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# On special cases of local Möbius equations

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**Abstract.** An overwiev of certain differential equations as special cases of local Möbius equations is given.

## 1. Introduction

In analysis, mostly the existence of a solution to a differential equation on a certain domain is argued. But in geometry, one can also argue the existence of a domain manifold for a differential equation to possess a nontrivial solution. This may be considered as an analytic characterization (or representation) of a manifold by a differential equation if this manifold serves as a unique domain for this differential equation to possess a nontrivial solution in a certain class of manifolds. In the literature, some characterizations of rank one symmetric Riemannian manifolds by differential equations can be found. For example, some known characterizations of Euclidian spheres, complex projective spaces and quaternionic projective spaces by differential equations can be found in [9], [10], [6], [4], [14], [13], [3], [8],[1], and also a survey of these results can be found in [5].

It seems that one of the most significant example of such characterizations of Euclidian spheres is a result of OBATA [9], that is, a necessary and sufficient condition for a connected, complete, $n\,(\geq 2)$-dimensional

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Riemannian manifold $(M, g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonconstant function $f$ on $M$ satisfying the (tensorial) differential equation $H_f + \lambda f g = 0$, where $H_f$ is the Hessian form of $f$ on $(M, g)$. In other words, the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, on a connected, complete, Riemannian manifold $(M, g)$ has a nontrivial solution if and only if its domain $(M, g)$ is the Euclidean sphere of radius $1/\sqrt{\lambda}$. Also, in this particular example, on the domain connected, complete Riemannian manifolds $(M, g)$, the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, can be considered as an analytic (or representative) characterization of Euclidean spheres. As well, if we take the trace of the differential equation $H_f + \lambda f g = 0$ on an $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ with respect to $g$ then we obtain another differential (in fact an eigenvalue) equation $\Delta f = -n\lambda f$ on $(M, g)$, where $\Delta f$ is the trace of $H_f$ with respect to $g$. It is shown in [9] that, if $(M, g)$ is connected, compact, Einstein $n\,(\geq 2)$-dimensional Riemannian manifold with constant scalar curvature $\tau$ and there exist a nonconstant function $f$ on $M$ satisfying $\Delta f = -n\lambda f$ then $\lambda \leq -\frac{\tau}{n-1}$, and in particular $\lambda = \frac{\tau}{n-1}$ if and only if $(M, g)$ isometric with the Euclidian sphere of radius $\sqrt{\frac{n(n-1)}{\tau}}$. Also, in [6], there is stated another differential equation (which is equivalent to $H_f + \lambda f g = 0, \lambda \neq 0$) on connected, complete Riemannian manifolds $(M, g)$ characterizing Euclidian spheres by the existence of a nontrivial solution to that differential equation. More precisely, it is shown that, a necessary and sufficient condition for a connected, complete, $n\,(\geq 2)$-dimensional Riemannian manifold to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the differential equation $(\nabla^2 Z)(\cdot\,, \cdot) + \lambda g(Z, \cdot)\cdot = 0$ on $(M, g)$. That is, the differential equation $\nabla^2 Z(\cdot\,, \cdot) + \lambda g(Z, \cdot)\cdot = 0$, $\lambda > 0$, on a connected, complete, $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ has a nontrivial solution if and only if its domain $(M, g)$ is the Euclidean sphere of radius $1/\sqrt{\lambda}$. Hence, in the class of domain connected, complete Riemannian manifolds $(M, g)$, the differential equation $\nabla^2 Z(\cdot\,, \cdot) + \lambda g(Z, \cdot)\cdot = 0$, $\lambda > 0$, also serves as an analytic (or representative) characterization of Euclidean spheres. If we take the trace of the differential equation $\nabla^2 Z(\cdot\,, \cdot) + \lambda g(Z, \cdot)\cdot = 0$ on an $n\,(\geq 2)$-dimensional Riemannian manifold

