Publ. Math. Debrecen
67/1-2 (2005), 199–208

On a functional inequality related to the stability problem for the Gołąb–Schinzel equation

By JACEK CHUDZIAK (Rzeszów)

Abstract. We determine all unbounded continuous functions satisfying the inequality

$$|f(x+yf(x))-f(x)f(y)|\le\varepsilon\quad\text{for } x,y\in\mathbb{R},$$

where ε is a fixed positive real number. As a consequence we obtain that in the class of continuous functions the Gołąb–Schinzel functional equation is superstable.

1. Introduction

The Gołąb–Schinzel functional equation

$$f(x+yf(x))=f(x)f(y)\quad\text{for } x,y\in\mathbb{R},\tag{1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is the unknown function, is one of the most intensively studied equations of the composite type. Some information concerning (1), recent results, applications and numerous references one can find in [1]–[6] and [8]–[12]. At the 38th International Symposium on Functional Equations (2000, Noszvaj, Hungary) R. Ger raised, among others, the problem of Hyers–Ulam stability of (1) (see [7]). Motivated by this problem, we consider the inequality

$$|f(x+yf(x))-f(x)f(y)|\le\varepsilon\quad\text{for } x,y\in\mathbb{R},\tag{2}$$

Mathematics Subject Classification: 39B22, 39B82.
Key words and phrases: Gołąb–Schinzel equation, stability.

where ε is a fixed positive real number. We determine all unbounded continuous solutions of (2). As a consequence we obtain that in the class of continuous functions the equation (1) is superstable.

2. Auxiliary results

For the proof of our main results we need few lemmas.

Lemma 1. *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies* (2). *Then:*

(i) *either $f(0) = 1$ or f is bounded;*

(ii)
$$|f(x + yf(x)) - f(y + xf(y))| \leq 2\varepsilon \quad \textit{for } x, y \in \mathbb{R}; \tag{3}$$

(iii) *if f is bounded above then f is bounded.*

Proof. (i) Putting $y = 0$ in (2), we get $|f(x)||1-f(0)| \leq \varepsilon$ for $x \in \mathbb{R}$. Whence either $f(0) = 1$ or f is bounded.

(ii) This follows immediately from (2).

(iii) Suppose that f is unbounded. Then there exists a sequence $(x_n : n \in \mathbb{N})$ of real numbers such that $\lim_{n\to\infty} |f(x_n)| = \infty$. Using (2) we obtain that $f(x_n + x_n f(x_n)) \geq f(x_n)^2 - \varepsilon$ for $n \in \mathbb{N}$. Consequently f is unbounded above. □

Lemma 2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying* (2). *Fix a $z \in \mathbb{R} \setminus \{0\}$ and define the function $\psi_z : \mathbb{R} \to \mathbb{R}$ by*

$$\psi_z(x) = x + zf(x) \quad \textit{for } x \in \mathbb{R}. \tag{4}$$

(i) *If ψ_z is bounded then*

$$f(x) = 1 - \frac{x}{z} \quad \textit{for } x \in \mathbb{R}. \tag{5}$$

(ii) *If $f(z) = 0$ and ψ_z is unbounded below (above), then*

$$|f(x)| \leq \varepsilon \quad \textit{for } x \in (-\infty, z] \quad (x \in [z, \infty), \textit{resp.}). \tag{6}$$

(iii)
$$\psi_z^{n+1}(z) = \psi_z^n(z) + zf(\psi_z^n(z)) \quad \textit{for } n \in \mathbb{N}. \tag{7}$$

(iv) *If there exists a $q := \lim_{n\to\infty} \psi_z(z)^n$, then $f(q) = 0$.*

Proof. (i) Assume that ψ_z is bounded. From (4) it follows that

$$f(x) = \frac{1}{z}(\psi_z(x) - x) \quad \text{for } x \in \mathbb{R},$$

so using (2), one can obtain

$$\frac{1}{z^2}\Big|z\psi_z\Big(x + \frac{y}{z}(\psi_z(x) - x)\Big) - \psi_z(x)\psi_z(y) + x(\psi_z(y) - z)\Big| \le \varepsilon$$

for $x, y \in \mathbb{R}$. Since ψ_z is bounded, this means that $\psi_z(y) - z = 0$ for $y \in \mathbb{R}$, which implies (5).

