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Inner approximation-solvability of nonlinear equations

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Abstract. Result on the inner approximation-solvability are generalized to the case of uniformly ϕ-monotone operators in an A-properness setting.

1. Introduction

It seems that PETRYSHYN [4] was the first who, in a series of publications, studied the approximation-solvability problem: For what type of a linear or nonlinear mapping A, is it possible to construct a solution x of the equation

$$Ax = b$$

as a strong limit of solutions x_n of simpler finite-dimensional equations

$$A_n x_n = E_n^* b \ ?$$

The technique, thus, evolved from Petryshyn's investigations is that of an approximation-properness (A-properness). It turned out that the importance of A-properness of A was not only limited to the approximation-solvalibility of the equation $Ax = b$, but extends to the results relating to the Galerkin type methods for linear and nonlinear operator equations with more recent works on strongly monotone and accretive operators and other mappings. For the details on A-proper maps and approximation-solvabilty, see ([1], [4], [5], [7], [11], [13]).

Here our motivation is to extend the results on the inner approximation-solvability [13] of the equations $Ax = b$ to the case of uniformly ϕ-monotone operators $A : X \to X^*$ from a real separable reflexive Banach space X to is dual X^*. The obtained results include a number of significant results as special cases.

Definition 1.1. Let X and Y be real Banach spaces, and $\phi : X \to Y$ from X into Y be such that
(i) $\phi(X)$ is dense in Y; and
(ii) for each $x \in X$ and each $t \geq 0$, $\phi(tx) = t\phi(x)$.

A mapping $A : X \to Y^*$ from X into Y^* (dual of Y) is said to be *uniformly phi-monotone* if, for all $x, y \in X$,

$$\langle Ax - Ay, \phi(x-y)\rangle \geq c\,(\|x-y\|)\,\|\phi(x-y)\|,$$

where $c(r)$ is some gauge function, and $\langle\cdot\,,\cdot\rangle$ is the pairing between Y^* and Y.

Note that when $Y = X$ (reflexive) and $\phi = I$, A is just uniformly monotone, and when $Y = X^*$, the uniform ϕ-monotonicity of A coincides with strong K-monotonicity ([4], [6]).

2. Inner approximation scheme

Let X be a real separable reflexive Banach space with $\dim X = \infty$, and $A : X \to X^*$ from X into its dual X^* be uniformly ϕ-monotone and continuous.

Let (X_n) be a Galerkin scheme in X such that

$$X_n = \operatorname{span}\{e_{1n}, \ldots, e_{n'n}\}, \quad n = 1, 2, \ldots\ .$$

Let $E_n : X_n \to X$ be the embedding operator corresponding to $X_n \subseteq X$ (i.e., $E_n x = x$ for all $x \in X_n$). We construct the operator $R_n : X \to X_n$ in such a manner that, for each $x \in X$, there exists at least an element $R_n x \in X_n$ such that

$$\|x - R_n x\| = \operatorname{dist}(x, X_n)\,.$$

We consider the operator equation

$$Ax = b \quad (x \in X, b \in X^*) \tag{1}$$

along with the approximate equations

$$A_n x_n = E_n^* b \quad (x_n \in X_n, \quad n = 1, 2, \ldots) \tag{2}$$

corresponding to the following approximation scheme $\{X_n, E_n, R_n, X_n^*, E_n^*\}$ represented by the accompanying diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\ \phi\ } & X & \xrightarrow{\ A\ } & X^* \\ {\scriptstyle E_n}\big\uparrow & & {\scriptstyle R_n}\big\downarrow\big\uparrow{\scriptstyle E_n} & & \big\downarrow{\scriptstyle E_n^*} \\ X_n & \xrightarrow{R_n \phi E_n} & X_n & \xrightarrow{\ A_n\ } & X_n^* \end{array} \tag{3}$$

where all the operators $A_n = E_n^* A E_n$ are continuous.

For $n = 1, 2, \ldots$, the approximate equation (2) is equivalent to Galerkin equations

$$\langle A x_n, e_{jn}\rangle = \langle b, e_{jn}\rangle \tag{4}$$

for $x_n \in X_n$, $j = 1, \dots, n'$.

