Publ. Math. Debrecen
67/1-2 (2005), 145–154

On weakly symmetric spaces with semi-symmetric metric connection

By S. AYNUR UYSAL (Istanbul) and R. ÖZLEM LALEOĞLU (Istanbul)

Abstract. The notions of weakly symmetric and weakly projective symmetric spaces were introduced by TAMÁSSY and BINH [1] and an example of the modified form of weakly symmetric Riemannian spaces was constructed by U. C. DE and S. BANDYOPADHYAY, [2]. The object of this paper is to introduce the modified form of weakly symmetric spaces with semi-symmetric metric connection with an illustrative example.

1. Introduction

A non-flat Riemannian space V_n $(n > 2)$ is called weakly symmetric if its Riemannian curvature tensor R_{hijk} satisfies the condition

$$\nabla_l R_{hijk} = a_l R_{hijk} + b_h R_{lijk} + c_i R_{hljk} + d_j R_{hilk} + e_k R_{hijl}, \tag{1.1}$$

where a b, c, d, e are 1-forms (non-zero simultaneously) and '∇_l' denotes the covariant differentiation with respect to the Riemannian connection ∇.

The one-forms a, b, c, d, e are called associated one-forms of the space and an n-dimensional space of this kind is denoted by $(WS)_n$. In [2], a reduction in the defining equation of a $(WS)_n$ is obtained in the following simpler form:

$$\nabla_l R_{hijk} = a_l R_{hijk} + b_h R_{lijk} + b_i R_{hljk} + d_j R_{hilk} + d_k R_{hijl}.$$

Mathematics Subject Classification: 53B05.
Key words and phrases: weakly symmetric space, semi-symmetric metric connection.

Let D denote the semi-symmetric metric connection over V_n with coefficients

$$\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + w_j \delta^i_k - w_k \delta^i_j, \quad D_k g_{ij} = 0, \tag{1.2}$$

where 'D_k' denotes the covariant differentiation with respect to the semi-symmetric metric connection.

The curvature tensor L^h_{ijk} of the manifold V_n is defined by

$$L^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^a_{ik} \Gamma^h_{aj} - \Gamma^a_{ij} \Gamma^h_{ak}, \quad \left(\partial_k = \frac{\partial}{\partial x^k} \right). \tag{1.3}$$

Substituting (1.2) in (1.3) we obtain the following equation for the curvature tensor L^h_{ijk} of V_n with semi-symmetric metric connection:

$$L^h_{ijk} = R^h_{ijk} - \delta^h_i (P_{jk} - P_{kj}) + \delta^h_k P_{ji} - \delta^h_j P_{ki}, \tag{1.4}$$

where $P_{jk} = \nabla_j w_k - w_j w_k$ and R^h_{ijk} is the Riemannian curvature tensor of the space.

Multiplying (1.4) by g_{ah} and interchanging the indices a and h, the above equation can be converted into the covariant form

$$L_{hijk} = R_{hijk} - g_{ih}(P_{jk} - P_{kj}) + g_{kh} P_{ji} - g_{jh} P_{ki}. \tag{1.5}$$

Interchanging the indices j and k and taking $k = j$ in (1.5) leads to the identities

$$L_{hijk} = -L_{hikj}, \tag{1.6}$$

and

$$L_{hijj} = 0 \tag{1.7}$$

respectively. An n-dimensional, $(n > 2)$, weakly symmetric space with semi-symmetric metric connection, $((WS)_n, D)$ for short, is a non-flat space satisfying the condition

$$D_l L_{hijk} = a_l L_{hijk} + b_h L_{lijk} + c_i L_{hljk} + d_j L_{hilk} + e_k L_{hijl}, \tag{1.8}$$

and it has the coefficients (1.2), where L_{hijk} is the curvature tensor of the space and a, b, c, d, e are 1-forms (non-zero simultaneously). By using the following method, which is also used in [2], we come to the conclusion that the five associated 1-forms cannot be all different. Moreover, we can state that the associated one forms d and e are identically equal to each other.

