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Semihomogeneous topological spaces

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Abstract. We introduce semihomogeneity as a generalization of homogeneity. Several results are included discussing some relations between the semihomogeneity of a space and some generated topologies. Various counter examples relevant to the relations are given. We study semihomogeneous components which form a partition of any space and we study some of their properties. Finally, we introduced a product theorem concerning semihomogeneous spaces.

1. Introduction

Throughout this paper by a space we mean a topological space. As defined by Sierpiński in 1920, a space (X,τ) is *homogeneous* [20] if for any two points $x, y \in X$ there exists a homeomorphism $f : (X,\tau) \to (X,\tau)$ such that $f(x) = y$. Afterwards, various types of homogeneity were studied by [2]–[5], [9]–[11], [16], [18] and others. In [1], [14], we fuzzified some types of homogeneity. In this paper we study semihomogeneity as a generalization of homogeneity. Let $\backsim$ (see [13]) be a relation defined on X by $x \backsim y$ if there is a homeomorphism $f : (X,\tau) \to (X,\tau)$ such that $f(x) = y$. This relation turns out to be an equivalence relation on X whose equivalence classes C_x will be called *homogeneous components* determined by $x \in X$. Homogeneous components preserved under homeomorphisms and indeed

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homogeneous subspaces of X. It is clear that (X, τ) is homogeneous iff it has only one homogeneous component.

Let (X, τ) be a space and $A \subseteq X$. We will denote the complement of A in X, the closure of A, the interior of A and the relative topology on A by $X - A = A^c$, $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and τ_A respectively. A is called *regular open* if $\mathrm{Int}(\mathrm{Cl}(A)) = A$. A is *semiopen* [15] if there exists an open set O such that $O \subseteq A \subseteq \mathrm{Cl}(O)$, this is equivalent to say that $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. $SO(X, \tau)$ will denote the family of all semiopen sets in X. The topology on X with the subbase $SO(X, \tau)$ is called *the semi open set generated topology of* (X, τ) [19] and is denoted by τ_ψ. The complement of a semiopen set is called *semiclosed* [6]. The *semiclosure* of A [6], denoted by $s\,\mathrm{Cl}(A)$, is the smallest semiclosed set that contains A. The *semiinterior* of A [6], denoted by $\mathrm{sInt}(A)$, is the largest semiopen set contained in A. A is called an *α-set* [17] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The family of all *$\alpha$-sets* in a space (X, τ), is denoted by τ^α is again a topology on X which is finer than τ. A space (X, τ) is *extremally disconnected* [22] if for each $O \in \tau$, $\mathrm{Cl}(O) \in \tau$. A space (X, τ) is *semiregular* [23] if for each $O \in \tau$ and $x \in O$, there exists a regular open set R such that $x \in R \subseteq O$, i.e., the class of all regular open sets form a base for the topology τ. As a generalization of semiregularity Sivaraj [21] defined *s-semiregular* spaces as follows. A space (X, τ) is *s-semiregular* if (X, τ) is semiregular at all points which are semiclosed in X. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is *semicontinuous* [15] if $f^{-1}(V) \in SO(X, \tau_1)$ for all $V \in \tau_2$. f is *irresolute* [7] if $f^{-1}(A) \in SO(X, \tau_1)$ for all $A \in SO(Y, \tau_2)$. f is *presemiopen* [7] if $f(A) \in SO(Y, \tau_2)$ for all $A \in SO(X, \tau_1)$. f is *semihomeomorphism* [7] if f is bijective, irresolute and presemiopen.

Throughout this paper, R, N will denote the set of real numbers and the set of natural numbers respectively. Let X be any set. By $\tau_{\mathrm{disc}}, \tau_{\mathrm{ind}}$, $\tau_{\mathrm{l.r}}$, and $\tau_{\mathrm{r.r}}$ (in the case $X = R$), we mean the discrete, the indiscrete, the left ray, and the right ray topologies, respectively. By $P(X)$ we mean the power set of X.

