Publ. Math. Debrecen
67/1-2 (2005), 93–102

Soluble groups with many 2-generator torsion-by-nilpotent subgroups

By NADIR TRABELSI (Sétif)

Abstract. We prove in this paper that a finitely generated soluble group in which every infinite subset contains a pair of distinct elements x, y such that $\langle x, y\rangle$ is torsion-by-nilpotent (respectively, $\langle x, x^y\rangle$ is Chernikov-by-nilpotent), is itself torsion-by-nilpotent (respectively, finite-by-nilpotent).

1. Introduction and results

Following a question of Erdős, B. H. Neumann proved in [18] that a group is centre-by-finite if, and only if, every infinite subset contains a commuting pair of distinct elements. Since this result, problems of similar nature have been the object of many papers (for example [1]–[7], [10], [15]–[17], [21]–[23]). In particular, in [15] Lennox and Wiegold considered the class (Ω, ∞) of groups in which every infinite subset contains two distinct elements generating an Ω-group, where Ω is a given class of groups. They characterised finitely generated soluble groups which belong to (Ω, ∞) when Ω is the class of polycyclic, or nilpotent, or coherent groups. Here we will consider the class (Ω, ∞), when Ω is the class $\mathcal{TN}$ of torsion-by-nilpotent groups, or the class $\mathcal{CN}$ of Chernikov-by-nilpotent groups, and we will prove the following results:

Mathematics Subject Classification: 20F16.
Key words and phrases: finitely generated soluble groups, finite-by-nilpotent groups, infinite subsets, Chernikov groups, torsion groups.
I would like to thank the referee whose comments improved the exposition of this paper.

Theorem 1. *Let G be a finitely generated soluble group in the class $(\mathcal{TN},\infty)$. Then G is torsion-by-nilpotent.*

Let k be a positive integer and let $\mathcal{N}_k$ be the class of nilpotent groups of class at most k. In [2], Abdollahi and Taeri proved that a finitely generated metabelian group G is in $(\mathcal{N}_k,\infty)$ if, and only if, $G/Z_k(G)$ is finite; and a finitely generated soluble group G is in the class $(\mathcal{N}_k,\infty)$, if and only if, G belongs to $\mathcal{FN}_k^{(2)}$, where $\mathcal{F}$ is the class of finite groups and $\mathcal{N}_k^{(2)}$ denotes the class of groups whose 2-generated subgroups are nilpotent of class at most k. Also let $\mathcal{E}_k$ be the class of k-Engel groups. In [16], Longobardi proved that if G is a finitely generated locally graded group in the class $(\mathcal{E}_k,\infty)$, then G belongs to $\mathcal{FE}_k$. Combining the results of [2], [16], and Theorem 1, we shall obtain the following consequences.

Corollary 2. *Let k be a positive integer.*

(i) *A finitely generated soluble group G is in the class $(\mathcal{TN}_k,\infty)$ if and only if G belongs to $\mathcal{TN}_k^{(2)}$.*

(ii) *A finitely generated metabelian group G is in the class $(\mathcal{TN}_k,\infty)$ if and only if G belongs to $\mathcal{TN}_k$.*

(iii) *A finitely generated soluble group G is in the class $(\mathcal{TE}_k,\infty)$ if and only if G belongs to $\mathcal{TE}_k$.*

In the Chernikov-by-nilpotent case, we weaken the hypothesis by considering the class $(\mathcal{CN},\infty)^*$ of groups in which every infinite subset contains two distinct elements x, y such that $\langle x, x^y\rangle$ is in $\mathcal{CN}$. More precisely, we will prove the following result:

Theorem 3. *Let G be a finitely generated soluble group in the class $(\mathcal{CN},\infty)^*$. Then G is finite-by-nilpotent.*

Note that Theorem 3 improves the result of [22, Proposition 2], where it is proved that a finitely generated soluble group in the class $(\mathcal{FN},\infty)$ is finite-by-nilpotent.

