Publ. Math. Debrecen
67/1-2 (2005), 209–252

Finsleroid space with angle and scalar product

By G. S. ASANOV (Moscow)

Abstract. A systematic approach has been developed to encompass the Minkowski-type extension of Euclidean geometry such that a one-vector anisotropy is permitted, retaining simultaneously the concept of angle. For the respective geometry, the Euclidean unit ball is to be replaced by the body which is convex and rotund and is found on assuming that its surface (the indicatrix extending the unit sphere) is a space of constant positive curvature. We have called the body the *Finsleroid* in view of its intrinsic relationship with the metric function of Finsler type. The main point of the present paper is the angle coming from geodesics through the cosine theorem, the underlying idea being to derive the angular measure from the solutions to the geodesic equation which prove to be obtainable in simple explicit forms. The substantive items concern geodesics, angle, scalar product, and perpendicularity.

1. Introduction and synopsis of new conclusions

The Euclidean geometry is simple and totally spherically symmetric, and corresponds well to our ordinary everyday experience and intuition, while the Finsler or Banach–Minkowski geometries [1]–[9] are much more extended and sophisticated constructions that may serve to reflect various anisotropic scenarios. When a single vector is distinguished geometrically to be the only isotropic direction in extending the Euclidean geometry, the sphere may not be regarded as an exact carrier of the unit-vector image.

Mathematics Subject Classification: 53A35, 53B40, 53B50.
Key words and phrases: Finsler space, metric space, indicatrix, geodesics.

So under respective conditions one may expect that some directionally-anisotropic figure should be substituted with the sphere. To this end we shall use the Finsleroid which, being convex and rotund, is not, however, a second-order figure. The constant positive curvature is the fundamental property of the Finsleroid.

The present paper develops and elaborates in much detail the related Finsleroid-geometry (initiated by the author earlier in [10]–[12]) in the direction of evidencing the concepts of angle and scalar product. No special knowledge of Banach–Minkowski or Finsler geometries is assumed.

It will be recollected that, despite the fact that in geometry one certainly needs to use not only length but also angle and scalar product, various known attempts to introduce the concept of angle in the Minkowski or Finsler spaces were steadily encountered with drawback positions:

"Therefore no particular angular measure can be entirely natural in Minkowski geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance". (Busemann [2], p. 279.)

"Unfortunately, there exists a number of distinct invariants in a Minkowskian space all of which reduce to the same classical euclidean invariant if the Minkowskian space degenerates into a euclidean space. Consequently, distinct definitions of the trigonometric functions and of angles have appeared in the literature concerning Minkowskian and Finsler spaces". (Rund [3], p. 26)

A short but profound review of the respective attempts can be found in Section 1.7 of the book [4]. The fact that the attempts have never been unambiguous seems to be due to a lack of the proper tools. For the opinion was taken for granted that the angle ought to be defined or constructed in terms of the basic Finslerian metric tensor (and whence ought to be explicated from the initial Finslerian metric function). Let us doubt the opinion from the very beginning. Instead, we would like to raise alternatively the principle that the angle is a concomitant of the geodesics (and not of the metric function proper). The angle is determined by two vectors (instead of one vector in case of the length) and actually implies using a due extension of the Finslerian metric function to a two-vector metric function (to a scalar product). Below, the principle is applying to the Finsleroid space in a systematic way.

In the sequel the abbreviations FMF, FMT, and FHF will be used for the Finsleroid metric function, the associated Finslerian metric tensor, and the associated Finslerian Hamiltonian function, respectively. The notation E_g^{PD} will be applied to the Finsleroid space, with the upperscripts "PD" meaning "positive-definite". The characteristic parameter g may take on the values between -2 and 2; at $g = 0$ the space is reduced to become an ordinary Euclidean one.