The following sequence of propositions will be useful in the sequel.

Proposition 1.1 ([21]). *Every homeomorphism is a semihomeomorphism but not conversely.*

Proposition 1.2 ([21]). *If (X, τ_1) and (Y, τ_2) are s-semiregular spaces, then every semihomeomorphism $f : (X, \tau_1) \to (Y, \tau_2)$ is a homeomorphism.*

Proposition 1.3 ([21]). *A function $f : (X, \tau) \to (X, \tau)$ is a semihomeomorphism iff $f : (X, \tau^\alpha) \to (X, \tau^\alpha)$ is a homeomorphism.*

Proposition 1.4 ([17]). *For any space (X, τ), $(\tau^\alpha)^\alpha = \tau^\alpha$.*

Proposition 1.5 ([8]). *Let (X, τ) be a space. Then $(X, (\tau_\psi)_\psi)$ is extremally disconnected and all finite repetitions of the semi open set generated topology process starting with $(X, (\tau_\psi)_\psi)$ gives $(X, (\tau_\psi)_\psi)$.*

Proposition 1.6 ([17]). *A space (X, τ) is an extremally disconnected iff $SO(X, \tau)$ is a topology on X.*

Proposition 1.7. *If (X, τ) is an extremally disconnected space and $A \in SO(X, \tau_\psi)$, then there exists $U \in \tau_\psi$ such that $U \subseteq A \subseteq s\,\mathrm{Cl}(U)$.*

(Follows from Proposition 1.6.)

2. Semihomogeneous spaces

Definition 2.1. A space (X, τ) is said to be semihomogeneous if for any two points $x, y \in X$ there exists a semihomeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(x) = y$.

Theorem 2.2. *Every homogeneous space is a semihomogeneous.*

Proof. Proposition 1.1. □

The following example shows that the converse of Theorem 2.2 is not true. It also shows that extremally disconnected semihomogeneous space need not to be a homogeneous space.

Example 2.3. The subspace $(N, \tau_{r.r})$ of the right ray topology is a semihomogeneous space that is not homogeneous.

Proof. It is easy to see that $V \in SO(N, \tau_{r.r}) - \{\varnothing\}$ iff V contains a non empty open set. Let $n, m \in N$ then it is not difficult to see that the function $f : (N, \tau_{r.r}) \to (N, \tau_{r.r})$ where $f(n) = m$, $f(m) = n$ and $f(x) = x$ for all $x \notin \{n, m\}$ is a semihomeomorphism. Therefore, N

is semihomogeneous. On the other hand, if $f : N \to N$ is a bijection for which $f(1) = 2$, then $f(N - \{1\}) = N - \{2\}$, so f is not a homeomorphism. Hence $(N, \tau_{r.r})$ is not homogeneous. □

Theorem 2.4. *If (X, τ) is s-semiregular space, then (X, τ) is homogeneous iff (X, τ) is a semihomogeneous space.*

Proof. Proposition 1.2. □

Theorem 2.5. *Let (X, τ) be a space. Then the following are equivalent.*

(i) (X, τ) *is semihomogeneous.*

(ii) (X, τ^α) *is homogeneous.*

(iii) (X, τ^α) *is semihomogeneous.*

Proof. Propositions 1.3 and 1.4. □

Lemma 2.6. *If $f : (X, \tau_1) \to (Y, \tau_2)$ is a semihomeomorphism, then $f : (X, (\tau_1)_\psi) \to (Y, (\tau_2)_\psi)$ is a homeomorphism.*

Proof. Let B be a basic open set in $(Y, (\tau_2)_\psi)$, then $B = \bigcap_{i=1}^n A_i$ where $A_i \in SO(Y, \tau_2)$, then $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(A_i) \in (\tau_1)_\psi$ since $f^{-1}(A_i) \in SO(X, \tau_1)$ for all $i = 1, 2, \ldots, n$. Therefore, $f : (X, (\tau_1)_\psi) \to (Y, (\tau_2)_\psi)$ is a continuous function. Similarly, we can show that $f : (X, (\tau_1)_\psi) \to (Y, (\tau_2)_\psi)$ is an open function. □

Theorem 2.7. *Given a space (X, τ) and consider the following statements.*

(a) (X, τ) *is semihomogeneous.*

(b) (X, τ_ψ) *is homogeneous.*

(c) (X, τ_ψ) *is semihomogeneous.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) Lemma 2.6.