Let k be a positive integer and let $\mathcal{E}_k(\infty)$ be the class of groups in which every infinite subset contains two distinct elements x, y such that $[x,_k y] = 1$. In [1], Abdollahi proved that a finitely generated metabelian group G is in $\mathcal{E}_k(\infty)$ if, and only if, $G/Z_k(G)$ is finite, and if G is a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$, then there exists an integer $c = c(k)$, depending only on k, such that $G/Z_c(G)$ is finite. Note that $(\mathcal{N}_k, \infty)^*$ is contained in $\mathcal{E}_{k+1}(\infty)$. Combining the results of [1], [2], [16] and Theorem 3, we shall obtain the following consequences.

Corollary 4. *Let k be a positive integer.*

(i) *If G is a finitely generated soluble group in the class $(\mathcal{CN}_k, \infty)^*$, then there is an integer $c = c(k)$, depending only on k, such that $G/Z_c(G)$ is finite.*

(ii) *A finitely generated metabelian group is in the class $(\mathcal{CN}_k, \infty)^*$ if and only if $G/Z_{k+1}(G)$ is finite.*

Corollary 5. *Let k be a positive integer.*

(i) *A finitely generated soluble group G is in the class $(\mathcal{CN}_k, \infty)$ if and only if G belongs to $\mathcal{FN}_k^{(2)}$.*

(ii) *A finitely generated metabelian group G is in the class $(\mathcal{CN}_k, \infty)$ if and only if $G/Z_k(G)$ is finite.*

(iii) *A finitely generated soluble group G is in the class $(\mathcal{CE}_k, \infty)$ if and only if G belongs to $\mathcal{FE}_k$.*

2. Proof of the results

To prove our theorems, we will use recent results of Endimioni and Traustasson [9] on torsion-by-nilpotent groups.

Lemma 6. *Let $c > 0$ be an integer and let G be a group in $\mathcal{N}_c\mathcal{T}$. If G belongs to $(\mathcal{TN}, \infty)$ then it is in $(\mathcal{TN}_c, \infty)$.*

Proof. Let $x, y \in G$ such that $\langle x, y\rangle \in \mathcal{TN}$. Clearly $\langle x, y\rangle$ belongs also to $\mathcal{N}_c\mathcal{T}$ and the set of its torsion elements is a subgroup T. Hence $\langle x, y\rangle/T$ is a torsion-free nilpotent group which belongs to $\mathcal{N}_c\mathcal{T}$. It follows from [19, Lemma 6.33] that $\langle x, y\rangle/T \in \mathcal{N}_c$, so $\langle x, y\rangle \in \mathcal{TN}_c$. Consequently, if G belongs to $(\mathcal{TN}, \infty)$, then it is in $(\mathcal{TN}_c, \infty)$. □

Lemma 7. *Let G be a soluble group in the class $(\mathcal{TN}, \infty)$. If G is abelian-by-torsion then it is torsion-by-abelian.*

Proof. By Lemma 6, G belongs to $(\mathcal{TA},\infty)$, where $\mathcal{A}$ denotes the class of abelian groups. First of all, we show that the set of torsion elements of G is a subgroup. Let $x, y \in G$ be two elements of finite order. Then $H = \langle x, y\rangle$ is a finitely generated soluble group which belongs to $\mathcal{AT}$, so it is abelian-by-finite. Clearly we may assume H infinite. Therefore H has a torsion-free normal abelian subgroup A of finite index. Let $1 \neq a \in A$ and let $h \in H$, then the subset $\{a^i h : i > 0\}$ is infinite. By the property $(\mathcal{TA},\infty)$, there are two distinct positive integers i, j such that $\langle a^i h, a^j h\rangle \in \mathcal{TA}$, so $\langle a^{i-j}, a^i h\rangle \in \mathcal{TA}$. Hence $[a^{i-j}, a^i h]^m = 1$ for some positive integer m. Since A is abelian and normal in H we obtain $[a, h]^{(i-j)m} = 1$, and this gives $[a, h] = 1$ as A is torsion-free. It follows that A is contained in the centre of H. So H is a centre-by-finite group. Thus, by a result of Schur [19, Theorem 4.12], H' is finite and therefore H is a finitely generated finite-by-abelian group. This contradicts the fact that H is infinite. Consequently, H is a finite group, so xy^{-1} is of finite order. This means that the elements of finite order in G form a subgroup T, as claimed. Now G/T is a torsion-free group in the class $(\mathcal{TA},\infty)$. So G/T belongs to $(\mathcal{A},\infty)$. It follows by the result of B. H. Neumann [18] that G/T is centre-by-finite. Thus G/T is finite-by-abelian and, therefore, G is torsion-by-abelian, as required. □