Interchanging the indices j and k in (1.8) we obtain

$$D_l L_{hikj} = a_l L_{hikj} + b_h L_{likj} + c_i L_{hlkj} + d_k L_{hilj} + e_j L_{hikl}. \qquad (1.9)$$

Now, adding (1.8) and (1.9) and using (1.6), we get

$$(d_j - e_j) L_{hilk} + (d_k - e_k) L_{hilj} = 0$$

or

$$A_j L_{hilk} + A_k L_{hilj} = 0 \qquad (1.10)$$

where $A_j = d_j - e_j$. We want to show that $A_j = 0$ $(j = 1, 2, \ldots, n)$. Suppose on the contrary there exists a fixed index q for which $A_q \neq 0$. Putting $j = l = q$ in (1.10), with the help of (1.7) we have $A_q L_{hiqk} = 0$, which implies that $L_{hiqk} = 0$ for all h, i, k. Next, putting $k = q$ in (1.10) we obtain $A_j L_{hilq} + A_q L_{hilj} = 0$ which means that $L_{hilj} = 0$ for all h, i, j, l, since $L_{hiqk} = 0$ for all h, i, k and $A_q \neq 0$. Then the space is flat, contradicting our hypothesis. Hence $A_j = 0$ for all j, which implies that $d_j = e_j$ for all j. Now in virtue of the above process we can state that the condition (1.9) can always be expressed in the following form:

$$D_l L_{hijk} = a_l L_{hijk} + b_h L_{lijk} + c_i L_{hljk} + d_j L_{hilk} + d_k L_{hijl}. \qquad (1.11)$$

2. An example of weakly symmetric spaces with semi-symmetric metric connection

In this section, we construct an example of weakly symmetric spaces with semi-symmetric metric connection.

We define the metric g in $((WS)_n, D)$ by the formula, [2], [3]

$$ds^2 = \varphi (dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \quad (n \geq 4) \qquad (2.1)$$

where $[k_{\alpha\beta}]$ is a symmetric non-singular matrix consisting of constants. The function φ, which is independent of x^n, will be determined with some assumptions such that the condition (1.11) is satisfied. Here and throughout this section each Latin index runs over $1, 2, \ldots, n$ and each Greek index runs over $2, 3, \ldots, n-1$.

Also we define the vector components w_h, which are contained in the formula of coefficients of the connection D, as

$$w_h = \begin{cases} \psi(x^1), & \text{for } h = 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

where ψ is a continuous function of x^1 defined on the interval $I = [a, b]$, [5]. The function ψ will be determined precisely such that the condition (1.11) is satisfied for certain values of the associated 1-forms a, b, c, d for the $((WS)_n, D)$ space, which has the metric (2.1). Since the above process is equivalent to determining the metric (2.1) and the coefficients of the connection completely, the construction of our example will be completed.

Due to the metric (2.1), the only non-vanishing components of the Christoffel symbols and the Riemannian curvature tensors R_{hijk} are the followings, [4]:

$$\left\{ \begin{array}{c} \beta \\ 11 \end{array} \right\} = -\frac{1}{2}k^{\beta\alpha}\varphi_{.\alpha}, \quad \left\{ \begin{array}{c} n \\ 11 \end{array} \right\} = \frac{1}{2}\varphi_{.1}, \quad \left\{ \begin{array}{c} n \\ 1\alpha \end{array} \right\} = \frac{1}{2}\varphi_{.\alpha}$$
$$\text{and} \qquad R_{1\alpha\beta 1} = \frac{1}{2}\varphi_{.\alpha\beta} \tag{2.3}$$

respectively, where (.) denotes the partial differentiation with respect to coordinates, and $[k^{\alpha\beta}]$ is the inverse matrix of $[k_{\alpha\beta}]$.

With the assumption (2.2) we obtain the non-zero components of Γ^h_{ij} and P_{ij} as follows:

$$\Gamma^{\beta}_{11} = \left\{ \begin{array}{c} \beta \\ 11 \end{array} \right\} + w_1\delta^{\beta}_1 - w_1\delta^{\beta}_1 = \left\{ \begin{array}{c} \beta \\ 11 \end{array} \right\}$$

$$\Gamma^{\beta}_{1\alpha} = \left\{ \begin{array}{c} \beta \\ 1\alpha \end{array} \right\} + w_1\delta^{\beta}_{\alpha} - w_{\alpha}\delta^{\beta}_1 = \begin{cases} \psi(x^1), & \text{for } \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