Below Section 2 gives an account of the notation and conventions for the space E_g^{PD} and introduces the initial concepts and definitions that are required. The space is constructed by assuming an axial symmetry and, therefore, incorporates a single preferred direction, which we shall often refer as the Z-axis. After preliminary introducing a characteristic quadratic form B, which is distinct from the Euclidean sum of squares by entrance of a mixed term (see equation (2.22)), we define the FMF K for the space E_g^{PD} by the help of the formulae (2.30)–(2.33). A characteristic feature of the formulae is the occurrence of the function "arctan". Next, we calculate basic tensor quantities of the space. There appears a remarkable phenomenon, which essentially simplifies all the constructions, that the associated Cartan tensor occurs being of a simple algebraic structure (see equations (2.66)–(2.67)). In particular, the phenomenon gives rise to a simple structure of the associated curvature tensor (equation (2.69)). As well as in the Euclidean geometry the locus of the unit vectors issuing from fixed point of origin is the unit sphere, in the E_g^{PD}-geometry under development the locus is the boundary (surface) of the Finsleroid. We call the boundary the Finsleroid Indicatrix. It can rigorously be proved that the Finsleroid Indicatrix is a closed, regular, and strongly convex (hyper)surface. The value of the curvature depends on the parameter g according to the simple law (2.73). The determinant of the associated FMT is strongly positive in accordance with equations (2.64)–(2.65).

In Section 3 we shortly indicate how the consideration can conveniently be converted into the co-approach. The explicit form of the associated Finsleroid Hamiltonian metric function is entirely similar to the form of the FMF K up to the substitution of $-g$ with g.

Section 4 gathers together the lucid facts concerning details of the form of the Finsleroids and co-Finsleroids. The Finsleroid is a generalization of the unit ball and may be visualized as comprising a deformed surface of revolution. Its form essentially depends on the value of the characteristic parameter, g. Under changing the sign of g, the Finsleroid turns up with respect to its equatorial section. When $|g| \to 2$, the Finsleroid is extending ultimately tending in its form more and more to the cone. The form and all the properties of the co-Finsleroid are essentially similar to that of the Finsleroid of the opposite sign of the parameter g. Various Maple9-designed figures have been presented to elucidate patterns and details, and to make this Finsleroid-framework a plausible one in methodological as well educational respects, – which also show all the basic features of the Finsleroids.

The E_g^{PD}-space has an auxiliary quasi-Euclidean structure, which is deeply inherent in the development. Section 5 introduces for the E_g^{PD}-space the quasi-Euclidean map under which the Finsleroid goes into the unit ball. The quasi-Euclidean space is simple in many aspects, so that relevant transformations make reduce various calculations and may provide one with constructive ideas. The induced quasi-Euclidean metric tensor (which is not of a Finslerian type) is simply a linear combination of the Euclidean metric tensor and the product of two unit vectors. The quasi-Euclidean space is not flat, but proves to be conformally-flat.

Section 6 is devoted to reviewing the key and basic concepts determined by geodesics and angle. For the space under study, the geodesics should be obtained as solutions to the equations (2.88)–(2.89) of Chapter 2 through well-known arguments. Surprisingly, the equation admits a simple and explicit general solution, which in turn admits explicating the angle by stipulating that the Cosine Theorem of ordinary form be rigorously valid. The respective scalar product ensues. The solution with fixed points, as well as the initial-date solution, are explicitly presented. An essential non-euclidean feature is that the E_g^{PD}-geodesic curves are not flat in general.

Paper ends with Section 7 in which we note that the angle α obtained in the Finsleroid Geometry under study has the following remarkable property: if the consideration is restricted to the $(N = 2)$-dimensional

Finsleroid–Minkowski plane, then α *is equal to the respective Landsberg angle*. We are able to prove that the two-dimensional criterion for the strong convexity of indicatrix works fine. The Cartan scalar proves to be the negative of the characteristic parameter g applied. The E_g^{PD}-Generalized Trigonometric Functions are appeared.

2. Bases

Suppose we are given an N-dimensional centered vector space V_N with some point "O" being the origin. Denote by R the vectors constituting the space, so that $R \in V_N$ and it is assumed that R is issued from the point "O". Any given vector R assigns a particular direction in V_N. Let us fix a member $R_{(N)} \in V_N$, introduce the straight line e_N oriented along the vector $R_{(N)}$, and use this e_N to serve as a R^N-coordinate axis in V_N. In this way we get the topological product

$$V_N = V_{N-1} \times e_N \tag{2.1}$$

together with the separation

$$R = \{\mathbf{R}, R^N\}, \quad R^N \in e_N \quad \text{and} \quad \mathbf{R} \in V_{N-1}. \tag{2.2}$$

For convenience, we shall frequently use the notation

$$R^N = Z \tag{2.3}$$

and

$$R = \{\mathbf{R}, Z\}. \tag{2.4}$$

Also, we introduce a Euclidean metric

$$q = q(\mathbf{R}) \tag{2.5}$$

over the $(N-1)$-dimensional vector space V_{N-1}.