(ii) Assume that $f(z) = 0$ and ψ_z is unbounded above. Since ψ_z is continuous and $\psi_z(z) = z$, we have $[z, \infty) \subset \psi_z(\mathbb{R})$. Moreover, taking in (2) $y = z$, we obtain $|f(\psi_z(x))| \le \varepsilon$ for $x \in \mathbb{R}$. Hence we get (6). In the case when ψ_z is unbounded below, the proof is analogous.

(iii) This follows immediately from (4).

(iv) This results at once form (iii). □

Lemma 3. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2) and $I \in \{(-\infty, 0), (0, \infty)\}$. If there is a $z \in I$ with $f(z) = 0$, then $f_{|I}$ is bounded above.*

Proof. We present the proof in the case $I = (0, \infty)$ only. Assume that $f(z) = 0$ for some $z \in (0, \infty)$. Let a function ψ_z be defined by (4). If ψ_z is bounded above (say, by a constant p), then from (4) it results that $f(x) \le \frac{p-x}{z}$ for $x \in \mathbb{R}$. Hence $f_{|(0,\infty)}$ is bounded above. If ψ_z is unbounded above, then according to Lemma 2(ii), we get $f(x) \le \varepsilon + \max\{f(t) : t \in [0, z]\}$ for $x \in (0, \infty)$, so again $f_{|(0,\infty)}$ is bounded above. □

Lemma 4. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2) and $I \in \{(-\infty, 0), (0, \infty)\}$. Then either $f_{|I}$ is bounded above or there exists a $k \in I$ such that*

$$f(x) \ge kx \quad \text{for } x \in I. \tag{8}$$

Proof. Similarly as in the proof of the previous lemma, we consider only the case $I = (0, \infty)$. Suppose that $f_{|(0,\infty)}$ is unbounded above. Then from Lemma 1(i) and Lemma 3 it follows that $f(0) = 1$ and $f(x) \ne 0$ for $(0, \infty)$. Hence, by the continuity of f,

$$f(x) > 0 \quad \text{for } x \in (0, \infty). \tag{9}$$

We divide the remaining part of the proof into two steps.

Step 1. We show that

$$f(x) \geq 1 \quad \text{for } x \in (0, \infty). \tag{10}$$

Suppose that (10) does not hold. Whence, according to (9), there is a $z \in (0, \infty)$ such that $f(z) \in (0, 1)$. Let the function ψ_z be defined by (4). Consider a sequence $(\psi_z^n(z) : n \in \mathbb{N})$. According to (7) and (9), we obtain that the sequence is strictly increasing. Moreover, it is unbounded. Indeed, if it were bounded, then it would exist $q := \lim_{n\to\infty} \psi_z^n(z)$. Hence, by Lemma 2(iv), $f(q) = 0$, which contradicts to (9). Now, we define a sequence of intervals $(I_n : n \in \mathbb{N} \cup \{0\})$ as follows: $I_0 := [0, z]$, $I_n := [\psi_z^{n-1}(z), \psi_z^n(z)]$ for $n \in \mathbb{N}$. Since the sequence $(\psi_z^n(z) : n \in \mathbb{N})$ is unbounded, we get

$$\bigcup_{n=1}^{\infty} I_n = [0, \infty). \tag{11}$$

Furthermore, for every $n \in \mathbb{N} \cup \{0\}$, we have

$$f(x) \leq Mf(z)^n + \varepsilon \sum_{i=0}^{n-1} f(z)^i \quad \text{for } x \in I_n, \tag{12}$$