A word of caution: Here and in what follows operators $E_n : X_n \to X$ and $E_n^* : X^* \to X_n^*$ are lenear and continuous. The symbols " $\to$ " and " $\xrightarrow{\omega}$ " shall denote strong and weak convergence, respectively.

Definition 2.1. An approximation scheme $\pi = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ represented by diagram (3) is an admissible inner approximation scheme iff

(C1) X and X^* are infinite-dimensional normed spaces over field $\mathbb{K}$ (complex).
(C2) X_n and X_n^* are normed spaces over $\mathbb{K}$ with $\dim X_n = \dim X_n^* < \infty$ for all n.
(C3) $E_n : X_n \to X$ and E_n^* are linear and continuous with $\sup \|E_n\| < \infty$ and $\sup \|E_n^*\| < \infty$.
(C4) $R_n : X \to X_n$ is defined in the sense of the compatibility condition

$$\lim_{n\to\infty} \|E_n R_n x - x\|_X = 0 \quad \text{for all} \quad x \in X.$$

Definition 2.2. The equation $Ax = b$ is said to be solvable if it has a solution for each $b \in X^*$.

Unique Approximation-Solvability. The equation $Ax = b$ is said to be uniquely approximation-solvable if, for each $b \in X^*$,

(i) $Ax = b$, $x \in X$, has a unique solution.
(ii) For each $n \geq n_0$, the approximate equation $E_n^* A E_n x_n = E_n^* b$, $x_n \in X_n$, has a unique solution.
(iii) The sequence (x_n) converges to the solution x of the equation $Ax = b$ in the sense that

$$\lim_{n\to\infty} \|E_n x_n - x\|_X = 0.$$

Definition 2.3. The operator $A : X \to X^*$ is said to be approximation-proper (abbreviated as A-proper) with respect to the approximation scheme $\pi = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ if the following holds: Let (n') be any subsequence of the sequence of natural numbers. If $(x_{n'})$ is a sequence with $x_{n'} \in X_{n'}$ for all n', and if

$$\lim_{n\to\infty} \|A_{n'} x_{n'} - E_n^* b\|_{X_{n'}^*} = 0 \quad \text{for fixed} \quad b \in X^*$$

and $\sup \|x_{n'}\|_{X_{n'}} < \infty$, then there exists a subsequence $(x_{n''})$ such that

$$\lim_{n\to\infty} \|E_{n''} x_{n''} - x\|_X = 0$$

and $Ax = b$.

Definition 2.4. The approximation scheme $\pi = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is said to be consistent if, for all $x \in X$,

$$\lim_{n\to\infty} \|E_n^* A x - A_n R_n x\|_{X_n^*} = 0.$$

Definition 2.5. The approximation scheme π is said to be stable if there is an n_0 such that

$$\|A_n x - A_n y\|_{X_n^*} \geq c\left(\|x-y\|_{X_n}\right)$$

for all $x, y \in X_n$ and $n \geq n_0$.

Before we wind up this section, we state the following theorem of Petryshyn [4], crucial to our approximation-solvability:

Lemma 2.6. *If the approximation scheme $\pi = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is an admissible inner approximation scheme with consistency and stability, then the equation*

$$Ax = b, \quad x \in X,$$

is uniquely approximation-solvable for each $b \in X^*$ iff A is A-proper.

3. Inner approximation-solvabilty

Now we are just about ready to describe our main results on the solvability (approximation-solvability).

Theorem 3.1. *Suppose that the operator $A : X \to X^*$ is uniformly ϕ-monotone and continuous from a real separable reflexive Banach space X $(\dim X < \infty)$ into its dual X^*. Then the operator equation*

$$Ax = b \quad (x \in X)$$

has a unique solution every $b \in X^*$.

For $\phi = I$, Theorem 3.1 reduces to the following corollary:

Corollary 3.2. *If $A : X \to X^*$ is uniformly monotone and continuous, then the equation*

$$Ax = b \quad (x \in X)$$

has a unique solution for every $b \in X^*$.