$(M, g)$ with respect to $g$ we obtain another differential (in fact, an eigenvalue) equation $\Delta Z = -\lambda Z$ on $(M, g)$, where $\Delta Z$ is the trace of the second covariant differential $\nabla^2 Z$ of $Z$ with respect to $g$. The subject of [4] is differential (in fact, the eigenvalue) equation $\Delta Z = -\lambda Z$ on a connected, compact, Einstein $n \geq 2$-dimensional Riemannian manifold of constant scalar curvature of $\tau$. Mainly, it is shown that, a necessary and sufficient condition for a connected, compact, Einstein $n\,(\geq 2)$-dimensional manifold $(M, g)$ with $\tau > 0$ to be isometric with an Euclidian sphere of radius $\sqrt{\frac{n(n-1)}{\tau}}$ is the existence of a nonzero vector field $Z$ on $(M, g)$ satisfying the differential equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$. Also, it is shown that, the differential equations $\Delta f = -\frac{\tau}{n-1} f$ and $\Delta Z = -\frac{\tau}{n(n-1)} Z$ are "equivalent" on a connected, compact, Einstein $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ of constant scalar curvature $\tau > 0$, provided that $\dim M = n \geq 3$.

In this paper, we state another differential equation, which is a slight generalization of an equation given by Obata [10], characterizing Euclidean spheres. We also give an overview of the above differential equations as special cases of local Möbius equations. In fact, the idea underlying this paper is to characterize (or represent) Riemannian manifolds analytically by a differential equation on certain classes of Riemannian manifolds determined by mild geometric/topological assumptions.

## 2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let $Z$ be a vector field on an $n$-dimensional Riemannian manifold $(M, g)$ with Levi–Civita connection $\nabla$. The second covariant differential $\nabla^2 Z$ of $Z$ is defined by

$$(\nabla^2 Z)(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$

where $X, Y \in \Gamma TM$. We define the Laplacian $\Delta Z$ of $Z$ on $(M, g)$ to be the trace of $\nabla^2 Z$ with respect to $g$, that is,

$$\Delta Z = \operatorname{trace} \nabla^2 Z = \sum_{i=1}^{n} (\nabla^2 Z)(X_i, X_i),$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal frame for $TM$.

Also, if $Z$ is a vector field on a Riemannian manifold $(M, g)$ then the affinity tensor $L_Z\nabla$ of $Z$ is defined by

$$(L_Z\nabla)(X, Y) = L_Z\nabla_X Y - \nabla_{L_Z X} Y - \nabla_X L_Z Y,$$

where $L_Z$ is the Lie derivative with respect to $Z$ and $X, Y \in \Gamma TM$. (See, for example page 109 of [12]). We define the tension field $\Box Z$ of $Z$ on $(M, g)$ to be the trace of $L_Z\nabla$ with respect to $g$, that is,

$$\Box Z = \operatorname{trace} L_Z\nabla = \sum_{i=1}^{n}(L_Z\nabla)(X_i, X_i),$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal frame for $TM$.

By a straightforward computation, it can be shown by using the torsion-free property of $\nabla$ that

$$(L_Z\nabla)(X, Y) = (\nabla^2 Z)(X, Y) + R(Z, X)Y$$

(see page 110 of [12]) and hence,

$$\Box Z = \Delta Z + \widehat{\mathrm{Ric}}(Z),$$

where $R$ is the curvature tensor of $(M, g)$, $\widehat{\mathrm{Ric}}$ is the Ricci operator of $(M, g)$ and $X, Y \in \Gamma TM$. (Also see page 40 of [15].)

## 3. A characterization of Euclidean spheres

Now we will give an overview of the differential equations $H_f + \lambda f g = 0$ and $(\nabla^2 Z)(\cdot\,,\cdot) + \lambda g(Z, \cdot)\cdot = 0$ on a Riemannian manifold $(M, g)$, where $\lambda \in \mathbb{R}$.

*Definition 3.1.* A function $f$ on an $n$-dimensional Riemannian manifold $(M, g)$ is said to satisfy the local Möbius equation on $(M, g)$ if $H_f = \frac{\Delta f}{n} g$. (For example, see [11].)

Note that, at the same time, if $f$ is also an eigenfunction of $\Delta$, that is, $\Delta f = -\lambda f$, then the local Möbius equation reduces to the equation $H_f + \lambda f g = 0$ on $(M, g)$. We also have a similar situation for vector fields.