where $M := \sup\{f(x) : x \in [0, z]\}$. In fact, for $n = 0$ (12) trivially holds (we adopt the convention $\sum_{i=0}^{-1} = 0$). If (12) occurs for a $n \in \mathbb{N} \cup \{0\}$, then taking an $x \in I_{n+1} = [\psi_z^n(z), \psi_z^{n+1}(z)]$ and using the continuity of ψ_z, we obtain that $x = \psi_z(t)$ for some $t \in I_n$. Whence, in view of (2) and (12) (for n), we obtain

$$\begin{aligned} f(x) = f(\psi_z(t)) = f(t + zf(t)) &\leq f(t)f(z) + \varepsilon \\ &\leq Mf(z)^{n+1} + \varepsilon \sum_{i=0}^{n} f(z)^i. \end{aligned}$$

Now, using (12), for every $n \in \mathbb{N} \cup \{0\}$, we have

$$f(x) \leq Mf(z)^n + \varepsilon \sum_{i=0}^{\infty} f(z)^i \leq M + \frac{\varepsilon}{1 - f(z)} \quad \text{for } x \in I_n.$$

Thus, in view of (11), $f_{|[0,\infty)}$ is bounded above, which yields a contradiction.

STEP 2. Since $f_{|(0,\infty)}$ is unbounded above, there is a $p \in (0,\infty)$ with $f(p) > 1+\varepsilon$. Define the function $h_p : [0,\infty) \to \mathbb{R}$ by $h_p(x) = p + xf(p)$ for $x \in [0,\infty)$. Consider a sequence $(h_p^n(p) : n \in \mathbb{N})$ and note that

$$h_p^n(p) = p\sum_{i=0}^{n} f(p)^i \quad \text{for } n \in \mathbb{N}. \tag{13}$$

Hence, the sequence $(h_p^n(p) : n \in \mathbb{N})$ is strictly increasing and unbounded. Let $I_0 := [0,p]$ and $I_n := [h_p^{n-1}(p), h_p^n(p)]$ for $n \in \mathbb{N}$. Then (11) occurs. Furthermore, using (10), similarly as in the previous step, one can show that for every $n \in \mathbb{N} \cup \{0\}$

$$f(x) \ge f(p)^n - \varepsilon\sum_{i=0}^{n-1} f(p)^i \quad \text{for } x \in I_n. \tag{14}$$

Fix an $x \in (0,\infty)$. In view of (11), $x \in I_n$ for some $n \in \mathbb{N} \cup \{0\}$. Hence $x \le h_p^n(p)$, so according to (13) and (14), we get

$$\begin{aligned}\frac{f(x)}{x} &\ge \frac{f(p)^n - \varepsilon\sum_{i=0}^{n-1} f(p)^i}{h_p^n(p)} \ge \frac{1-\varepsilon\sum_{i=1}^{\infty} f(p)^{-i}}{p\sum_{i=0}^{\infty} f(p)^{-i}}\\ &= \frac{f(p)-(1+\varepsilon)}{pf(p)} > 0.\end{aligned}$$

Therefore (8) holds with $k := \frac{f(p)-(1+\varepsilon)}{pf(p)} > 0$. □

Lemma 5. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is an unbounded continuous function satisfying* (2). *Then either*

$$f(x) \le M \quad \textit{for } x \in (-\infty,0] \tag{15}$$

and

$$f(x) \ge kx \quad \textit{for } x \in (0,\infty) \tag{16}$$

with some $M \in \mathbb{R}$ and $k \in (0,\infty)$; or

$$f(x) \ge sx \quad \textit{for } x \in (-\infty,0) \tag{17}$$

and

$$f(x) \le M \quad for\ x \in [0,\infty) \tag{18}$$

with some $M \in \mathbb{R}$ and $s \in (-\infty,0)$.