Theorem 3.3. *Let X be a real separable reflexive Banach space with $\dim X = \infty$. Let $\{X_n, E_n, R_n, X_n^*, E_n^*\}$ be the approximation-scheme for (X, X^*) represented by the diagram (3), and let $\phi : X \to X$ be weakly continuous with*

$$R_n \phi x = \phi x \quad \text{for all } x \in X_n \text{ and each } n.$$

If $A : X \to X^*$ is uniniformly ϕ-monotone and continuous, then, for each $b \in X^*$, the operator equation

$$Ax = b \quad (x \in X)$$

is uniquely approximation-solvable.

Corollary 3.4. *Under the assumptions of Theorem 3.3, if $A : X \to X^*$ is uniformly ϕ-monotone and continuous, and $C : X \to X^*$ is compact, then, for each real $\lambda \neq 0$, the operator*

$$\lambda(A + C) : X \to X^*$$

is A-proper.

For $\phi = I$, Theorem 3.3 reduces to the following corollary:

Corollary 3.5 ([13], Prop. 34.9). *Under the assumptions of Theorem 3.3, if $A : X \to X^*$ is uniformly monotone and continuous, then the equation*

$$Ax = b \quad (x \in X)$$

is uniquely approximation-solvable for every $b \in X^*$.

Corollary 3.6. *For $X = H$ (Hilbert space), $\phi = I$ and $R_n = E_n^* = P_n$ (orthogonal projection operator), Theorem 3.3 reduces to* ([13], *Theorem* 34. B*).*

For the sake of simplicity, from now on the symbol $\|\cdot\|$ shall denote all norms $\|\cdot\|_X, \|\cdot\|_{X^*}, \|\cdot\|_{X_n}$ and $\|\cdot\|_{X_n^*}$ in the respective spaces X, X^*, X_n and X_n^*.

Proof of theorem 3.1. Since A is uniformly ϕ-monotone, i.e.,

$$\text{(C5)} \qquad \langle Ax - Ay, \phi(x - y)\rangle \geq c(\|x - y\|)\|\phi(x - y)\|$$

for all $x, y \in X$, it is immediate that A is injective. Let us take $k(r) = c(r) - \|A(0)\|$. Then, for $x \in X$, we have

$$\begin{aligned}\langle Ax, \phi x\rangle &= \langle Ax - A(0), \phi x\rangle + \langle A(0), \phi x\rangle \\ &\geq \langle Ax - A(0), \phi x\rangle - \langle A(0), \phi x\rangle \geq c(\|x\|)\|\phi x\| - \|A(0)\|\ \|\phi x\| \\ &= \big[c(\|x\|) - \|A(0)\|\big]\ \|\phi x\| = k(\|x\|)\|\phi x\|,\end{aligned}$$

and so $\|Ax\| \geq k(\|x\|)$ for $x \neq 0$. For each $M > 0$, therefore, there exists $h(M)$ such that $\|Ax\| \leq M$, implying $\|x\| \leq h(M)$. Thus, A^{-1} carries bounded subsets of $R(A)$, the range of A, into bounded subsets of X, and is continuous from $R(A)$ into X. By the Brouwer theorem on invariance of domain, $R(A)$ is open.

To this end, it only remains to show that $R(A)$ is closed. To show this, let $Ax_m \to b$ as $m \to \infty$. Thus, (Ax_m) is a Cauchy sequence, and by the condition (C5), we obtain

$$\begin{aligned}c(\|x_m - x_n\|)\,\|\phi\,(x_m - x_n)\| &\leq \langle Ax_m - Ax_n, \phi\,(x_m - x_n)\rangle \\ &\leq \|Ax_m - Ax_n\|\ \|\phi\,(x_m - x_n)\|,\end{aligned}$$

and so

$$c(\|x_m - x_n\|) \leq \|Ax_m - Ax_n\| \to 0 \quad \text{as} \quad m, n \to \infty.$$

This, in turn, implies that

$$\|x_m - x_n\| \to 0.$$

Hence, (x_m) is also a Cauchy sequence, and so $x_m \to x$ as $m \to \infty$. Since A is continuous, we find $Ax = b$, implying $b \in R(A)$.

Thus, the non-empty set $R(A)$ is both open and closed and, therefore, $R(A) = X^*$ and A is bijective. This completes the proof.