(b) $\Rightarrow$ (c) Theorem 2.2. □

The following example shows in Theorem 2.7 that (b) $\not\Rightarrow$ (a).

Example 2.8. Let $X = R$ with the topology $\tau = \{U \subseteq R : 0 \notin U\} \cup \{U \subseteq R : 0 \in U \text{ and } R - U \text{ is finite}\}$. Then $\tau_\psi = \tau_{\text{disc}}$ and so (X, τ_ψ) is semihomogeneous. On the other hand, (X, τ) is not semihomogeneous since there is no semihomeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(1) = 0$.

About validity of the implication (c) $\Rightarrow$ (b) in Theorem 2.7, we raise the following question.

Question 2.9. Let (X, τ) be a space for which (X, τ_ψ) is semihomogeneous. Is it true that (X, τ_ψ) homogeneous?

Concerning Question 2.9, if τ is a semi open set generated topology, then we have the following result.

Theorem 2.10. *Let (X, τ) be a space where τ is a semi open set generated topology. If (X, τ_ψ) is semihomogeneous then (X, τ_ψ) homogeneous?*

PROOF. Theorem 2.7 and Proposition 1.5. □

If (X, τ) is extremally disconnected, then the three statements in Theorem 2.7 are equivalent as the following result says.

Theorem 2.11. *If (X, τ) is an extremally disconnected space, then the following are equivalent.*

(a) (X, τ) *is semihomogeneous.*

(b) (X, τ_ψ) *is homogeneous.*

(c) (X, τ_ψ) *is semihomogeneous.*

PROOF. According to Theorem 2.7, it is sufficient to show (c) $\Rightarrow$ (a). Suppose that (X, τ) is an extremally disconnected space with (X, τ_ψ) is semihomogeneous and let $x, y \in X$. Since (X, τ_ψ) is semihomogeneous, there exists a semihomeomorphism $f : (X, \tau_\psi) \to (X, \tau_\psi)$ such that $f(x) = y$. Let $A \in SO(X, \tau)$, then $A \in \tau_\psi$. Thus $A \in SO(X, \tau_\psi)$ and hence $f^{-1}(A) \in SO(X, \tau_\psi)$. So, by Proposition 1.7, there exists $U \in \tau_\psi$ such that $U \subseteq f^{-1}(A) \subseteq s\,\mathrm{Cl}(U) \subseteq \mathrm{Cl}(U)$. Now by Proposition 1.6, there exists $W \in \tau$ such that $W \subseteq U \subseteq \mathrm{Cl}(W)$. Therefore, $W \subseteq U \subseteq f^{-1}(A) \subseteq \mathrm{Cl}(U) \subseteq \mathrm{Cl}(W)$ which means that $f^{-1}(A) \in SO(X, \tau)$ and hence f is irresolute. Similarly we can show that f is presemiopen. Therefore, $f : (X, \tau) \to (X, \tau)$ is a semihomeomorphism which takes x to y and hence (X, τ) is semihomogeneous. $\square$

Definition 2.12. Let (X, τ) be a space. A non empty open subset $A \subseteq X$ is called a minimal open set in X if the relative topology on A, $\tau_A = \tau_{\text{ind}}$, i.e., A has no proper non empty open subset.

Definition 2.13. Let (X, τ) be a space. A non empty semiopen set $A \subseteq X$ is called a minimal semiopen set in X if whenever $B \in SO(X, \tau)$ and $\varnothing \subseteq B \subseteq A$, $B = \varnothing$ or $B = A$, i.e., A has no proper non empty semiopen subset.