$$\Gamma^{\beta}_{\alpha 1} = \left\{ \begin{array}{c} \beta \\ \alpha 1 \end{array} \right\} + w_{\alpha}\delta^{\beta}_1 - w_1\delta^{\beta}_{\alpha} = \begin{cases} -\psi(x^1), & \text{for } \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

$$\Gamma^{n}_{11} = \left\{ \begin{array}{c} n \\ 11 \end{array} \right\} + w_1\delta^{n}_1 - w_1\delta^{n}_1 = \left\{ \begin{array}{c} n \\ 11 \end{array} \right\}$$

$$\Gamma^n_{1\alpha} = \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} + w_1\delta^n_\alpha - w_\alpha\delta^n_1 = \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\}$$

$$\Gamma^n_{1n} = \left\{ \begin{matrix} n \\ 1n \end{matrix} \right\} + w_1\delta^n_n - w_n\delta^n_1 = \psi(x^1)$$

$$\Gamma^n_{\alpha 1} = \left\{ \begin{matrix} n \\ \alpha 1 \end{matrix} \right\} + w_\alpha\delta^n_1 - w_1\delta^n_\alpha = \left\{ \begin{matrix} n \\ \alpha 1 \end{matrix} \right\}$$

$$\Gamma^n_{n1} = \left\{ \begin{matrix} n \\ n1 \end{matrix} \right\} + w_n\delta^n_1 - w_1\delta^n_n = -\psi(x^1). \tag{2.4}$$

$$P_{11} = \nabla_1 w_1 - w_1^2 = \frac{\partial w_1}{\partial x^1} - w_a\left\{ \begin{matrix} a \\ 11 \end{matrix} \right\} - w_1^2 = \psi'(x^1) - \psi^2(x^1). \tag{2.5}$$

Also one can easily show that the only non-zero component of $D_l P_{11}$ is

$$D_1 P_{11} = \psi''(x^1) - 2\psi\psi'(x^1). \tag{2.6}$$

For the metric (2.1), if we consider $k_{\alpha\beta}$ as $\delta_{\alpha\beta}$ and $\varphi = k_{\alpha\beta}x^\alpha x^\beta e^{x^1} e^{-\int_a^{x^1} 2\psi(t)dt}$, $x^1 \in I = [a,b]$, we obtain

$$\varphi = \sum_{\alpha=2}^{n-1} x^\alpha x^\alpha e^{x^1} e^{-\int_a^{x^1} 2\psi(t)dt}, \qquad x^1 \in I. \tag{2.7}$$

Hence

$$\varphi_{.\alpha\alpha} = 2e^{x^1} e^{-\int_a^{x^1} 2\psi(t)dt} \quad \text{and} \quad \varphi_{.\alpha\beta} = 0\ldots \quad \text{for } \alpha \neq \beta. \tag{2.8}$$

It follows from (2.3) and (2.8) that the only non-zero components of R_{hijk} are

$$R_{1\alpha\alpha 1} = e^{x^1} e^{-\int_a^{x^1} 2\psi(t)dt}. \tag{2.9}$$

Also, by means of (2.4) and (2.9) it can be easily shown that the only non-zero components of $D_l R_{hijk}$ are

$$D_1 R_{1\alpha\alpha 1} = e^{x^1} e^{-\int_a^{x^1} 2\psi(t)dt}, \tag{2.10}$$

which are equal to $R_{1\alpha\alpha 1}$.

We get the non-zero components of the curvature tensor L_{hijk} and their covariant derivatives with respect to the connection D as follows, [5]:

$$
\begin{aligned}
L_{111n} &= R_{111n} - g_{11}(P_{1n} - P_{n1}) + g_{n1}P_{11} - g_{11}P_{n1} = P_{11} \\
L_{11n1} &= -L_{111n} = -P_{11} \\
L_{1\alpha 1\alpha} &= R_{1\alpha 1\alpha} - g_{\alpha 1}(P_{1\alpha} - P_{\alpha 1}) + g_{\alpha 1}P_{1\alpha} - g_{11}P_{\alpha\alpha} = R_{1\alpha 1\alpha} \\
L_{1\alpha\alpha 1} &= -L_{1\alpha 1\alpha} = -R_{1\alpha 1\alpha} = R_{1\alpha\alpha 1} \\
L_{\alpha 11\alpha} &= R_{\alpha 11\alpha} - g_{1\alpha}(P_{1\alpha} - P_{\alpha 1}) + g_{\alpha\alpha}P_{11} - g_{1\alpha}P_{\alpha 1} = R_{\alpha 11\alpha} + P_{11} \\
L_{\alpha 1\alpha 1} &= -L_{\alpha 11\alpha} = -R_{\alpha 11\alpha} - g_{\alpha\alpha}P_{11} = R_{\alpha 1\alpha 1} - P_{11},
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
D_k L_{111n} &= D_k P_{11} = \begin{cases} D_1 P_{11}, & k = 1 \\ 0, & k \neq 1 \end{cases} \\
D_k L_{11n1} &= -D_k P_{11} = \begin{cases} -D_1 P_{11}, & k = 1 \\ 0, & k \neq 1 \end{cases} \\
D_k L_{1\alpha 1\alpha} &= D_k R_{1\alpha 1\alpha} = \begin{cases} D_1 R_{1\alpha 1\alpha}, & k = 1 \\ 0, & k \neq 1 \end{cases} \\
D_k L_{1\alpha\alpha 1} &= D_k R_{1\alpha\alpha 1} = \begin{cases} D_1 R_{1\alpha\alpha 1}, & k = 1 \\ 0, & k \neq 1 \end{cases} \\
D_k L_{\alpha 11\alpha} &= D_k R_{\alpha 11\alpha} + D_k P_{11} = \begin{cases} D_1 R_{\alpha 11\alpha} + D_1 P_{11}, & k = 1 \\ 0, & k \neq 1 \end{cases} \\
D_k L_{\alpha 1\alpha 1} &= D_k R_{\alpha 1\alpha 1} - D_k P_{11} = \begin{cases} D_1 R_{\alpha 1\alpha 1} - D_1 P_{11}, & k = 1 \\ 0, & k \neq 1. \end{cases}
\end{aligned}
\tag{2.12}
$$

Since $D_l R_{1\alpha\alpha 1} = 0$ and $D_l P_{11} = 0$ for $l \neq 1$, we observe that $D_l L_{hijk} = 0$ for $l \neq 1$.

$$\text{Let} \quad a_i = \begin{cases} \frac{1}{3}, & i = 1 \\ 0, & i \neq 1 \end{cases}, \quad b_i = c_i = 0, \ \forall i, \quad d_i = \begin{cases} \frac{2}{3}, & i = 1 \\ 0, & i \neq 1. \end{cases} \tag{2.13}$$

In order to verify the condition (1.8), it is sufficient to check the relations:

(A) $D_n L_{1111} = a_n L_{1111} + b_1 L_{n111} + c_1 L_{1n11} + d_1 L_{11n1} + d_1 L_{111n}$

(B) $D_\alpha L_{111\alpha} = a_\alpha L_{111\alpha} + b_1 L_{\alpha 11\alpha} + c_1 L_{1\alpha 1\alpha} + d_1 L_{11\alpha\alpha} + d_\alpha L_{111\alpha}$

(C) $D_1 L_{111n} = a_1 L_{111n} + b_1 L_{111n} + c_1 L_{111n} + d_1 L_{111n} + d_n L_{1111}$

(D) $D_\alpha L_{11\alpha 1} = a_\alpha L_{11\alpha 1} + b_1 L_{\alpha 1\alpha 1} + c_1 L_{1\alpha\alpha 1} + d_\alpha L_{11\alpha 1} + d_1 L_{11\alpha\alpha}$

(E) $D_1 L_{11n1} = a_1 L_{11n1} + b_1 L_{11n1} + c_1 L_{11n1} + d_n L_{1111} + d_1 L_{11n1}$

(F) $D_1 L_{11nn} = a_1 L_{11nn} + b_1 L_{11nn} + c_1 L_{11nn} + d_n L_{111n} + d_n L_{11n1}$

(G) $D_\alpha L_{1\alpha 11} = a_\alpha L_{1\alpha 11} + b_1 L_{\alpha\alpha 11} + c_\alpha L_{1\alpha 11} + d_1 L_{1\alpha\alpha 1} + d_1 L_{1\alpha 1\alpha}$