Proof of theorem 3.3. We prove the theorem by an application of Lemma 2.6, i.e., we need to show first that the approximation scheme $\pi = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is an admissible inner approximation scheme. Since $\|E_n\| = 1$ (and hence $\|E_n^*\| = 1$) and (X_n) is a Galerkin scheme, $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for all $x \in X$. This implies that $\|R_n x - x\| \to 0$ as $n \to \infty$, and thus compatibility condition (C4) is satisfied.

Consistency. Since A is continuous and compatibility condition (C4) is satisfied, the consistency condition is as follows: Since

$$\|AE_nR_nx - Ax\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|E_n^*\| = 1,$$

this implies that

$$\begin{aligned}\|E_n^*Ax - A_nR_nx\| &= \|E_n^*Ax - E_n^*AE_nR_nx\| \\ &\leq \|E_n^*\| \; \|Ax - AE_nR_nx\| \to 0\end{aligned}$$

as $n \to \infty$

Stability. For all $x, y \in X_n$, we have

$$\begin{aligned}&\|A_nx - A_ny\| \; \|\phi(x-y)\| \geq \langle A_nx - A_ny, \phi(x-y)\rangle \\ &= \langle E_n^*AE_nx - E_n^*AE_ny, \phi(x-y)\rangle = \langle Ax - Ay, E_n\phi(x-y)\rangle \\ &= \langle Ax - Ay, \phi(x-y)\rangle \geq c(\|x-y\|) \; \|\phi(x-y)\|,\end{aligned}$$

and consequently,

$$\|A_nx - A_ny\| \geq c(\|x-y\|) \quad \text{for all} \quad x, y \in X_n.$$

A-properness. Let $\sup \|x_n\| < \infty$ for all $x_n \in X_n$ and

$$\text{(A1)} \qquad \|A_nx_n - E_n^*b\| = \|E_n^*Ax_n - E_n^*b\| \to 0$$

as $n \to \infty$ for all n. Since X is reflexive, there exists a subsequence (again) denoted by (x_n) such that

$$x_n \xrightarrow{\omega} x \quad \text{in } X \text{ as } n \to \infty.$$

This means, we need only to show

$$x_n \to x \quad \text{in } X \text{ as } n \to \infty, \text{ and } Ax = b.$$

Since $\|R_n x - x\| \to 0$ as $n \to \infty$, $x_n \xrightarrow{\omega} x$ as $n \to \infty$ implies that

$$x_n - R_n x \xrightarrow{\omega} 0 \quad \text{as } n \to \infty.$$

Thus, $\phi(x_n - R_n x) \xrightarrow{\omega} 0$ since ϕ is weakly continuous, and $b - AR_n x \to b - Ax$ since $\|x - R_n x\| \to 0$ as $n \to \infty$.

From Condition (C5) and above arguments, we obtain, for $x_n \in X_n$ as above, as $n \to \infty$,

$$\begin{aligned}
&c\left(\|x_n - R_n x\|\right)\|\phi(x_n - R_n x)\| \le \langle Ax_n - AR_n x, E_n \phi(x_n - R_n x)\rangle \\
&= \langle E_n^* Ax_n - E_n^* AR_n x, \phi(x_n - R_n x)\rangle \\
&= \langle E_n^* Ax_n - E_n^* b + E_n^* b - E_n^* AR_n x, \phi(x_n - R_n x)\rangle \\
&= \langle E_n^* Ax_n - E_n^* b, \phi(x_n - R_n x)\rangle + \langle E_n^* b - E_n^* AR_n x, \phi(x_n - R_n x)\rangle \\
&= \langle E_n^* Ax_n - E_n^* b, \phi(x_n - R_n x)\rangle + \langle b - AR_n x, E_n \phi(x_n - R_n x)\rangle \\
&= \langle E_n^* Ax_n - E_n^* b, \phi(x_n - R_n x)\rangle + \langle b - AR_n x, \phi(x_n - R_n x)\rangle \to 0
\end{aligned}$$

This also implies that $\|x_n - R_n x\| \to 0$ as $n \to \infty$, and consequently, $x_n - R_n x \to 0$ as $n \to \infty$. Since $\|E_n\| < \infty$, it implies that

$$\|E_n x_n - E_n R_n x\| \to 0 \quad \text{as } n \to \infty,$$

and the compatibility condition (C4) (in Definition 2.1) implies

$$\|E_n R_n x - x\| \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\|E_n x_n - x\| \to 0 \quad \text{as } n \to \infty.$$

Now it only remains to show $Ax = b$. Since $x_n - R_n x \to 0$ as $n \to \infty$, implies that $x_n \to x$ as $n \to \infty$, and A is continuous, we obtain $Ax = b$, that is, A is A-proper.

