tag{11}$$

CASE $a = -1$ and $b = 1$, or $a = -\infty$ and $b = \infty$.

In this case, the fixed points of ϕ are -1, 0 and 1. Notice that F is a bijection on I and so is ϕ. Thus ϕ is an increasing homeomorphism of I onto I.

Theorem 11.2.5 in [4] is applicable, giving the construction of g. A fundamental region for g is $[x_0, \phi(x_0)]$ on $[-1, 1]$, or $[x_0, \phi(x_0)] \cup [\phi(y_0), y_0]$ on $\mathbb{R}$, where $-1 < x_0 < 0$ and $y_0 < -1$ are arbitrarily chosen.

Case $-1 < a < 0$, $0 < b < 1$.

From $\mathrm{ran}(\phi) = \mathrm{ran}(g^{[2]}) \subset \mathrm{ran}(g) \subset [a, b]$ we get $[\phi(a), \phi(b)] \subset [g(b), g(a)] \subset [a, b]$. Letting

$$[c, d] = \mathrm{ran}(g), \quad \text{where } c = g(b),\ d = g(a),$$

we note their relative positions in $\mathbb{R}$:

$$-1 < a \le c \le \phi(a) < 0 < \phi(b) \le d \le b < 1.$$

Consider the sequences

$$a, \phi(a), \phi^{[2]}(a), \ldots, \phi^{[\ell]}(a), \ldots$$

and

$$d, \phi(d), \phi^{[2]}(d), \ldots, \phi^{[\ell]}(d), \ldots$$

which are strictly increasing and strictly decreasing, respectively. They tend to 0, the unique fixed point of ϕ. Observe that g maps each term in the first sequence to a corresponding term in the second sequence, i.e.,

$$g(\phi^{[\ell]}(a)) = \phi^{[\ell]}(d) \quad \ell = 0, 1, 2, \ldots \tag{12}$$

as g and ϕ commute. Moreover, each term in the second sequence is mapped by g to a shifted term in the first sequence, i.e.,

$$g(\phi^{[\ell]}(d)) = \phi^{[\ell+1]}(a) \quad \ell = 0, 1, 2, \ldots \tag{13}$$

The interval $[a, \phi(a)]$ is a fundamental region for g as we shall explain.

(1) The initial g_0 maps $[a, \phi(a)]$ homeomorphically onto $[\phi(d), d]$, $g_0(a) = d$, $g_0(\phi(a)) = \phi(d)$ and $d \in [\phi(b), b]$.

(2) g_0 determines g elsewhere as follows:

Step 1. For each $\ell \ge 1$ and $x \in [\phi^{[\ell]}(a), \phi^{[\ell+1]}(a)]$, there exists a unique $y \in [a, \phi(a)]$ such that $\phi^{[\ell]}(y) = x$. We have $g(x) = g(\phi^{[\ell]}(y)) = \phi^{[\ell]}(g(y))$. Thus $g(x) = \phi^{[\ell]}(g_0(y))$.

Step 2. For each $\ell \ge 0$ and $x \in [\phi^{[\ell+1]}(d), \phi^{[\ell]}(d)]$, there exists a unique $y \in [\phi^{[\ell]}(a), \phi^{[\ell+1]}(a)]$ such that $g(y) = x$. We have $g(x) = g(g(y)) = \phi(y)$.

Step 3. For each $x \in [d, b]$, we see $\phi(x) \in [\phi(d), \phi(b)] \subset [\phi(d), d]$. Since g_0 maps $[a, \phi(a)]$ homeomorphically onto $[\phi(d), d]$, there exists a unique

$y \in [a, \phi(a)]$ such that $g(y) = \phi(x)$. Because $\phi(x) = g(g(x))$ and g is injective, we have $g(x) = y$.

So g maps $[d, b]$ homeomorphically onto $[c, \phi(a)]$, order reversing.

Step 4. $g(0) = 0$ according to Lemma 1.

Conversely, it is straight forward to check that if g_0 is a continuous order reversing map of the interval $[a, \phi(a)]$ onto the interval $[\phi(d), d]$ for some $d \in [\phi(b), b]$, then the above four steps well define a continuous extension g on $[a, b]$ which satisfies $g^{[2]} = \phi$.

4. Cases of even n

In this final section, we determine the continuous $(m-1)$-th order iterative roots of $F(x) = x^n$ on I when n is an even positive integer. According to Proposition 1, there are four types of $I = [a, b] \cap \mathbb{R}$ to cover: Case (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$.

Lemma 2. *Let $n \geq 2$ be even. Let g be a continuous $(m-1)$-th order iterative root of $F(x) = x^n$ on I. Then* (i) *g has no periodic points other than* 0 *and* 1*, and they are in fact fixed points of g whenever they are in I,* (ii) *g is strictly increasing on $I_+ := I \cap [0, +\infty[$, and is strictly decreasing on $I_- := I \cap\,]-\infty, 0]$.*

Proof. (i) If x_0 is a periodic point of g, then x_0 must be a fixed point of an iterate of F. For any integer $k \geq 1$ the function $F^{[k]}(x) = x^{n^k}$ has exactly two fixed points at 0 and 1. This proves that x_0 can only be 0 or 1. Conversely, every fixed point x_0 of F, if it is in I, is a periodic point of g. So 0 and 1 are indeed periodic points of g whenever they are in I. Having no third fixed point between 0 and 1, they cannot form a 2-cycle under the continuous g. This shows that 0 and 1 are in fact fixed points of g whenever they are in I.