Lemma 8. *Let G be a finitely generated abelian-by-nilpotent group with abelian Fitting subgroup A and let $x \in G$. Suppose that for each $a \in A$, there are integers $n \geq 0$, $m_1 > 0$ and $m_2 > 0$ such that $[a, x^{m_1},_n x^{m_2}] = 1$. Then there is a positive integer d, depending only on G, such that $x^d \in A$.*

Proof. Since G is a finitely generated abelian-by-nilpotent group, we may therefore apply a result of Lennox and Roseblade [14, Theorem B], which asserts that in a finitely generated abelian-by-nilpotent group G, there is a positive integer d, depending only on G, such that for all $i > 0$ and for all g in G the inclusion $C_G(g^i) \leq C_G(g^d)$ holds. We firstly show by induction on n that if a is an element of A satisfying the hypothesis of the lemma, then $[a,_{n+1} x^d] = 1$. If $n = 0$, then we have $[a, x^{m_1}] = 1$ hence $[a, x^d] = 1$, as desired. Now assume that $n > 0$ and $[a, x^{m_1},_n x^{m_2}] = 1$. So we obtain $[a, x^{m_1},_{n-1} x^{m_2}, x^d] = 1$. Now $\langle a, x\rangle$ being metabelian, it is easy to see that $[a, x^i, x^j] = [a, x^j, x^i]$ for any integers i, j. Thus we get that $[a, x^d, x^{m_1}, {}_{n-1} x^{m_2}] = 1$, and by the inductive hypothesis we obtain $[a, {}_{n+1} x^d] = 1$, as required.

Now consider the subgroup $K = \langle A, x\rangle$. Since G/A is nilpotent, K is subnormal in G. For every $y \in K$, there exist $a \in A$ and an integer r such that $y = x^r a$. As we have just shown, there is a positive integer d such that $[a, {}_{n+1} x^d] = 1$ for some non-negative integer n, so we have $[y, {}_{n+1} x^d] = [x^r a, {}_{n+1} x^d] = [a, {}_{n+1} x^d] = 1$. Thus x^d is a left Engel element of K. Since K is soluble, the set of its left Engel elements coincı̈des with its Hirsch–Plotkin radical A_1 [19, Theorem 7.34], so $x^d \in A_1$. Since K is subnormal in G, A_1 is a subnormal locally nilpotent subgroup in G. So A_1 is contained in the Hirsch–Plotkin radical of G [20, 12.1.4]. Now G is a finitely generated abelian-by-nilpotent group, so it satisfies the maximal condition on normal subgroups [12]. Therefore the Hirsch–Plotkin radical of G coïncides with its Fitting subgroup, hence $x^d \in A$ as claimed. □