The following lemmas will be needed in proving our next main result.

Lemma 2.14. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an injection, irresolute function. If A is a minimal semiopen set in X such that $f(A)$ is semiopen in Y, then $f(A)$ is a minimal semiopen subset of Y.*

Proof. Since A is a minimal semiopen set, $A \neq \varnothing$, so $f(A) \neq \varnothing$. Suppose that for some $B \in SO(Y, \tau_2) - \{\varnothing\}$, $B \subseteq f(A)$, then $f^{-1}(B) \subseteq f^{-1}(f(A))$. Since f is injective, $f^{-1}(f(A)) = A$. Also, since f is irresolute, $f^{-1}(B) \in SO(X, \tau_1)$. Since A is minimal semiopen, $f^{-1}(B) = A$. Thus, $f(A) = f(f^{-1}(B)) \subseteq B$ and hence $f(A) = B$. Therefore, $f(A)$ is a minimal semiopen subset of Y. $\square$

Lemma 2.15. *Let (X, τ) be a space and $A \subseteq X$. Then A is a minimal semiopen set iff A is a minimal open set.*

Proof. $\Rightarrow$) Since $\text{Int}(A) \subseteq A$ and $\text{Int}(A)$ is semiopen, $\text{Int}(A) = \varnothing$ or $\text{Int}(A) = A$. If $\text{Int}(A) = \varnothing$, then $A \subseteq \text{Cl}(\text{Int}(A)) = \text{Cl}(\varnothing) = \varnothing$ and so $A = \varnothing$, but $A \neq \varnothing$. Thus, $\text{Int}(A) = A$ and hence A is open.

To show that A is minimal open, suppose that $\varnothing \neq B \subseteq A$. Since B is semiopen and A is minimal semiopen we must have $A = B$. Hence A is a minimal open set.

$\Leftarrow$) Suppose that A is a minimal open set, then A is semiopen. To show that A is minimal semiopen, let B be a semiopen set for which $\varnothing \neq B \subseteq A$. Since $\text{Int}(B) \subseteq A$, $\text{Int}(B) = \varnothing$ or $\text{Int}(B) = A$. If $\text{Int}(B) = \varnothing$, then $B \subseteq \text{Cl}(\text{Int}(B)) = \text{Cl}(\varnothing) = \varnothing$. So $B = \varnothing$, but $B \neq \varnothing$. Thus, $\text{Int}(B) = A$, but $B \subseteq A$. Therefore, $B = A$. Hence A is a minimal semiopen set. $\square$

Lemma 2.16. *The semihomeomorphic image of a minimal open set is a minimal open set.*

Proof. Lemmas 2.14 and 2.15. □

Definition 2.17. Let (X, τ) be a space. A base for τ is called a partition base for τ if their elements form a partition on X.

Lemma 2.18. *Let (X, τ) be a space and let ß be a base for τ. Then ß is a partition base for τ iff their elements are minimal open subsets of X.*

Proof. Straightforward. □

Lemma 2.19. *Let (X, τ) be a semihomogeneous space and let G be the group of semihomeomorphisms from (X, τ) to itself. If A is a minimal open set in X, then ß $= \{h(A) : h \in G\}$ is a base for τ which consists of elements all of which is homeomorphic to one another.*

Proof. Let $h \in G$, then by Lemma 2.16, it follows that $h(A)$ is minimal open, also since $\tau_A = \tau_{\text{ind}}, \tau_{h(A)} = \tau_{\text{ind}}$, and the restriction $h|_A : A \to h(A)$ is a bijection, $A \cong h(A)$. Therefore, each element of ß is homeomorphic to any one another. To show that ß is a base for τ, let $x \in U \in \tau$ and choose $a \in A$. Since (X, τ) semihomogeneous, there exists $h \in G$ such that $h(a) = x$ and hence $x \in h(A)$. Therefore, since $h(A)$ is minimal open $h(A) \subseteq U$. Thus, ß is a base for τ. □

Proposition 2.20 ([12])**.** *Let (X, τ) be a space which contains a minimal open set. Then the following are equivalent.*

(a) *(X, τ) is a homogeneous space.*

(b) *(X, τ) has a partition base consisting of minimal open sets all of which is homeomorphic to one another.*

Now we are ready to state one of our main results.