(ii) With even $n \geq 2$, F is injective on $]-\infty, 0]$ and on $[0, +\infty[$. Repeating the arguments in the proof of Lemma 1, we see that g must be strictly monotonic on each of the two intervals $I \cap\,]-\infty, 0]$ and $I \cap [0, +\infty[$. Consider the four types of I.

Case (E_4): $a \in [1, \infty[$ and $b = \infty$. Subcase 1. Suppose $a > 1$. By (i), $g(a) \neq a$. The interval I being g-invariant, we must have $g(a) > a$. If g were strictly decreasing, we would have $a \leq g^{(2)}(a) < g(a)$. The compact interval $[a, g(a)]$ will then be g-invariant and must contain a fixed point of g. This is a contradiction to (i), that g has no fixed point in I when $a > 1$. So g is strictly increasing. Subcase 2. $a = 1$. By (i), $g(a) = a$. As I is g-invariant, g cannot be strictly decreasing.

Case (E_1): $a \in [-1, 0]$ and $b \in [a^n, 1]$. Subcase 1. Suppose $a < 0$. By (i), $g(0) = 0$. First, we observe that if g were strictly increasing on both I_+ and I_-, then it is strictly increasing on I. This would imply all its iterates, including F, are strictly increasing on I. But F is not strictly increasing. This contradiction shows that g cannot be strictly increasing on both I_+ and I_-. For similar reasons it cannot be strictly decreasing on both I_+ and I_-. Next, if g were strictly increasing on I_- and strictly decreasing on I_+, then g and therefore all its iterates will have their range included in I_-. This is a contradiction as F does not comply with this property. So we reached the conclusion that g is strictly increasing on I_+ and strictly decreasing on I_-. Subcase 2. Suppose $a = 0$. The argument for g strictly increasing is the same as that of Subcase 2 in Case (E_4).

Case (E_2): $a \in [-\infty, 0]$ and $b = \infty$. Subcase 1. Suppose $a < 0$. In this case, 0 is an interior point of I and the proof given above for Subcase 1 in Case (E_1) is also valid. Subcase 2. Suppose $a = 0$. The argument for an increasing g is the same as that of Subcase 2 in Case (E_4).

Case (E_3): $a = 1$ and $b = 1$. The statements are trivial. □

Lemma 3. *Let $n \geq 2$ be even. Let g be a continuous $(m-1)$-th orderiterative root of $F(x) = x^n$ on $I = [a, b] \cap \mathbb{R}$ which contains 0 as an interior point. Then g is even on the maximal subinterval of I symmetric about 0. Furthermore, it can be extended uniquely to a continuous $(m-1)$-th order iterative root $\bar{g}$ of F on the balanced interval $\bar{I} := [-c, c] \cap \mathbb{R}$ where $c = \max(|a|, b)$, with $\operatorname{ran}(\bar{g}) = \operatorname{ran}(g) \subset I$.*

Proof. Let $x \in I$ and assume $-x \in I$. By Lemma 2, g is minimized at the fixed point 0. So $y_1 := g(x)$ and $y_2 := g(-x)$ are both in I_+. We also have $g^{[m-2]}(y_1) = g^{[m-1]}(x) = F(x)$ and $g^{[m-2]}(y_2) = g^{[m-1]}(-x) = F(-x)$. Because F is even, it follows that $g^{[m-2]}(y_1) = g^{[m-2]}(y_2)$. The strict monotonicity of g on I_+ which is g-invariant yields $y_1 = y_2$. This proves the evenness of g, that $g(x) = g(-x)$ whenever both x and $-x$ are in I. The function $\bar{g} : \bar{I} \to \bar{I}$ given by

$$\bar{g}(x) = \begin{cases} g(x), & \forall x \in I \\ g(-x), & \forall x \in -I \end{cases}$$

is thus well defined. It is straight forward to check that on the interval $\bar{I} = I \cup (-I)$, $\bar{g}$ is again a continuous $(m-1)$-th iterative root of F. It is immediate from the definition of $\bar{g}$ that $\operatorname{ran}(\bar{g}) = \operatorname{ran}(g)$. The function $\bar{g}$ is clearly the unique even extension of g from I to $\bar{I}$. □

Combining the above two lemmas we arrive at the following conclusion for the Section. The convention $-(-\infty) = \infty$ will be in use.