*Definition 3.2.* A vector field $Z$ on a Riemannian manifold $(M, g)$ is said to satisfy the local Möbius equation on $(M, g)$ if $(\nabla^2 Z)(\cdot\,,\cdot) = g(\Delta Z, \cdot)\,\cdot$.

Hence, at the same time, if $Z$ is also an eigenvector field of $\Delta$, that is, $\Delta Z = -\lambda Z$, then the local Möbius equation reduces to the equation $(\nabla^2 Z)(\cdot\,,\cdot) + \lambda g(Z, \cdot)\,\cdot = 0$ on $(M, g)$. (Note that, on a connected, compact Riemannian manifold $(M, g)$, the Laplacian $\Delta$ is negative semi-definite on both spaces of functions and vector fields on $(M, g)$.) Thus, if $(M, g)$ is compact, eigenvalues of $\Delta$ are nonpositive both on spaces of functions and vector fields on $(M, g)$. (See [2] or [4]).)

It is known that, if a nonconstant function $f$ satisfies the local Möbius equation $H_f = \frac{\Delta f}{n} g$ on an $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ then, near each regular point $p \in M$ of $f$, $(M, g)$ can be expressed as a nontrivial warped product of an open Euclidean interval and a Riemannian manifold, where the warping function $\psi$ on the interval is a scalar multiple of $\|\nabla f\|$ (see for example, Theorem 5.2 of [11]), and in particular, if $(M, g)$ is connected and compact with constant scalar curvature $\tau$ then $\tau > 0$ and $(M, g)$ is isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$ (see Theorem 24 of [7]). Now we prove analogs of these results for the local Möbius equation $(\nabla^2 Z)(\cdot\,,\cdot) = g(\Delta Z, \cdot)\,\cdot$ on a Riemannian manifold $(M, g)$.

First we state an elementary lemma to be used in the proof of the main result of this paper which is Lemma 3.1 in [6]

**Lemma 3.3.** *Let $(M, g)$ be an $n$-dimensional Riemannian manifold. If $Z$ is a vector field on $(M, g)$ satisfying the Local Möbius equation*

$$(\nabla^2 Z)(X, Y) = g(\Delta Z, X)Y$$

*for all $X, Y \in \Gamma TM$ then,*

(i) *$R(X, Y)Z = -[g(\Delta Z, Y)X - g(X, \Delta Z)Y]$ for every $X, Y \in \Gamma TM$, and hence, $\widehat{\mathrm{Ric}}(Z) = -(n-1)\Delta Z$,*

(ii) *$\nabla \operatorname{div} Z = n\Delta Z$, and hence, $\nabla^2 \operatorname{div} Z = n\nabla\Delta Z$, where $\nabla^2 \operatorname{div} Z$ is the Hessian tensor of $\operatorname{div} Z$.*

Proof. (i) Let$X, Y \in \Gamma TM$. Then,

$$R(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = g(\Delta Z, X)Y - g(\Delta Z, Y)X$$
$$= -[g(\Delta Z, Y)X - g(\Delta Z, X)Y].$$

Hence,

$$g(\widehat{\mathrm{Ric}}, X) = g\left(\sum_{i=1}^{n} R(Z, X_i)X_i, X\right) = \sum_{i=1}^{n} g(R(Z, X_i)X_i, X)$$
$$= \sum_{i=1}^{n} R(Z, X_i, X_i, X) = \sum_{i=1}^{n} R(X_i, X, Z, X_i)$$
$$= \sum_{i=1}^{n} g(R(X_i, X)Z, X_i)$$
$$= \sum_{i=1}^{n} g(-g(\Delta Z, X)X_i + g(\Delta Z, X_i)X, X_i)$$
$$= -g(\Delta Z, X)\sum_{i=1}^{n} g(X_i, X_i) + \sum_{i=1}^{n} g(\Delta Z, X_i)g(X, X_i)$$
$$= -ng(\Delta Z, X) + g(\Delta Z, X) = (-n+1)g(\Delta Z, X)$$
$$= -(n-1)g(\Delta Z, X) = g(-(n-1)\Delta Z, X).$$