Now Lemma 2.6 is applicable to conclude the proof.

4. Application to numerical ranges

In this section, we apply the obtained results to the solvability of nonlinear equations involving the numerical range — a generalization of the Zarantonello numerical range to the case of the reflexive Banach space operator.

Definition 4.1. (Duality Mapping). We recall a continuous function $\mu : \mathbb{R}^+ = \{t : t \geq 0\} \to \mathbb{R}^+$ is called a gauge function if $\mu(0) = 0$, and μ is strictly increasing. Let X be a real reflexive Banach space and X^* its dual. We denote by $\langle f, x\rangle$ the duality pairing between the elements of $f \in X^*$ and $x \in X$. A mapping $J : X \to X^*$ is said to be a duality mapping between X and X^* with respect to a gauge function μ if

$$\text{(C6)} \qquad \langle Jx, x\rangle = \mu(\|x\|)\|x\|, \quad \text{and}$$

$$\text{(C7)} \qquad \|Jx\| = \mu(\|x\|) \quad \text{for} \quad x \in X.$$

Note that if $\mu(t) = t$, J is said to be a normalized duality mapping. If X^* is strictly convex, then J is uniquely determined by μ, and if X is also reflexive, then J is a single-valued demicontinuous mapping of X onto X^*, which is bounded and positively homogeneous. Furthermore, J is monotone and satisfies

$$\text{(C8)} \quad \langle Jx - Jy, x - y\rangle \geq (\mu(\|x\|) - \mu(\|y\|))(\|x\| - \|y\|) \qquad \text{for all } x, y \in X,$$

and

$$\text{(C8)}^* \quad \langle Jx - Jy, x - y\rangle = \langle Jx, x - y\rangle - \langle Jy, x - y\rangle \geq \geq \Big|\mu(\|x\|) - \mu(\|y\|)\Big|\|x - y\| \quad \text{for all } x, y \in X.$$

For J a normalized duality, (C8) reduces to

$$\text{(C9)} \qquad \langle Jx - Jy, x - y\rangle \geq (\|x\| - \|y\|)^2 \quad \text{for all } x, y \in X,$$

and

$$\text{(C9)}^* \qquad \langle Jx - Jy, x - y\rangle \geq \Big|\|x\| - \|y\|\Big|\|x - y\| \quad \text{for all } x, y \in X.$$

Definition 4.2 (Numerical Range). Let $A : X \to X^*$ be a mapping from a reflexive Banach space X to its dual X^* over the field $\mathbb{K}$ (real or complex). Let $J : X \to X^*$ be a strictly monotone normalized duality mapping. The set

$$V[A] = \left\{ \frac{\langle Ax - Ay, \phi(x - y)\rangle}{\langle Jx - Jy, \phi(x - y)\rangle} : x, y \in X, x \neq y \right\}$$

is called the numerical range of A. Here $\phi : X \to X$ is weakly continuous such that

(i) $\phi(X)$ is dense in X; and
(ii) for each $x \in X$ and each $t \geq 0$, $\phi(tx) = t\phi(x)$ and $\|\phi x\| = \|x\|$.

Clearly, $V[A]$ is a subset of $\mathbb{K}$, and $V[A]$ reduces to the Zarantonello numerical range when X is a Hilbert space and $J = \phi = I$ in the form [12]

$$N[A] = \left\{ \frac{(Ax - Ay, x - y)}{\|x - y\|^2} : x, y \in X, x \neq y \right\}$$

where $(\cdot\,,\cdot)$ is the standard inner product on X.

We describe some of the elementary properties of $V[A]$ in the following theorem:

Theorem 4.3. *Let $A, B : X \to X^*$ be mappings from a reflexive Banach space to its dual X^*, $J : X \to X^*$ strictly monotone normalized duality, and $\lambda \in \mathbb{K}$ (real or complex field). Then*

(i) $V[\lambda A] = \lambda V[A]$;

(ii) $V[A + B] \subseteq V[A] + V[B]$; *and*

(iii) $V[A - \lambda J] = V[A] - \{\lambda\}$.