(H) $D_1 L_{1\alpha 1\alpha} = a_1 L_{1\alpha 1\alpha} + b_1 L_{1\alpha 1\alpha} + c_\alpha L_{111\alpha} + d_1 L_{1\alpha 1\alpha} + d_\alpha L_{1\alpha 11}$

(I) $D_1 L_{1\alpha\alpha 1} = a_1 L_{1\alpha\alpha 1} + b_1 L_{1\alpha\alpha 1} + c_\alpha L_{11\alpha 1} + d_\alpha L_{1\alpha 11} + d_1 L_{1\alpha\alpha 1}$

(J) $D_\alpha L_{\alpha 111} = a_\alpha L_{\alpha 111} + b_\alpha L_{\alpha 111} + c_1 L_{\alpha\alpha 11} + d_1 L_{\alpha 1\alpha 1} + d_1 L_{\alpha 11\alpha}$

(K) $D_1 L_{\alpha 11\alpha} = a_1 L_{\alpha 11\alpha} + b_\alpha L_{111\alpha} + c_1 L_{\alpha 11\alpha} + d_1 L_{\alpha 11\alpha} + d_\alpha L_{\alpha 111}$

(L) $D_1 L_{\alpha 1\alpha 1} = a_1 L_{\alpha 1\alpha 1} + b_\alpha L_{11\alpha 1} + c_1 L_{\alpha 1\alpha 1} + d_\alpha L_{\alpha 111} + d_1 L_{\alpha 1\alpha 1}$.

With these choices of a_i, b_i, c_i and d_i as in any other case, the components of L_{hijk} and $D_l L_{hijk}$ vanish identically, and the relation (1.11) holds trivially. From the equations (C) and (E) we get

$$D_1 P_{11} = (a_1 + b_1 + c_1 + d_1) P_{11}, \tag{2.14}$$

from (H) and (I) we have

$$D_1 R_{1\alpha 1\alpha} = (a_1 + b_1 + d_1) R_{1\alpha 1\alpha}, \tag{2.15}$$

and from (K) and (L) we obtain

$$D_1 (R_{\alpha 11\alpha} + P_{11}) = (a_1 + c_1 + d_1)(R_{1\alpha 1\alpha} + P_{11}). \tag{2.16}$$

As a result of (1.6) and (2.13) other relations hold trivially. With the help of (2.9), (2.10) and (2.13), we conclude that (2.14), (2.15) and (2.16) are equivalent to

$$D_1 P_{11} = P_{11}. \tag{2.17}$$

Substituting (2.5) and (2.6) in (2.17), we obtain the differential equation

$$\psi'' - 2\psi\psi' = \psi' - \psi^2, \tag{2.18}$$

where ($'$) denotes the derivation with respect to x^1. By integration with respect to x^1, (2.18) reduces to

$$\psi' - \psi^2 = ce^{x^1}, \tag{2.19}$$

where c is an arbitrary constant. If we take $c = 1$ and use the transformation $\psi = -\frac{U'}{U}$, (2.19) turns into the linear differential equation

$$U'' + e^{x^1} U = 0. \tag{2.20}$$

By a further transformation $e^{x^1} = t^2$ (2.20) can be converted into the form

$$t^2 \ddot{U} + t\dot{U} + 4t^2 U = 0,$$

which is a Bessel differential equation, where $\dot{U} = \frac{\partial U}{\partial t}$ and $\ddot{U} = \frac{\partial^2 U}{\partial t^2}$. Solving this equation, we obtain the general solution of U in terms of Bessel functions as follows:

$$U = AJ_o(2t) + BY_o(2t), \tag{2.21}$$

where A and B are arbitrary constants.