(ii) Let $\{X_1, \dots, X_n\}$ be an adapted orthonormal frame near $p \in M$, that is, $\{X_1, \dots, X_n\}$ is an orthonormal frame in $TM$ with $(\nabla X_i)_p = 0$ for $i = 1, \dots, n$ and let $X \in \Gamma(TM)$. Then, at $p \in M$,

$$g(\nabla \operatorname{div} Z, X) = X(\operatorname{div} Z) = X\left[\sum_{i=1}^{n} g(\nabla_{X_i} Z, X_i)\right]$$
$$= \sum_{i=1}^{n} [Xg(\nabla_{X_i} Z, X_i)]$$
$$= \sum_{i=1}^{n} [g(\nabla_X \nabla_{X_i} Z, X_i) + g(\nabla_{X_i} Z, \nabla_X X_i)]$$

$$= \sum_{i=1}^{n} g((\nabla^2 Z)(X, X_i), X_i) - g(\nabla_{\nabla_X X i} Z, X_i)]$$

$$= \sum_{i=1}^{n} g(g(\Delta Z, X) X_i, X_i) = \sum_{i=1}^{n} g(\Delta Z, X) g(X_i, X_i)$$

$$= g(\Delta Z, X) \sum_{i=1}^{n} g(X_i, X_i) = n g(\Delta Z, X) = g(n\Delta Z, X).$$

Hence it follows that $\nabla \operatorname{div} Z = n\Delta Z$, and hence, $\nabla^2 \operatorname{div} Z = n\nabla\Delta Z$. □

*Remark 3.4.* Note that, a vector field $Z$ on an $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ satisfying the local Möbius equation $(\nabla^2 Z)(\cdot, \cdot) = g(\Delta Z, \cdot)\,\cdot$ on $(M, g)$ also satisfies the equation $\square Z + \frac{n-2}{n}\nabla \operatorname{div} Z = 0$ on $(M, g)$:

$$\begin{aligned}\square Z + \frac{n-2}{n}\nabla \operatorname{div} Z &= \Delta Z + \widehat{\mathrm{Ric}} Z + \frac{n-2}{n}\nabla \operatorname{div} Z \\ &= \frac{1}{n}\nabla \operatorname{div} Z - (n-1)\Delta Z + \frac{n-2}{n}\nabla \operatorname{div} Z \\ &= \frac{1}{n}\nabla \operatorname{div} Z - \frac{n-1}{n}\nabla \operatorname{div} Z + \frac{n-2}{n}\nabla \operatorname{div} Z \\ &= \frac{-n+2}{n}\nabla \operatorname{div} Z + \frac{n-2}{n}\nabla \operatorname{div} Z = 0.\end{aligned}$$

Hence, if $(M, g)$ is compact, then a vector field $Z$ satisfying the local Möbius equation $(\nabla^2 Z)(\cdot\,, \cdot) = g(\Delta Z, \cdot)\,\cdot$ on $(M, g)$ is a conformal vector field. (See page 47 of [15].)

**Lemma 3.5.** *Let $(M, g)$ be an $n\,(\geq 2)$-dimensional Riemannian manifold. If $Z$ is conformal vector field on $M$ satisfying the equation*

$$R(X, Y)Z = -[g(\Delta Z, Y)X - g(X, \Delta Z)Y]$$

*for every $X, Y \in \Gamma TM$, then $Z$ satisfies the local Möbius equation*

$$(\nabla^2 Z)(\cdot\,, \cdot) = g(\Delta Z, \cdot)\,\cdot$$

*on $(M, g)$.*

Proof. This can be easily obtained from $(L_Z\nabla)(X,Y) = \frac{1}{n}[g(\nabla \operatorname{div} Z, Y)X + g(X, \nabla \operatorname{div} Z)Y - g(X,Y)\nabla \operatorname{div} Z]$ for a conformal vector field $Z$ (see page 46 of [15]) and then by taking the trace of this equation to show that $\Delta Z = \frac{1}{n}\nabla \operatorname{div} Z$. Briefly,