With respect to an admissible coordinate basis $\{e_a\}$ in V_{N-1}, we obtain the coordinate representations

$$\mathbf{R} = \{R^a\} = \{R^1, \ldots, R^{N-1}\} \tag{2.6}$$

and

$$R = \{R^p\} = \{R^a, R^N\} \equiv \{R^a, Z\}, \tag{2.7}$$

together with

$$q(\mathbf{R}) = \sqrt{r_{ab}R^aR^b}; \tag{2.8}$$

the matrix (r_{ab}) is assumed to be symmetric and positive-definite. The indices $(a, b, \dots)$ and $(p, q, \dots)$ will be specified over the ranges $(1, \dots, N-1)$ and $(1, \dots, N)$, respectively; vector indices are up, co-vector indices are down; repeated up-down indices are automatically summed; δ will stand for the Kronecker symbol, such that $(\delta_{ab}) = \mathrm{diag}\,(1, 1, \dots)$. The variables

$$w^a = R^a/Z, \quad w_a = r_{ab}w^b, \quad w = q/Z, \tag{2.9}$$

where

$$w \in (-\infty, \infty), \tag{2.10}$$

are convenient whenever $Z \neq 0$. Sometimes we shall mention the associated metric tensor

$$r_{pq} = \{r_{NN} = 1,\ r_{Na} = 0,\ r_{ab}\} \tag{2.11}$$

meaningful over the whole vector space V_N.

Given a parameter g subject to ranging

$$-2 < g < 2, \tag{2.12}$$

we introduce the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \tag{2.13}$$

$$G = g/h, \tag{2.14}$$

$$g_+ = \frac{1}{2}g + h, \qquad g_- = \frac{1}{2}g - h, \tag{2.15}$$

$$g^+ = -\frac{1}{2}g + h, \qquad g^- = -\frac{1}{2}g - h, \tag{2.16}$$

so that

$$g_+ + g_- = g, \qquad g_+ - g_- = 2h, \tag{2.17}$$

$$g^+ + g^- = -g, \qquad g^+ - g^- = 2h, \tag{2.18}$$

$$(g_+)^2 + (g_-)^2 = 2, \tag{2.19}$$

$$(g^+)^2 + (g^-)^2 = 2. \tag{2.20}$$

The symmetry

$$g_+ \overset{g \to -g}{\Longleftrightarrow} -g_-, \qquad g^+ \overset{g \to -g}{\Longleftrightarrow} -g^- \tag{2.21}$$

holds.

The *characteristic quadratic form*

$$\begin{aligned} B(g;R) &:= Z^2 + gqZ + q^2 \\ &\equiv \frac{1}{2}\left[(Z + g_+q)^2 + (Z + g_-q)^2\right] > 0 \end{aligned} \tag{2.22}$$

is of the negative discriminant, namely

$$D_{\{B\}} = -4h^2 < 0, \tag{2.23}$$

because of equations (2.12) and (2.13). Whenever $Z \neq 0$, it is also convenient to use the quadratic form

$$Q(g;w) := B/(Z)^2, \tag{2.24}$$

obtaining

$$Q(g;w) = 1 + gw + w^2 > 0, \tag{2.25}$$

together with the function

$$E(g;w) := 1 + \frac{1}{2}gw. \tag{2.26}$$

The identity

$$E^2 + h^2w^2 = Q \tag{2.27}$$

can readily be verified. In the limit $g \to 0$, the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):

$$B\big|_{g=0} = r_{pq}R^pR^q. \tag{2.28}$$

Also

$$Q\big|_{g=0} = 1 + w^2. \tag{2.29}$$

In terms of this notation, we propose

Definition. The function $K(g;R)$ given by the formulae

$$K(g;R) = \sqrt{B(g;R)}\, J(g;R), \tag{2.30}$$

where

$$J(g;R) = \mathrm{e}^{\frac{1}{2}G\Phi(g;R)}, \tag{2.31}$$

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\left(\frac{q}{hZ} + \frac{G}{2}\right), \quad \text{if } Z \ge 0, \tag{2.32}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\left(\frac{q}{hZ} + \frac{G}{2}\right), \quad \text{if } Z \le 0, \tag{2.33}$$

is called the *Finsleroid metric function* (the FMF for short).