Proof of Theorem 1. Let G be a finitely generated soluble group in the class $(\mathcal{TN}, \infty)$. To prove that G is torsion-by-nilpotent, we proceed by induction on the derived length d of G. If $d = 1$ there is nothing to prove, so we can assume $d > 1$. By the inductive hypothesis, $G/G^{(d-1)}$ is torsion-by-nilpotent. Thus G is in the class $(\mathcal{AT})\mathcal{N}$, and by Lemma 7 it belongs to $\mathcal{T}(\mathcal{AN})$. Therefore, we may suppose G abelian-by-nilpotent, so G satisfies the maximal condition on normal subgroups [12] and $(\mathcal{TN}, \infty)$ is a quotient closed class, we may assume that G is a just-non-(torsion-by-nilpotent) group, that is, $G \notin \mathcal{TN}$ but every proper quotient of G is torsion-by-nilpotent. In [9, Corollary 1.3], it is proved that if H is a normal subgroup of a locally soluble group G such that H and G/H' are torsion-by-nilpotent, then G is torsion-by-nilpotent. It follows that every normal torsion-by-nilpotent subgroup of G is abelian. In particular, the Fitting subgroup A of G, is abelian. Moreover, it is easy to see that any normal torsion subgroup of G must be trivial. Thus A is torsion-free. Let $1 \neq a \in A$ and let xA be an element of infinite order in G/A. Then the subset $\{x^i a : i > 0\}$ is infinite. Hence there exist two positive integers i, j such that $\langle x^i a, x^j a\rangle$ is torsion-by-nilpotent. So $\langle x^i a, x^{i-j}\rangle$ is torsion-by-nilpotent. Then there is an integer $n \geq 0$ such that $\gamma_{n+1}(\langle x^i a, x^{i-j}\rangle)$ is a torsion group. If $n = 0$, then $\langle x^i a, x^{i-j}\rangle$ is a torsion group. So $(x^i a)^m = 1$ for some positive integer m. Hence $x^{im} \in A$, this is a contradiction and so

$n > 0$. Thus there is a positive integer m such that $[a,_n x^{i-j}]^m = 1$. Hence $[a,_n x^{i-j}] = 1$ as A is torsion-free. It follows by Lemma 8 that there exists a positive integer d such that $x^d \in A$, this is a contradiction and so G/A is a torsion group. Therefore G is abelian-by-finite, so by Lemma 7 G is torsion-by-abelian, a contradiction which completes the proof.

Proof of Corollary 2. Let k be a positive integer.

(i) If G is a finitely generated soluble group in $(\mathcal{TN}_k, \infty)$, then from Theorem 1, G is torsion-by-nilpotent. Thus G has a torsion subgroup T. Clearly G/T is in $(\mathcal{TN}_k, \infty)$, hence G/T being torsion-free is in $(\mathcal{N}_k, \infty)$. So by [2], $G/T \in \mathcal{FN}_k^{(2)}$. Consequently, $G \in \mathcal{TN}_k^{(2)}$, as required. It is easy to see that if G is in $\mathcal{TN}_k^{(2)}$, then it belongs to $(\mathcal{TN}_k, \infty)$.

(ii) If G is a finitely generated metabelian group in $(\mathcal{TN}_k, \infty)$, then as in (i) there is a torsion normal subgroup T such that G/T is a finitely generated metabelian group in $(\mathcal{N}_k, \infty)$. So by [2], $G/T \in \mathcal{FN}_k$. Thus $G \in \mathcal{TN}_k$, as required. The converse is obvious.

(iii) Let G be a finitely generated soluble group in the class $(\mathcal{TE}_k, \infty)$. Since soluble Engel groups are locally nilpotent [20, 12.3.3], G belongs to $(\mathcal{TN}, \infty)$. It follows, by Theorem 1, that G is torsion-by-nilpotent. Let T be the torsion subgroup of G. So G/T is a torsion-free group in the class $(\mathcal{TE}_k, \infty)$. We deduce that G/T is in $(\mathcal{E}_k, \infty)$. It follows, from [16], that G/T is in $\mathcal{FE}_k$. Thus G is in $\mathcal{TE}_k$. The converse is obvious.

Lemma 9. *Let G be a finitely generated soluble group in the class $(\mathcal{CN}, \infty)^*$. Then G is nilpotent-by-finite.*