Proof. According to Lemma 4, it is enough to show that exactly one of functions $f_{|(-\infty,0)}$ and $f_{|(0,\infty)}$ is unbounded above. From Lemma 1(iii), it follows that at least one of them is unbounded above. Suppose that both $f_{|(-\infty,0)}$ and $f_{|(0,\infty)}$ are unbounded above. Then, on account of Lemma 4, there exist $k \in (0,\infty)$ and $s \in (-\infty,0)$ such that (16) and (17) occur. Moreover, in virtue of Lemma 1(i), $f(0)=1$. Since f is continuous, it implies that there is a $d>0$ such that $f(x) \geq d$ for $x \in \mathbb{R}$. Fix an $x_0 \in \mathbb{R}$ with $f(x_0) > \frac{1+\varepsilon}{d}$. Then $f(x_0)f(-\frac{x_0}{f(x_0)}) > 1+\varepsilon$. On the other hand, in view of (2), we get

$$\begin{aligned}\left|1-f(x_0)f\left(-\frac{x_0}{f(x_0)}\right)\right| &= \left|f(0)-f(x_0)f\left(-\frac{x_0}{f(x_0)}\right)\right| \\ &= \left|f\left(x_0+\left(-\frac{x_0}{f(x_0)}\right)f(x_0)\right)-f(x_0)f\left(-\frac{x_0}{f(x_0)}\right)\right| \leq \varepsilon,\end{aligned}$$

which yields a contradiction. □

Lemma 6. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is an unbounded continuous function satisfying* (2). *Then there exists a $p \in \mathbb{R}$ such that $f(p)=0$.*

Proof. Suppose that $f(x) \neq 0$ for $x \in \mathbb{R}$. Since f is continuous and, in view of Lemma 1(i), $f(0)=1$, this implies that $f(x)>0$ for $x \in \mathbb{R}$. According to Lemma 5, either (15) and (16); or (17) and (18) hold. Since the proof in both cases is similar, assume that (15) and (16) occur. Then, on account of (16), we have $x-\frac{1}{k}f(x) \leq 0$ for $x \in (0,\infty)$. Hence, in view of (15) $f(x-\frac{1}{k}f(x)) \leq M$ for $x \in (0,\infty)$. On the other hand, from (16) it follows that $f(-\frac{1}{k}+xf(-\frac{1}{k})) \geq -1+kf(-\frac{1}{k})x$ for $x > \frac{1}{kf(-\frac{1}{k})}$. Thus $\lim_{x\to\infty}|f(-\frac{1}{k}+xf(-\frac{1}{k}))-f(x-\frac{1}{k}f(x))| = \infty$, which contradicts (3). □

3. Main results

Theorem 1. *A function $f:\mathbb{R}\to\mathbb{R}$ is an unbounded continuous solution of* (2) *if and only if there exists a non-zero real constant a such that either*

$$f(x) = 1+ax \quad \text{for } x \in \mathbb{R} \tag{19}$$

or

$$f(x) = \max\{1 + ax, 0\} \quad \textit{for } x \in \mathbb{R}. \tag{20}$$

Proof. It is obvious that for every non-zero real constant a, the function f given by (19) or (20), is an unbounded continuous solution of (2). Assume that f is an unbounded continuous function satisfying (2). Then, according to Lemma 1(i) and Lemma 6, $f(0) = 1$ and there is a $p \in \mathbb{R} \setminus \{0\}$ such that $f(p) = 0$. Assume that $p < 0$ (if $p > 0$, the proof is similar). Then, in view of Lemma 3 and 5, we have (15) and (16). Let $z := \max\{x \in (-\infty, 0] : f(x) = 0\}$ and ψ_z be given by (4). Then $z < 0$ and