$$\Box Z = \operatorname{tr}(L_Z\nabla) = \sum_{i=1}^{n}(L_Z\nabla)(X_i, X_i)$$

$$= \sum_{i=1}^{n}\frac{1}{n}[g(\nabla \operatorname{div} Z, X_i)X_i + g(\nabla \operatorname{div} Z, X_i)X_i - g(X_i, X_i)\nabla \operatorname{div} Z]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} g(\nabla \operatorname{div} Z, X_i)X_i + \sum_{i=1}^{n} g(\nabla \operatorname{div} Z, X_i)X_i - \sum_{i=1}^{n} g(X_i, X_i)\nabla \operatorname{div} Z\right]$$

$$= \frac{1}{n}[\nabla \operatorname{div} Z + \nabla \operatorname{div} Z - n\nabla \operatorname{div} Z] = \frac{1}{n}(2-n)\nabla \operatorname{div} Z = \Delta Z + \widehat{\operatorname{Ric}} Z,$$

which implies,

$$\Delta Z = \frac{2-n}{n}\nabla \operatorname{div} Z + \frac{n-1}{n}\nabla \operatorname{div} Z = \frac{1}{n}\nabla \operatorname{div} Z. \qquad \Box$$

**Theorem 3.6.** *Let $(M,g)$ be an $n\,(\geq 2)$-dimensional Riemannian manifold. If there exists a nonzero vector field $Z$ on $(M,g)$ satisfying the equation*

$$(\nabla^2 Z)(X,Y) = g(\Delta Z, X)Y$$

*for all $X, Y \in \Gamma TM$ and $\Delta Z$ is a conformal vector field on $(M,g)$ then, $(M,g)$ is a nontrivial warped product of an Euclidean interval and a Riemannian manifold near each point $p \in M$ with $(\Delta Z)_p \neq 0$, where the warping function $\psi$ on this interval is a scalar multiple of $\|\Delta Z\|$. In particular, if $(M,g)$ is connected and compact with constant scalar curvature $\tau$ then $\tau > 0$ and $(M,g)$ is isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$*

Proof. First note that, by Lemma 3.3, $\nabla\Delta Z$ is self-adjoint on $(M,g)$ and $\Delta \operatorname{div} Z = n \operatorname{div} \Delta Z$. Thus,

$$\nabla^2 \operatorname{div} Z = n\nabla\Delta Z = \operatorname{div}\Delta Z \operatorname{id} + n\sigma = \frac{\Delta \operatorname{div} Z}{n}\operatorname{id} + n\sigma,$$

where $\sigma$ is the traceless self-adjoint part of $\nabla\Delta Z$. But, since $\Delta Z$ is assumed to be a conformal vector field, $\sigma = 0$ (see page 173 of [12]), and it follows that $\nabla^2 \operatorname{div} Z = \frac{\Delta \operatorname{div} Z}{n} \operatorname{id}$. Hence, by Theorem 5.2 of [11], $(M, g)$ is a nontrivial warped product of an Euclidean interval and a Riemannian manifold near each point $p \in M$ with $\frac{1}{n}(\nabla \operatorname{div} Z)_p = (\Delta Z)_p \neq 0$, where the warping function $\psi$ on the interval is a scalar multiple of $\|\Delta Z\|$. In particular, if $(M, g)$ is connected and compact with constant scalar curvature $\tau$ then, by Theorem 24 of [7], $\tau > 0$ and $(M, g)$ is isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$. □

Finally we state another differential equation, which is a slight generalization of an equation given by Obata [10], characterizing Euclidean spheres. It is shown in [13] that, a necessary and sufficient condition for a connected, simply connected, complete $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonconstant function $f$ on $M$ satisfying the equation $(\nabla^2\nabla f)(X, Y) + \lambda[2g(\nabla f, X)Y + g(Y, \nabla f)X + g(X, Y)\nabla f] = 0$ for all $X, Y \in \Gamma TM$. In fact, we can replace $\nabla f$ with a nonzero vector field $Z$ in the above equation.