Other convenient forms for the function Φ are

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\left(\frac{L(g;R)}{hZ}\right), \quad \text{if } Z \ge 0, \tag{2.34}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\left(\frac{L(g;R)}{hZ}\right), \quad \text{if } Z \le 0, \tag{2.35}$$

where

$$L(g;R) = q + \frac{g}{2}Z, \tag{2.36}$$

and also

$$\Phi(g;R) = \arctan\frac{A(g;R)}{hq}, \tag{2.37}$$

where

$$A(g;R) = Z + \frac{1}{2}gq. \tag{2.38}$$

This FMF has been normalized to show the properties

$$-\frac{\pi}{2} \le \Phi \le \frac{\pi}{2} \tag{2.39}$$

and

$$\begin{aligned} &\Phi = \frac{\pi}{2}, \quad \text{if } q = 0 \text{ and } Z > 0;\\ &\Phi = -\frac{\pi}{2}, \quad \text{if } q = 0 \text{ and } Z < 0. \end{aligned} \tag{2.40}$$

We also have

$$\tan\Phi = \frac{A}{hq} \tag{2.41}$$

and

$$\Phi|_{Z=0} = \arctan\frac{G}{2}. \tag{2.42}$$

It is often convenient to use the sign indicator ϵ_Z for the argument Z:

$$\begin{gathered}\epsilon_Z = 1, \quad \text{if } Z > 0; \qquad \epsilon_Z = -1, \quad \text{if } Z < 0; \\ \epsilon_Z = 0, \quad \text{if } Z = 0.\end{gathered} \tag{2.43}$$

Under these conditions, we introduce

Definition. The arisen space

$$E_g^{PD} := \{V_N = V_{N-1} \times e_N;\ R \in V_N;\ K(g;R);\ g\} \tag{2.44}$$

is called the E_g^{PD}*-space*, or alternatively the *Finsleroid space*.

The right-hand part of the definition (2.30) can be considered to be a function $\breve{K}$ of the arguments $\{g; q, Z\}$, such that

$$\breve{K}(g;q,Z) = K(g;R). \tag{2.45}$$

We observe that

$$\breve{K}(g;q,-Z) \neq \breve{K}(g;q,Z), \qquad \text{unless} \quad g = 0. \tag{2.46}$$

Instead, the function $\breve{K}$ shows the property of gZ*-parity*

$$\breve{K}(-g;q,-Z) = \breve{K}(g;q,Z). \tag{2.47}$$

The $(N-1)$-space reflection invariance holds true

$$K(g;R) \overset{R^a \leftrightarrow -R^a}{\Longleftrightarrow} K(g;R) \tag{2.48}$$

(such an operation does not influence the quantity q).

It is frequently convenient to rewrite the representation (2.30) in the form

$$K(g;R) = |Z|V(g;w), \tag{2.49}$$

whenever $Z \neq 0$, with the *generating metric function*

$$V(g;w) = \sqrt{Q(g;w)}\, j(g;w). \tag{2.50}$$

We have

$$j(g;w) = J(g;1,w).$$

Using (2.25) and (2.31)–(2.35), we obtain

$$V' = wV/Q, \qquad V'' = V/Q^2, \tag{2.51}$$

$$(V^2/Q)' = -gV^2/Q^2, \quad (V^2/Q^2)' = -2(g+w)V^2/Q^3, \tag{2.52}$$

$$j' = -\frac{1}{2}gj/Q, \tag{2.53}$$

and also

$$\frac{1}{2}(V^2)' = wV^2/Q, \qquad \frac{1}{2}(V^2)'' = (Q-gw)V^2/Q^2, \tag{2.54}$$

$$\frac{1}{4}(V^2)''' = -gV^2/Q^3, \tag{2.55}$$

together with

$$\Phi' = -h/Q, \tag{2.56}$$

where the prime ($'$) denotes the differentiation with respect to w.