$$f(x) > 0 \quad \text{for } x \in (z, 0). \tag{21}$$

If ψ_z is bounded then, in virtue of Lemma 1(iv), f has the form (19) with $a := -\frac{1}{z}$. Assume that ψ_z is unbounded. If ψ_z were unbounded above, then in virtue of Lemma 2(ii), we would have $|f(x)| \leq \varepsilon$ for $x \in [z, \infty)$, which contradicts to (16). Whence ψ_z is unbounded below and bounded above (say, by a constant w). Consequently, in view of (4) and Lemma 2(ii), we have

$$f(x) \geq \frac{w - x}{z} \quad \text{for } x \in \mathbb{R} \tag{22}$$

and

$$|f(x)| \leq \varepsilon \quad \text{for } x \in (-\infty, z]. \tag{23}$$

We divide the remaining part of the proof into three steps.

Step 1. We prove that

$$\lim_{x \to \infty} \frac{f(x)}{x} = -\frac{1}{z}. \tag{24}$$

Suppose that (24) does not hold. Then, according to (22), there are a constant $t > 0$ and a sequence $(x_n : n \in \mathbb{N})$ of positive real numbers such that $\lim_{n\to\infty} x_n = \infty$ and

$$\frac{f(x_n)}{x_n} > -\frac{1}{z} + t \quad \text{for } n \in \mathbb{N}. \tag{25}$$

Since $z < \frac{z}{1-tz} < 0$, according to (21), we get $f\left(\frac{z}{1-tz}\right) > 0$. Thus

$$\lim_{n \to \infty} \left(\frac{z}{1 - tz} + x_n f\left(\frac{z}{1 - tz} \right) \right) = \infty,$$

so in virtue of (16), we obtain $\lim_{n\to\infty} f\left(\frac{z}{1-tz} + x_n f\left(\frac{z}{1-tz}\right)\right) = \infty$. On the other hand, in view of (25), we have

$$x_n + \frac{z}{1-tz} f(x_n) < x_n + \frac{z}{1-tz}\left(-\frac{1}{z} + t\right) x_n = 0 \quad \text{for } n \in \mathbb{N}.$$

Hence, using (15), we get $f\left(x_n + \frac{z}{1-tz} f(x_n)\right) \le M$ for $n \in \mathbb{N}$. Consequently,

$$\lim_{n\to\infty} \left| f\left(\frac{z}{1-tz} + x_n f\left(\frac{z}{1-tz}\right)\right) - f\left(x_n + \frac{z}{1-tz} f(x_n)\right)\right| = \infty,$$

which contradicts to (3).

STEP 2. We show that

$$f(x) = 1 - \frac{x}{z} \quad \text{for } x \in (z, \infty). \tag{26}$$

Fix a $y \in (z, \infty)$. From (2) and (24) it follows that

$$\lim_{x\to\infty} \frac{f(x + yf(x))}{x} = \lim_{x\to\infty} \frac{f(x)}{x} f(y) = -\frac{1}{z} f(y). \tag{27}$$

and

$$\lim_{x\to\infty} \left(1 + y\frac{f(x)}{x}\right) = 1 - \frac{y}{z} \neq 0.$$

Thus $\lim_{x\to\infty} x\left(1 + y\frac{f(x)}{x}\right) = \lim_{x\to\infty}(x + yf(x)) = \infty$, so according to (24) and (27), we obtain

$$-\frac{1}{z} = \lim_{x\to\infty} \frac{f(x + yf(x))}{x + yf(x)} = \lim_{x\to\infty} \frac{\frac{f(x+yf(x))}{x}}{1 + y\frac{f(x)}{x}} = \frac{f(y)}{y - z}.$$

Hence $f(y) = 1 - \frac{y}{z}$, which proves (26).