Theorem 2.21. *Let (X, τ) be a space which contains a minimal open set. Then the following are equivalent.*

(a) *(X, τ) is a homogeneous space.*

(b) *(X, τ) is a semihomogeneous space.*

(c) *(X, τ) has a partition base consisting of minimal open sets all of which is homeomorphic to one another.*

Proof. (a) $\Rightarrow$ (b) Follows by Theorem 2.2.

(b) $\Rightarrow$ (c) Let A be a minimal open set and let ß $= \{h(A) : h \in G\}$ where G is the group of semihomeomorphisms from (X, τ) to itself. Then by Lemmas 2.18 and 2.19, it follows that ß is the required base.

(c) $\Rightarrow$ (a) Follows by Proposition 2.20. $\square$

Corollary 2.22. *If (X, τ) is a homogeneous space which contains a minimal open set, then every semihomeomorphism $f : (X, \tau) \to (Y, \tau)$ is a homeomorphism.*

Proof. By Theorem 2.21, it is not difficult to show that (X, τ) is a semiregular space. Therefore, by Proposition 1.2, we conclude that $f : (X, \tau) \to (X, \tau)$ is a homeomorphism. $\square$

The following example shows that the condition 'contains a minimal open set' on the space (X, τ) in Corollary 2.22 cannot be dropped. Moreover, it also shows that in a homogeneous space semihomeomorphisms need not to be homeomorphisms.

Example 2.23. Let $X = R$ with the topology $\tau = \tau_{l.r}$. For $p, q \in R$, define $f : (X, \tau) \to (X, \tau)$ by $f(x) = (x - p) + q$, then it is easy to see that f is a homeomorphism with $f(p) = q$. Therefore, (X, τ) is a homogeneous space. Let $g : (X, \tau) \to (X, \tau)$ be given by $g(0) = 1, g(1) = 0$ and $g(x) = x$ elsewhere. Then g is a semihomeomorphism but not a homeomorphism.

The following example shows that the condition 'homogeneous' on the space (X, τ) in Corollary 2.22 cannot be dropped.

Example 2.24. Let $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$, then the space (X, τ) contains a minimal open set. Define $f : (X, \tau) \to (X, \tau)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is a semihomeomorphism while f is not a homeomorphism.

3. Components and products

Definition 3.1. Let (X, τ) be a space. We define the relation $\widetilde{s}$ on X as follows: For $x, y \in X$, $x\widetilde{s}y$ iff there exists a semihomeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(x) = y$.

Theorem 3.2. *The above relation is an equivalence relation.*

Proof. Straightforward. □

According to Theorem 3.2, the above relation induces a partition on X into equivalence classes, and it leads us to the following definition.

Definition 3.3. A subset of a space (X, τ) which has the form $sC_x = \{y \in X : x\widetilde{s}y\}$ is called a semihomogeneous component determined by $x \in X$.

Remark 3.4. A space (X, τ) semihomogeneous iff it has exactly one component.

It is known that homogeneous components are preserved under homeomorphisms. The following result says that semihomogeneous components are preserved under semihomeomorphisms.

Theorem 3.5. *If $f : (X, \tau) \to (X, \tau)$ is a semihomeomorphism, then $f(sC_x) = sC_x$ for any semihomogeneous component sC_x.*

Proof. Let $y \in f(sC_x)$, then there exists $t \in sC_x$ such that $y = f(t)$ and so $t\widetilde{s}y$. Since $t \in sC_x, t\widetilde{s}x$. Therefore, $y\widetilde{s}x$ and hence $y \in sC_x$.