Proof. The assertions (i) and (ii) follow directly from the definition. To prove (iii), if $x, y \in D(A + J) = D(A) \cap D(J) \neq \emptyset$ with $x \neq y$, we have

$$\frac{\langle (A - \lambda J)x - (A - \lambda J)y, \phi(x - y)\rangle}{\langle Jx - Jy, \phi(x - y)\rangle}$$
$$= \frac{\langle Ax - Ay, \phi(x - y)\rangle - \lambda\langle Jx - Jy, \phi(x - y)\rangle}{\langle Jx - Jy, \phi(x - y)\rangle} = V[A] - \{\lambda\}.$$

Now we apply the obtained results in the preceding section to the solvability (approximation-solvability) of the equation

$$Ax - \lambda Jx = b \quad \text{for fixed} \quad \lambda \in \mathbb{K}.$$

Theorem 4.4. *Let $A : X \to X^*$ be continuous from a separable real Banach space $X(\dim X < \infty)$ to its dual X^*. If X and X^* are locally uniformly convex, $J : X \to X^*$ is normalized duality, and the number $\lambda \in \mathbb{K}$ is at a positive distance from the numerical range of A, $V[A]$, that is*

$$d = \mathrm{dist}(\lambda, V[A]) > 0,$$

then, for each $b \in X^*$, the equation

$$Ax - \lambda Jx = b, \quad x \in X,$$

has a unique solution.

Theorem 4.5. *Suppose X $(\dim X = \infty)$ is a separable reflexive complex Banach space with dual X^*, and $A : X \to X^*$ is continuous mapping. If X and X^* are locally uniformly convex, $J : X \to X^*$ is normalized duality, and the number $\lambda \in \mathbb{K}$ is at a positive distance from the numerical range of $A, V[A]$, that is,*

$$d = \operatorname{dist}(\lambda, V[A]) > 0,$$

then, for each $b \in X^*$, the equation

$$Ax - \lambda Jx = b, \quad x \in X,$$

is uniquely approximation-solvable.

Corollary 4.6 ([12]), Theorem II). *If X is a separable Hilbert space, and $J = \phi = I$ in Theorems 4.4 and 4.5, then, for each $b \in X$, the equation*

$$Ax - \lambda x = b, \quad x \in X,$$

has a unique solution.

The inverse operator

$$(A - \lambda I)^{-1} : X \to X$$

is Lipschitz continuous, and if $A(0) = 0$, then the Zarantonello numerical range $N[A]$ contains the eigenvalues of A.

If, in addition, $\dim X = \infty$, then, for each $b \in X$, the equation

$$Ax - \lambda x = b, \quad x \in X,$$

is uniquely approximation-solvable.

Proof of theorem 4.4. The proof follows from an application of Theorem 3.1. To achieve this, we need the following key stability condition: For $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned} &|\langle (A-\lambda J)x - (A-\lambda J)y, \phi(x-y)\rangle| = \\ &\qquad = |\langle Ax - Ay, \phi(x-y)\rangle - \lambda \langle Jx - Jy, \phi(x-y)\rangle| \\ &\qquad = \left| \frac{\langle Ax - Ay, \phi(x-y)\rangle}{\langle Jx - Jy, \phi(x-y)\rangle} - \lambda \right| \, |\langle Jx - Jy, \phi(x-y)\rangle| \\ &\qquad \geq d \operatorname{Re}\langle Jx - Jy, \phi(x-y)\rangle \\ &\qquad \geq d \Big| \, \|x\| - \|y\| \, \Big| \, \|\phi(x-y)\|. \end{aligned} \tag{C10}$$

This implies that

$$\|(A-\lambda J)x - (A-\lambda J)y\| \geq d \Big| \|x\| - \|y\| \Big|$$

for all $x, y \in X$.

Now, the rest of the proof can be imitated from the proof of Theorem 3.1.

PROOF OF THEOREM 4.5. The proof follows from Theorem 3.3 on approximation-solvability.

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