Proposition 4. *Let $F(x) = x^n$, $n \geq 2$ even, $I = [a, b] \cap \mathbb{R}$ F-invariant, and let $g : I \to I$ denote a continuous $(m-1)$-th order iterative root of F.* (i) *When $a \geq 0$, g exists* (for every $m > 2$) *and its general construction is given in the same manner as in the discussions in Section 3.1.* (ii) *When $a < 0$ and $-a \leq b$, g exists. Its restriction to I_+, dentoted by g_*, is a root of $F|_{I_+}$ and g is the even extension of g_* to I. Conversely, for every root g_* of $F|_{I_+}$ whose general construction is covered in* (i), *the even extension $g(x) := g_*(|x|)$ to I (which is not necessarily the balanced $\bar{I}$) is a root on I.* (iii) *When $a < 0$ and $-a > b > a^n$, g exists. It is the restriction, to I, of some root $\bar{g} : [a, -a] \to [a, -a]$ of F on $[a, -a]$ satisfying the extra range condition* $\operatorname{ran}(\bar{g}) \subset [0, b]$. (iv) *When $a < 0$ and $-a > b = a^n$, g does not exist* (for every $m > 2$).

Proof. Recall that I belongs to one of the following types: (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$.

Case 1. Suppose $a \geq 0$.

This is the case where $I = I_+$. According to Lemma 2, g is strictly increasing having 0 and 1 as its only fixed points provided they are in I. Just like the discussions in section 3.1, its existence as well as its general construction are given by Theorem 11.2.2 in [4].

Case 2. Suppose $a < 0$.

This is the case where 0 is an interior point of I. According to Lemma 3, g is even and has an extension $\bar{g} : \bar{I} \to \bar{I}$, $\bar{I} = I \cup (-I)$.

Furthermore, the extension satisfies the range condition

$$\operatorname{ran}(\bar{g}) \subset I. \tag{14}$$

Subcase 1. Suppose $-a \le b$.

In this subcase, $x \in I_-$ implies $-x \in I_+$. The consideration of the extension $\bar{g}$ is not crucial. By Lemma 2, $g_* := g|_{I_+}$ is an increasing root of F on I_+. The existence and general construction of g_* are attended in Case 1. On the interval I_-, g is determined by its evenness $g(x) = g_*(-x)$.

Subcase 2. Suppose $-a > b$.

This can only occur within Case (E_1), with $a \in \,]-1,0[$ and $b \in [a^n, -a[$. We first attend the general construction of $\bar{g}_* := \bar{g}|_{\bar{I}_+}$ which is strictly increasing. The interval $\bar{I}_+$ is $[0,c]$ where $c = -a$. While Theorem 11.2.2 in [4] gives the general construction of all roots $\bar{g}_*$ mapping $[0,c]$ into $[0,c]$, we must confine ourselves to those meeting the stronger range condition (14) which requires that $\bar{g}_*$ maps $[0,c]$ into $[0,b]$. Its initialization on $[0,c]$ is, in part, based on the following:

The sequence

$$c_j := \bar{g}_*^{[j]}(c), \quad j = 0, \dots, m-1 \tag{15}$$

satisfies

$$1 > c = c_0 > c_1 > c_2 > \cdots > c_{m-2} > c_{m-1} = c^n > 0 \tag{16}$$

and

$$c_1 \le b. \tag{17}$$

Subcase 2.1. Suppose $a \in \,]-1,0[$ and $b = a^n$.

For $m > 2$ there is no sequence satisfying (16)–(17). So F has no root.

Subcase 2.2. Suppose $a \in \,]-1,0[$ and $b > a^n$. Then the conditions (16)–(17) can be met by some sequence, $\bar{g}_*$ exists, and a fundamental region for its construction is $[c_{m-2}, c_0]$. It determines g by

$$g(x) = \bar{g}_*(|x|) \quad \forall x \in I. \tag{18}$$

□

Acknowledgements. The authors are grateful to the referees for their helpful comments and suggestions.

References

[1] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* **61** (2001), 1–48.

[2] R. Cheng, A. Dasgupta, B. R. Ebanks, L. F. Kinch, L. M. Larson and R. B. McFadden, When does $f^{-1} = 1/f$?, *Amer. Math. Monthly* **105** (1998), 8: 704–717.

[3] R. Euler and J. Foran, On functions whose inverse is their reciprocal, *Math. Mag.* **54** (1981), 185–189.

[4] M. Kuczma, B. Choczewski and R. Ger, Iterative functional equations, Encyclopedia of Math. & Its Applications Vol. 32, *Cambridge University Press*, *Cambridge*, 1990.

[5] A. Mukherjea and J. S. Ratti, On a functional equation involving iterates of a bijection on the unit interval, *Nonlinear Anal.* **7** (1983), 899–908.

[6] W. Zhang, Solutions of equivariance for a polynomial-like iterative equation, *Proc. Roy. Soc. Edinburgh* **130A** (2000), 1153–1163.

CHE TAT NG
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO N2L 3G1
CANADA

WEINIAN ZHANG
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN 610064
CHINA

E-mail: matzwn@sohu.com

(Received August 8, 2003; revised March 31, 2004)