**Theorem 3.7.** *Let $(M, g)$ be a connected, simply connected, complete $n\,(\geq 2)$-dimensional Riemannian manifold. Then, a necessary and sufficient condition for $(M, g)$ to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonzero vector field $Z$ on $M$ satisfying the equation*

$$(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Y, Z)X + g(X, Y)Z] = 0$$

*for all $X, Y \in \Gamma TM$.*

Proof. We will show that $\nabla \operatorname{div} Z$ also satisfies this equation. First, it can be similarly shown as in the proof of Lemma 3.3 that $\nabla \operatorname{div} Z = -2\lambda(n+1)Z$ and hence,

$$\nabla^2 \operatorname{div} Z = -2\lambda(n+1)\nabla Z = -2\lambda(n+1)\left[\frac{\operatorname{div} Z}{n}\operatorname{id} + \sigma\right],$$

where $\sigma$ is the traceless self-adjoint part of $\nabla Z$. Also,

$$(\nabla\sigma)(X, Y) = (\nabla^2 Z)(X, Y) - \nabla\left(\frac{\operatorname{div} Z}{n}\operatorname{id}\right)(X, Y)$$

$$= -\frac{1}{n(n+1)} g(X, \nabla \operatorname{div} Z)Y + \frac{1}{2(n+1)} g(\nabla \operatorname{div} Z, Y)X$$
$$+ \frac{1}{2(n+1)} g(X, Y)\nabla \operatorname{div} Z$$

for all $X, Y \in \Gamma TM$. Thus,

$$(\nabla^2 \nabla \operatorname{div} Z)(X, Y) = -2\lambda(n+1) \left[ \frac{g(X, \nabla \operatorname{div} Z)}{n} Y + (\nabla\sigma)(X, Y) \right]$$
$$= -\lambda[2g(X, \nabla \operatorname{div} Z)Y + g(\nabla \operatorname{div} Z, Y)X + g(X, Y)\nabla \operatorname{div} Z]$$

for all $X, Y \in \Gamma TM$. Hence, by Theorem A of [13], the necessary and sufficient condition of the Theorem follows for $f = \operatorname{div} Z$. □

Note that, the differential equation $(\nabla^2 Z)(X, Y) + \lambda[2g(Z, X)Y + g(Z, Y)X + g(X, Y)Z] = 0$, $\lambda > 0$, can also be considered as an analytic characterization (or representative) of Euclidean spheres in the class of connected, simply connected, complete Riemannian manifolds by Theorem 3.7.

*Remark 3.8.* Considering differential equations $(\nabla^2 Z)(X, Y) + \lambda g(Z, X)Y = 0$ and $(\nabla^2 Z)(X, Y) + \lambda[2g(X, Z)Y + g(Z, Y)X + g(X, Y)Z] = 0$ for $\lambda > 0$ on the $n\,(\geq 2)$-dimensional Euclidean sphere of radius $1/\sqrt{\lambda}$, intuitively, the first differential equation corresponds to the first eigenvalue of the Laplacian (that is, $\Delta \operatorname{div} Z = -n\lambda \operatorname{div} Z$) and the latter differential equation corresponds to the second eigenvalue of the Laplacian (that is, $\Delta \operatorname{div} Z = -2\lambda(n+1) \operatorname{div} Z$) on the Euclidean sphere of radius $1/\sqrt{\lambda}$. In fact, a vector field $Z$ satisfying the latter differential equation is necessarily a projective vector field (see Proposition 2.1 in [13]). Also a vector field $Z$ satisfying the first differential equation is necessarily a conformal vector field (see Remark 3.5 in [4]). For further discussion of the latter differential equation, see [13].

Note that, if a nonzero vector field $Z$ on a Riemannian manifold $(M, g)$ satisfies the equation $(\nabla^2 Z)(X, Y) + \lambda[2g(X, Z)Y + g(Z, Y)X + g(X, Y)Z] = 0$, $\lambda \in \mathbb{R}$, then $Z$ also satisfies the eigenvalue equation $\Delta Z = -\lambda(n+3)Z$ on $(M, g)$. Hence, as before, we can generalize this equation to a local Möbius equation on $(M, g)$ by $(\nabla^2 Z)(X, Y) = \frac{1}{n+3}[2g(X, \Delta Z)Y +$

$g(\Delta Z, Y)X + g(X, Y)\Delta Z]$. Thus, in particular, if a solution $Z$ of the above local Möbius equation is also a solution of the eigenvalue equation $\Delta Z = -\lambda(n+3)Z$, then the above local Möbius equation reduces to the equation $(\nabla^2 Z)(X, Y) + \lambda[2g(X, Z)Y + g(Z, Y)X + g(X, Y)Z] = 0$ on $(M, g)$. The proof of the Lemma below can be given similar to the proof of Lemma 3.3.