Proof. Let G be a finitely generated soluble group in the class $(\mathcal{CN}, \infty)^*$. By [8, Corollary 2] G is nilpotent-by-finite if, and only if, for each 2-generator subgroup H, the factor group H/H'' is nilpotent-by-finite. It follows that we may assume G metabelian. Since $(\mathcal{CN}, \infty)^*$ is a quotient closed class of groups and finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [19, Lemma 6.17], that we may suppose that G is a just-non-(nilpotent-by-finite) group. In [13, Lemma 2.1] it is proved that the Fitting subgroup A of G is therefore abelian and either A is torsion-free, or it is an elementary abelian p-group of infinite rank for some prime p. Let $1 \neq a \in A$ and let xA be an element of infinite order in G/A. Then the subset $\{x^i a : i > 0\}$ is infinite. Hence there exist two positive integers i, j such that $\langle (x^i a)^{x^j a}, x^i a\rangle = \langle [x^j a, x^i a], x^i a\rangle$ is Chernikov-by-nilpotent. Using the facts that A is abelian and normal in G we have $[x^j a, x^i a] = [x^j, a][a, x^i] = [a, x^{-j}]^{x^j}[a, x^i] = [a, x^i x^{-j}]^{x^j} = [a^{x^j}, x^{i-j}]$. Set $H = \langle [a^{x^j}, x^{i-j}], x^i a\rangle$, then there is an integer $n \geq 0$ such that $\gamma_{n+1}(H)$ is a Chernikov group. On the other hand $\gamma_2(H)$ is contained in A as G is metabelian. If $n = 0$, then H is finite since Chernikov groups are locally finite. So $(x^i a)^m = 1$ for some positive integer m. Hence $x^{im} \in A$, this is a contradiction and so $n > 0$. It follows that $\gamma_{n+1}(H)$ is a Chernikov subgroup of A.

Suppose that A is torsion-free. Then $\gamma_{n+1}(H) = 1$ and hence $\big[[a^{x^j}, x^{i-j}],_n x^i a\big] = 1$, so $[a, x^{i-j},_n x^i] = 1$. By Lemma 8 there is, therefore, a positive integer d such that $x^d \in A$, and this contradicts the fact that xA is of infinite order.

It follows that we may assume that A is an elementary abelian p-group. So $\gamma_{n+1}(H)$ is a Chernikov and an elementary abelian p-group, hence finite. Thus H is finite-by-nilpotent, so H is nilpotent-by-finite. Therefore there exists a positive integer m such that $[[a^{x^j}, x^{i-j}],_{n+1} (x^i a)^m] = 1$, so $[a, x^{i-j},_{n+1} x^{im}] = 1$. This gives, by Lemma 8, that $x^d \in A$, for some positive integer d, a contradiction which completes the proof. □

Corollary 10. *Let G be a finitely generated soluble group. Then, $G \in (\mathcal{CN}, \infty)^*$ if and only if $G \in (\mathcal{FN}, \infty)^*$.*

Proof. Let G be a finitely generated soluble group in the class $(\mathcal{CN}, \infty)^*$. By Lemma 9, G is nilpotent-by-finite. So G satisfies max, the maximal condition on subgroups. Since Chernikov groups are locally finite, it follows that G is in the class $(\mathcal{FN}, \infty)^*$. □

Lemma 11. *Let G be a finitely generated abelian-by-finite group in the class $(\mathcal{FN}, \infty)^*$. Then G is finite-by-nilpotent.*

Proof. Let A be a normal abelian subgroup of finite index in G. Since G is finitely generated, we may assume that A is torsion-free. Let $x \in G$ and let $a \in A$ of infinite order. Then the subset $\{a^i x : i > 0\}$ is infinite. So there are two positive integers i, j such that $\langle [a^j x, a^i x], a^i x\rangle \in \mathcal{FN}$. Hence $\langle [a^{j-i}, x]^x, a^i x\rangle \in \mathcal{FN}$, and therefore $\langle [a^{j-i}, x], x a^i\rangle \in \mathcal{FN}$. Thus there exist two positive integers m, n such that $[a^{j-i}, x,_n x a^i]^m = [a, x,_n x a^i]^{(j-i)m} = [a, x,_n x]^{(j-i)m} = 1$. Since A is torsion-free, we obtain

$[a,_{n+1} x] = 1$. It follows that a is a right Engel element of G. Since G satisfies max, the set of its right Engel elements coïncides with a term of the upper central series [20, 12.3.7]. Hence $A \leq Z_k(G)$ for some integer $k > 0$. So $G/Z_k(G)$ is finite and this gives that G is finite-by-nilpotent [11]. □