Substituting (2.21) in $\psi = -\frac{U'}{U}$ and remembering that $t = e^{x^1/2}$ and $J'_m(z) = \frac{z'}{2}[J_{m-1}(z) + J_{m+1}(z)]$, $Y'_m(z) = \frac{z'}{2}[J_{m-1}(z) + Y_{m+1}(z)]$, we have

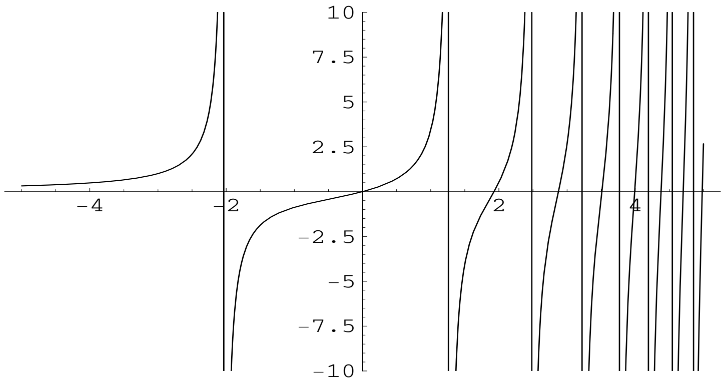
$$\psi(x^1) = \frac{e^{\frac{x^1}{2}}\left[J_1\left(2\,e^{\frac{x^1}{2}}\right) + c_1 Y_1\left(2\,e^{\frac{x^1}{2}}\right)\right]}{J_o\left(2\,e^{\frac{x^1}{2}}\right) + c_1 Y_o\left(2\,e^{\frac{x^1}{2}}\right)}, \quad \left(x^1 \in I\right),$$

where $c_1 = \frac{B}{A}$, J_o and J_1 are *Bessel* J functions and Y_o and Y_1 are *Bessel* Y-functions.

By determining $\psi(x^1)$ precisely we have completely determined the φ and w, therefore the coefficients of the connection and the metric (2.1), and relatively the whole space, such that the condition of being a $((WS)_n, D)$ space is satisfied. Hence the construction of our example of a weakly symmetric spaces with semi-symmetric metric connection is completed.

Although we achieved our aim, as a special case we will investigate the particular solution $\psi_0(x^1)$ satisfying the initial condition $\psi(0) = 0$, in order to give an example for the choice of the continuity interval of the function $\psi(x^1)$.

We can easily see that the particular solution which satisfies the initial condition $\psi(0) = 0$ is obtained with $c_1 = -\frac{J_1(2)}{Y_1(2)}$. This particular solution $\psi_0(x^1)$ has the following graph:



We see the apparent vertical asymptotes near $x^1 = -2$ and $x^1 = 1$. These vertical asymptotes, which are $x^1 \approx -2.0253$ and $x^1 \approx 1.27081$, correspond to the zeros of denominator of $\psi_0(x^1)$. Then, for this particular solution the interval I, which is the interval of continuity of the function $\psi(x^1)$, can be chosen to be any closed interval such that $\psi_0(x^1)$ is continuous on it, for example $[-2, 1]$ or $[a, b]$, such that $a < b < d$ where $d = x^1$ is the smallest asymptote of $\psi(x^1)$.

Finally, we have determined the function $\psi(x^1)$, and by the way the coefficients of the connection, such that the defining equation of weakly symmetric spaces is satisfied for the space $((WS)_n, D)$, which has the metric (2.1).

References

[1] L. Tamássy and T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc. J. Bolyai* **50** (1989), 663–670.

[2] U. C. De and Somnath Bandyopadhyay, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **54** (1999), 377–381.

[3] U.C. De and Bimal Krishna Mazumder, On pseudo Ricci symmetric spaces, *Tensor, N. S.* **60** (1998), 135–138.

[4] W. Roter, On conformally related conformally recurrent metrics 1, some general results, *Colloquium Mathematicum* **47** (1982), 39–46.

[5] S. Aynur Uysal and R. Özlem Doğan, On recurrent spaces with semi-symmetric singh connection, *Indian Journal of Pure and Applied Mathematics* **34**(7) (2003), 1061–1065.

S. AYNUR UYSAL
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND LETTERS
ISTANBUL TECHNICAL UNIVERSITY
ISTANBUL 80660
TURKEY

E-mail: uysal@itu.edu.tr

R. ÖZLEM LALEOĞLU
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND LETTERS
ISTANBUL TECHNICAL UNIVERSITY
ISTANBUL 80660
TURKEY

E-mail: doganroj@itu.edu.tr

(Received November 13, 2003; revised February 13, 2004)