Conversely, let $y \in sC_x$, then $y\widetilde{s}x$. Choose t such that $f(t) = y$. Since f is a semihomeomorphism, $y\widetilde{s}t$. Therefore, $t \in sC_x$ and $f(t) = y$. □

Theorem 3.6. *Let (X, τ) be a space and let sC_x be any semihomogenous component of (X, τ). Then either sC_x is semiopen or $s\,\mathrm{Int}(sC_x) = \varnothing$.*

Proof. Suppose that $s\,\mathrm{Int}(sC_x) \neq \varnothing$ and let $y \in sC_x$. Choose $t \in s\,\mathrm{Int}(sC_x)$. Since $y, t \in sC_x$, then there exists a semihomeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(t) = y$. Therefore, $y \in f(s\,\mathrm{Int}(sC_x)) \subseteq f(sC_x) = sC_x$ by Theorem 3.5 and hence sC_x is semiopen. □

Theorem 3.7. *If A is a clopen semihomogeneous subspace of X such that $A \cap sC_x \neq \varnothing$, then $A \subseteq sC_x$.*

Proof. Let $a \in A$. Choose $y \in A \cap sC_x$. Since A is semihomogeneous, then there exists a semihomeomorphism $f : (A, \tau_A) \to (A, \tau_A)$ such that $f(a) = y$. Define $g : (X, \tau) \to (X, \tau)$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in X - A. \end{cases}$$

Then g is a semihomeomorphism and so $a\widetilde{s}y$. Since $y \in sC_x$, we conclude that $a \in sC_x$. □

Theorem 3.8. *If (X, τ) is a space and sC_x is a semihomogeneous component, then the subspace topology (sC_x, τ_{sC_x}) is semihomogeneous.*

Proof. Let $x_1, x_2 \in sC_x$, then there exist two semihomeomorphisms $f_1, f_2 : (X, \tau) \to (X, \tau)$ such that $f_1(x) = x_1$ and $f_2(x) = x_2$. Let f be the restriction of $f_2 \circ f_1^{-1}$ to sC_x. Then by Theorem 3.5, $f : (sC_x, \tau_{sC_x}) \to (sC_x, \tau_{sC_x})$ is a semihomeomorphism with $f(x_1) = x_2$. □

Theorem 3.9. *Let (X, τ) be a space and $x \in X$, then $C_x \subseteq sC_x$.*

Proof. Proposition 1.1. □

In Example 2.3, we showed that $sC_1 = N$. In fact, it is not difficult to show that $C_x = \{x\}$ for each $x \in N$. Therefore, the equality in Theorem 3.5 does not hold in general. Example 2.3 is also an example of a space which has one semihomogeneous component but infinitely many homogeneous components. However, every semihomogeneous component has a partition consisting of homogeneous components as the following result says.

Theorem 3.10. *Let (X, τ) be a space and $x \in X$, then $sC_x = \bigcup_{a \in sC_x} C_a$.*

Proof. Straightforward. □

The following result concerning the product of two homogeneous spaces is known and follows easily.

Lemma 3.11. *The product of two homogeneous spaces is again homogeneous.*

For semihomogeneous spaces we propose the following question.

From now on τ will denote the product topology of τ_1 and τ_2.

Question 3.12. If (X, τ_1) and (Y, τ_2) are semihomogeneous spaces, is it true that $(X \times Y, \tau)$ semihomogeneous?

We are going to solve Question 3.12 partially. For this reason we need the following two lemmas.

Lemma 3.13. *Let $f : (X, \tau_1) \to (X, \tau_1)$ and $g : (Y, \tau_2) \to (Y, \tau_2)$ be two semi continuous functions and Let $h : (X \times Y, \tau) \to (X \times Y, \tau)$ where $h(x, y) = (f(x), g(y))$. If (X, τ_1) and (Y, τ_2) are extremallly disconnected, then $h(\mathrm{Cl}(O)) \subseteq \mathrm{Cl}(h(O))$ whenever $O \in \tau$.*