Proof of Theorem 3. Let G be a finitely generated soluble in the class $(\mathcal{CN}, \infty)^*$. It follows, from Lemma 9 and Corollary 10, that G is a nilpotent-by-finite group in the class $(\mathcal{FN}, \infty)^*$. Then G satisfies max. It is proved in [9, Theorem 1.1] that if Ω is a class of groups which is closed under taking subgroups and quotients and if all metabelian groups of Ω are torsion-by-nilpotent, then all soluble groups of Ω are torsion-by-nilpotent. So, by taking Ω to be the class of groups in $(\mathcal{FN}, \infty)^*$ which satisfy max, we may assume G metabelian. Since G is a finitely generated nilpotent-by-finite group, there is a normal torsion-free subgroup H such that $H \in \mathcal{N}_c$ and $|G/H| = d$ for some positive integers c, d. We prove that $G \in \mathcal{FN}$ by induction on c. From Lemma 11, this is true if $c = 1$. Assume that $c > 1$. Clearly $G/\gamma_c(H) \in \mathcal{N}_{c-1}\mathcal{F}$, so by the inductive hypothesis we have that $G/\gamma_c(H) \in \mathcal{FN}$. Thus there are two positive integers m, n such that $(\gamma_{n+1}(G))^m \leq \gamma_c(H)$, so $[(\gamma_{n+1}(G))^m, H] = 1$. Now $\gamma_{n+1}(G)$ is abelian as G is metabelian. Hence $[(\gamma_{n+1}(G))^m, H] = [\gamma_{n+1}(G), H]^m = 1$, and this gives $[\gamma_{n+1}(G), H] = 1$ since H is torsion-free. It follows that $[H,_n G] \leq \gamma_c(H)$. It is proved in [9, Lemma 2.1] that if H, K are normal subgroups of a group G and if for some integer $n > 0$ we have $[H,_n G] \leq K$, then for any integer $c > 0$ we have $[\gamma_c(H),_{c(n-1)+1} G] \leq [K,_{c-1} H]$. By taking $K = \gamma_c(H)$, we obtain $[\gamma_c(H),_{c(n-1)+1} G] \leq [\gamma_c(H),_{c-1} H] \leq \gamma_{c+1}(H) = 1$. It follows that $[\gamma_c(H),_{c(n-1)+1} G] = 1$, and this means that $\gamma_c(H) \leq Z_{c(n-1)+1}(G)$. Since $G/\gamma_c(H) \in \mathcal{FN}$, then $G/Z_{c(n-1)+1}(G) \in \mathcal{FN}$, which implies that $G \in \mathcal{FN}$, as required.

Proof of Corollary 4. Let k be a positive integer and let G be a finitely generated soluble group in $(\mathcal{CN}_k, \infty)^*$. From Theorem 3, G is finite-by-nilpotent. Thus G contains a normal finite subgroup H such that G/H is nilpotent and finitely generated, so its torsion subgroup T/H is finite, and consequently T is finite. Clearly G/T is in $(\mathcal{CN}_k, \infty)^*$, so G/T, being torsion-free, is in $(\mathcal{N}_k, \infty)^*$. Since $(\mathcal{N}_k, \infty)^*$ is contained in $\mathcal{E}_{k+1}(\infty)$, we can deduce that:

(i) G/T is a finitely generated soluble group in $\mathcal{E}_{k+1}(\infty)$, so by [1, Theorem 3], there exists an integer $c = c(k)$, depending only on k, such that $(G/T)/Z_c(G/T)$ is finite. So, by [11, Theorem 1] we obtain that $\gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T$ is finite. Since T is finite, it follows that $\gamma_{c+1}(G)$ is finite. Thus by [11, 1.5] we get that $G/Z_c(G)$ is finite.

(ii) G/T is a finitely generated metabelian group in $\mathcal{E}_{k+1}(\infty)$, so by [1, Theorem 2], $(G/T)/Z_{k+1}(G/T)$ is finite. Hence by [11, Theorem 1] we obtain that $\gamma_{k+2}(G/T) = \gamma_{k+2}(G)T/T$ is finite. Since T is finite, it follows that $\gamma_{k+2}(G)$ is finite. So by [11, 1.5] we deduce that $G/Z_{k+1}(G)$ is finite.