**Lemma 3.9.** *Let $(M, g)$ be an $n$-dimensional Riemannian manifold. If $Z$ is a vector field on $(M, g)$ satisfying the equation*

$$(\nabla^2 Z)(X, Y) = \frac{1}{n+3}[2g(X, \Delta Z)Y + g(\Delta Z, Y)X + g(X, Y)\Delta Z]$$

*for all $X, Y \in \Gamma TM$ then,*

(a) $R(X, Y)Z = -\frac{1}{n+3}[g(X, \Delta Z)Y - g(Y, \Delta Z)X]$ *for all* $X, Y \in \Gamma TM$, *and hence,* $\widehat{\mathrm{Ric}}(Z) = -\frac{n-1}{n+3}\Delta Z$,

(b) $\nabla \operatorname{div} Z = 2\frac{n+1}{n+3}\Delta Z$, *and hence,* $\nabla^2 \operatorname{div} Z = 2\frac{n+1}{n+3}\nabla\Delta Z$, *where* $\nabla\nabla \operatorname{div} Z$ *is the Hessian tensor of* $\operatorname{div} Z$.

*Remark 3.10.* Note that, if a nonzero vector field $Z$ on an $n\,(\geq 2)$-dimensional Riemannian manifold $(M, g)$ satisfies the local Möbius equation $(\nabla^2 Z)(X, Y) = \frac{1}{n+3}[2g(X, \Delta Z)Y + g(\Delta Z, Y)X + g(X, Y)\Delta Z]$ then, by using Lemma 3.9, it can be shown that $(L_Z\nabla)(X, Y) = \frac{1}{n+3}[g(\nabla \operatorname{div} Z, Y)X + g(X, \nabla \operatorname{div} Z)Y]$ for all $X, Y \in \Gamma TM$. That is, $Z$ is a projective vector field on $(M, g)$. Conversely, if $Z$ is a projective vector field on an $n\,(\geq 2)$-dimensional Riemannian manifold with $R(X, Y)Z = -\frac{1}{n+3}[g(X, \Delta Z)Y - g(\Delta Z, Y)X]$ for all $X, Y \in \Gamma TM$, then $Z$ satisfies the local Möbius equation $(\nabla^2 Z)(X, Y) = \frac{1}{n+3}[2g(X, \Delta Z)Y + g(\Delta Z, Y)X + g(X, Y)\Delta Z]$. This can be easily obtained from $(L_Z\nabla)(X, Y) = \frac{1}{n+1}[g(\nabla \operatorname{div} Z, Y)X + g(X, \nabla \operatorname{div} Z)Y]$ for a projective vector field $Z$ (see page 183 of [12]) and then by taking the trace of this equation to show that $\Delta Z = \frac{n+3}{2(n+1)}\nabla \operatorname{div} Z$.

Also by Lemma 3.9, we have the following analog of Theorem 3.6. The proof of the Theorem below can be given similar to the proof of Theorem 3.6.

**Theorem 3.11.** *Let $(M,g)$ be an $n\,(\geq 2)$-dimensional Riemannian manifold. If there exists a nonzero vector field $Z$ on $(M,g)$ satisfying the equation*

$$(\nabla^2 Z)(X,Y) = \frac{1}{n+3}[2g(X,\Delta Z)Y + g(\Delta Z,Y)X + g(X,Y)\Delta Z]$$

*for all $X, Y \in \Gamma TM$ and $\Delta Z$ is a conformal vector field on $(M,g)$ then, $(M,g)$ is a nontrivial warped product of an Euclidean interval and a Riemannian manifold near each point $p \in M$ with $(\Delta Z)_p \neq 0$, where the warping function $\psi$ on this interval is a scalar multiple of $\|\Delta Z\|$. In particular, if $(M,g)$ is connected and compact with constant scalar curvature $\tau$ then $\tau > 0$ and $(M,g)$ is isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$.*

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