Proof. Let $O \in \tau$ and $(x, y) \in h(\mathrm{Cl}(O))$. Let $U \times V$ be any basic open set in the product topology $X \times Y$ for which $(x, y) \in U \times V$. Let $(x_\circ, y_\circ) \in \mathrm{Cl}(O)$ such that $f(x_\circ) = x$ and $g(y_\circ) = y$. Since f and g are semi continuous, $f^{-1}(U) \in SO(X, \tau_1)$ and $g^{-1}(V) \in SO(Y, \tau_2)$ and so there are $U_\circ \in \tau_1$ and $V_\circ \in \tau_2$ such that $U_\circ \subseteq f^{-1}(U) \subseteq \mathrm{Cl}(U_\circ)$ and $V_\circ \subseteq g^{-1}(V) \subseteq \mathrm{Cl}(V_\circ)$. Thus, $(x_\circ, y_\circ) \in \mathrm{Cl}(U_\circ) \times \mathrm{Cl}(V_\circ)$. Since (X, τ_1) and (Y, τ_2) are extremally disconnected spaces, $\mathrm{Cl}(U_\circ \times V_\circ) = \mathrm{Cl}(U_\circ) \times \mathrm{Cl}(V_\circ) \in \tau$. Since $(x_\circ, y_\circ) \in \mathrm{Cl}(O)$, $O \cap \mathrm{Cl}(U_\circ \times V_\circ) \neq \varnothing$ and hence $O \cap (U_\circ \times V_\circ) \neq \varnothing$. Therefore, $O \cap h^{-1}(U \times V) = O \cap (f^{-1}(U) \times g^{-1}(V)) \neq \varnothing$ and hence $h(O) \cap (U \times V) \neq \varnothing$. This completes the proof. □

Lemma 3.14. *Let $f : (X, \tau_1) \to (X, \tau_1)$ and $g : (Y, \tau_2) \to (Y, \tau_2)$ be two semihomeomorphisms and Let $h : (X \times Y, \tau) \to (X \times Y, \tau)$ where $h(x, y) = (f(x), g(y))$. If (X, τ_1) and (Y, τ_2) are extremally disconnected, then h is a semihomeomorphism.*

Proof. Let $G \in SO(X \times Y, \tau)$. Then there exists $O \in \tau$ such that $O \subseteq G \subseteq \mathrm{Cl}(O)$ and so $h^{-1}(O) \subseteq h^{-1}(G) \subseteq h^{-1}(\mathrm{Cl}(O))$. It is not difficult to see that h is a semi continuous function and so $h^{-1}(O) \in SO(X \times Y, \tau)$. Thus, there exists $W \in \tau$ such that $W \subseteq h^{-1}(O) \subseteq \mathrm{Cl}(W)$. But by Lemma 3.13, $h^{-1}(\mathrm{Cl}(O)) \subseteq \mathrm{Cl}(h^{-1}(O))$. Therefore, $W \subseteq h^{-1}(O) \subseteq h^{-1}(G) \subseteq \mathrm{Cl}(h^{-1}(O)) = \mathrm{Cl}(W)$. This completes the proof that h is an irresolute. Similarly we can show that the function is h^{-1} irresolute. □

Theorem 3.15. *The product of two semihomogeneous extremally disconnected spaces is again semihomogeneous.*

Proof. Let (X, τ_1) and (Y, τ_2) be two semihomogeneous spaces and let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Since (X, τ_1) and (Y, τ_2) are semihomogeneous, there are two semihomeomorphisms $f : (X, \tau_1) \to (X, \tau_1)$ and $g : (Y, \tau_2) \to (Y, \tau_2)$ such that $f(x_1) = x_2$ and $g(y_1) = y_2$. Define $h : (X \times Y, \tau) \to (X \times Y, \tau)$ by $h(x, y) = (f(x), g(y))$. Then $h(x_1, y_1) = (x_2, y_2)$ and by Lemma 3.14, h is a semihomeomorphism. This proves that $(X \times Y, \tau)$ is semihomogeneous. □

Note that Example 2.3 shows that Theorem 3.15, does not follow from Lemma 3.11 directly.

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