Proof of Corollary 5. Note that if G is a finitely generated soluble group in the class $(\mathcal{CN}, \infty)$, then by Theorem 3 it satisfies max. Therefore Corollary 5 follows from Corollary 2 and the fact that finitely generated torsion soluble groups are finite.

References

[1] A. Abdollahi, Some Engel conditions on infinite subsets of certain groups, *Bull. Austral. Math. Soc.* **62** (2000), 141–148.

[2] A. Abdollahi and B. Taeri, A condition on finitely generated soluble groups, *Comm. Algebra* **27** (1999), 5633–5638.

[3] A. Abdollahi and N. Trabelsi, Quelques extensions d'un problème de Paul Erdős sur les groupes, *Bull. Belg. Math. Soc.* **9** (2002), 205–215.

[4] C. Delizia, On certain residually finite groups, *Comm. Algebra* **24** (1996), 3531–3535.

[5] C. Delizia, A. H. Rhemtulla and H. Smith, Locally graded groups with a nilpotence condition on infinite subsets, *J. Austral. Math. Soc. (series A)* **69** (2000), 415–420.

[6] G. Endimioni, Groups in which certain equations have many solutions, *Rend. Sem. Mat. Univ. Padova* **106** (2001), 77–82.

[7] G. Endimioni, Groups covered by finitely many nilpotent subgroups, *Bull. Austral. Math. Soc.* **50** (1994), 459–464.

[8] G. Endimioni, A characterization of nilpotent-by-finite groups in the class of finitely generated soluble groups, *Comm. in Algebra* **25** (1997), 1159–1168.

[9] G. Endimioni and G. Traustasson, On torsion-by-nilpotent groups, *J. Algebra* **241** (2001), 669–676.

[10] J. R. J. Groves, A conjecture of Lennox and Wiegold concerning supersoluble groups, *J. Austral. Math. Soc. Ser. A* **35** (1983), 218–220.

[11] P. Hall, Finite-by-nilpotent groups, *Proc. Cambridge Philos. Soc.* **52** (1956), 611–616.

[12] P. Hall, Finiteness conditions for soluble groups, *Proc. Lond. Math. Soc.* **4** (1954), 419–436.

[13] E. Hrushovski, P. H. Kropholler, A. Lubotzky and A. Shalev, Powers in finitely generated groups, *Trans. Amer. Math. Soc.* **348** (1996), 291–304.

[14] J. C. Lennox and J. E. Roseblade, Centrality in finitely generated soluble groups, *J. Algebra* **16** (1970), 399–435.

[15] J. C. Lennox and J. Wiegold, Extensions of a problem of Paul Erdős on groups, *J. Austral. Math. Soc. Ser. A* **31** (1981), 459–463.

[16] P. Longobardi, On locally graded groups with an Engel condition on infinite subsets, *Arch. Math.* **76** (2001), 88–90.

[17] P. Longobardi and M. Maj, Finitely generated soluble groups with an Engel condition on infinite subsets, *Rend. Sem. Mat. Univ. Padova* **89** (1993), 97–102.

[18] B. H. Neumann, A problem of Paul Erdős on groups, *J. Austral. Math. Soc. Ser. A* **21** (1976), 467–472.

[19] D. J. S. Robinson, Finiteness conditions and generalized soluble groups, *Springer-Verlag, Berlin, Heidelberg, New York*, 1972.

[20] D. J. S. Robinson, A course in the theory of groups, *Springer-Verlag, Berlin, Heidelberg, New York*, 1982.

[21] B. Taeri, A question of Paul Erdős and nilpotent-by-finite groups, *Bull. Austral. Math. Soc.* **64** (2001), 245–254.

[22] N. Trabelsi, Soluble groups with a condition on infinite subsets, *Algebra Colloq.* **9:4** (2002), 427–432.

[23] N. Trabelsi, Centre-by-metabelian groups with a condition on infinite subsets, *Publ. Mat.* **47** (2003), 451–457.

NADIR TRABELSI
DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES
UNIVERSITÉ FERHAT ABBAS
SÉTIF 19000
ALGÉRIE

E-mail: trabelsi_dz@yahoo.fr

(Received September 18, 2003; revised March 16, 2004)