STEP 3. We prove that

$$f(x) = 0 \quad \text{for } x \in (-\infty, z]. \tag{28}$$

For $x = z$ (28) trivially occurs. Fix a $y \in (-\infty, z)$. According to (2) and (23), we have $f(x+yf(x)) \le \varepsilon + \varepsilon^2$ for $x \in (-\infty, z]$. Moreover, using (26), we get

$$x + yf(x) = x + y\left(1 - \frac{x}{z}\right) = \left(1 - \frac{y}{z}\right) x + y < \left(1 - \frac{y}{z}\right) z + y = z < 0$$

for $x \in (z, \infty)$. Hence, in view of (15), $f(x + yf(x)) \le M$ for $x \in (z, \infty)$. Consequently, $f(x + yf(x)) \le \max\{\varepsilon + \varepsilon^2, M\}$ for $x \in \mathbb{R}$, so taking into account (3), we obtain that $f(y+xf(y)) \le \max\{3\varepsilon+\varepsilon^2, M+2\varepsilon\}$ for $x \in \mathbb{R}$. Now, if $f(y)$ were different form 0, we would have that f is bounded above, which contradicts to Lemma 1(iii). Therefore $f(y) = 0$, which proves (28).

Finally, from (26) and (28) it follows that f has the form (20) with $a := -\frac{1}{z}$, which completes the proof. □

It is easy to check that for every non-zero real constant a, the function f given by (19) or (20) is a continuous solution of (2). Therefore, we can reformulate Theorem 1 in the following way:

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying* (2), *then either f is bounded or f is a solution of* (1).

Remark 1. Note that the idea of the introduction of the function ψ_z (cf. (4)) to a given solution f of (1), as well as the idea of the determination of the set of all possible zeroes of f have already been used in the study of the Gołąb–Schinzel equation (cf. e.g. [5], [10], [11]).

Acknowledgements. The author wishes to express his gratitude to Professor Józef Tabor for several helpful comments during the preparation of the paper.

References

[1] J. Aczél and J. Dhombres, Functional equations in several variables, Encyclopedia of Mathematics and its Applications, v. 31, *Cambridge University Press*, 1989.

[2] N. Brillouët-Belluot, On some functional equations of Gołąb–Schinzel type, *Aequationes Math.* **42** (1991), 239–270.

[3] J. Brzdęk, Bounded solutions of the Gołąb–Schinzel functional equation, *Aequationes Math.* **59** (2000), 248–254.

[4] J. Brzdęk, On continuous solutions of a conditional Gołąb–Schinzel equation, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **138** (2001), 3–6 (2002).

[5] J. Brzdęk, On the continuous solutions of a generalization of the Gołąb–Schinzel equation, *Publ. Math. Debrecen* **63** (2003), 421–429.

[6] J. Chudziak, Continuous solutions of a generalization of the Gołąb–Schinzel equation, *Aequationes Math.* **61** (2001), 63–78.

[7] R. Ger, A collection of problems in stability theory, in: Report of Meeting, The Thirty-Eight International Symposium on Functional Equations, June 11–18, 2000, *Noszvaj, Hungary*, *Aequationes Math.* **61** (2001), 284.

[8] S. Gołąb and A. Schinzel, Sur l'équation fonctionnelle $f(x+yf(x)) = f(x)f(y)$, *Publ. Math. Debrecen* **6** (1959), 113–125.

[9] A. Mureńko, On solutions of the Gołąb–Schinzel equation, *Internat. J. Math. Math. Sci.* **27** (2001), 541–546.

[10] L. Reich, Über die stetigen Lösungen der Gołąb–Schinzel-Gleichung auf $\mathbb{R}$ und auf $\mathbb{R}_{\geq 0}$, *Österr. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.* II. **208** (1999), 165–170.

[11] L. Reich, Über die stetigen Lösungen der Gołąb–Schinzel-Gleichung auf $\mathbb{R}_{\geq 0}$, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **138** (2001), 7–12.

[12] M. Sablik, A conditional Gołąb–Schinzel equation, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **137** (200) (2001), 11–15.

JACEK CHUDZIAK
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF RZESZÓW
REJTANA 16 A
35-310 RZESZÓW
POLAND

E-mail: chudziak@univ.rzeszow.pl

(Received January 26, 2004; revised April 27, 2004)