Also,

$$(A(g;R))^2 + h^2q^2 = B(g;R) \tag{2.57}$$

and

$$(L(g;R))^2 + h^2Z^2 = B(g;R). \tag{2.58}$$

The simple results for these derivatives reduce the task of computing the components of the associated FMT to an easy exercise, indeed:

$$R_p := \frac{1}{2}\frac{\partial K^2(g;R)}{\partial R^p}:$$

$$R_a = r_{ab}R^b\frac{K^2}{B}, \qquad R_N = (Z+gq)\frac{K^2}{B}; \tag{2.59}$$

$$g_{pq}(g;R) := \frac{1}{2}\frac{\partial^2 K^2(g;R)}{\partial R^p \partial R^q} = \frac{\partial R_p(g;R)}{\partial R^q}:$$

$$g_{NN}(g;R)=[(Z+gq)^2+q^2]\frac{K^2}{B^2},\quad g_{Na}(g;R)=gqr_{ab}R^b\frac{K^2}{B^2},\tag{2.60}$$

$$g_{ab}(g;R)=\frac{K^2}{B}r_{ab}-g\frac{r_{ad}R^dr_{be}R^eZ}{q}\frac{K^2}{B^2}.\tag{2.61}$$

The reciprocal tensor components are

$$g^{NN}(g;R)=(Z^2+q^2)\frac{1}{K^2},\qquad g^{Na}(g;R)=-gqR^a\frac{1}{K^2},\tag{2.62}$$

$$g^{ab}(g;R)=\frac{B}{K^2}r^{ab}+g(Z+gq)\frac{R^aR^b}{q}\frac{1}{K^2}.\tag{2.63}$$

The determinant of the FMT given by equations (2.60)–(2.61) can readily be found in the form

$$\det(g_{pq}(g;R))=[J(g;R)]^{2N}\det(r_{ab})\tag{2.64}$$

which shows, on noting (2.31)–(2.33), that

$$\det(g_{pq})>0\quad\text{over all the space}\quad V_N.\tag{2.65}$$

The associated *angular metric tensor*

$$h_{pq}:=g_{pq}-R_pR_q\frac{1}{K^2}$$

proves to be given by the components

$$h_{NN}(g;R)=q^2\frac{K^2}{B^2},\qquad h_{Na}(g;R)=-Zr_{ab}R^b\frac{K^2}{B^2},$$

$$h_{ab}(g;R)=\frac{K^2}{B}r_{ab}-(gZ+q)\frac{r_{ad}R^dr_{be}R^e}{q}\frac{K^2}{B^2},$$

which entails

$$\det(h_{ab})=\det(g_{pq})\frac{1}{V^2}.$$

The use of the components of the Cartan tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.

Theorem 2.1. *The Cartan tensor associated with the FMF* (2.30) *is of the following special algebraic form:*

$$C_{pqr} = \frac{1}{N}\left(h_{pq}C_r + h_{pr}C_q + h_{qr}C_p - \frac{1}{C_sC^s}C_pC_qC_r\right) \tag{2.66}$$

with

$$C_tC^t = \frac{N^2}{4K^2}g^2. \tag{2.67}$$

Elucidating the structure of the respective curvature tensor

$$S_{pqrs} := (C_{tqr}C_p{}^t{}_s - C_{tqs}C_p{}^t{}_r) \tag{2.68}$$

results in the simple representation

$$S_{pqrs} = -\frac{C_tC^t}{N^2}(h_{pr}h_{qs} - h_{ps}h_{qr}). \tag{2.69}$$

Inserting here (2.67), we are led to

Theorem 2.2. *The curvature tensor of the space E_g^{PD} is of the special type*

$$S_{pqrs} = S^*(h_{pr}h_{qs} - h_{ps}h_{qr})/K^2 \tag{2.70}$$

with

$$S^* = -\frac{1}{4}g^2. \tag{2.71}$$

Definition. FMF (2.30) generates the *Finsleroid*

$$F_g^{PD} := \{R \in V_N : K(g;R) \le 1\}. \tag{2.72}$$

Definition. The *Finsleroid Indicatrix* I_g^{PD} is the boundary of the Finsleroid:

$$I_g^{PD} := \{R \in V_N : K(g;R) = 1\}. \tag{2.73}$$

Note. Since at $g = 0$ the space E_g^{PD} is Euclidean, then the body $F_{g=0}^{PD}$ is a unit ball and $I_{g=0}^{PD}$ is a unit sphere.

Recalling the known formula $R = 1 + S^*$ for the indicatrix curvature (see Section 1.2 in [4]), from (2.71) we conclude that

$$R_{\text{Finsleroid Indicatrix}} = h^2 = 1 - \frac{1}{4}g^2, \tag{2.74}$$

so that

$$0 < R_{\text{Finsleroid Indicatrix}} \le 1$$

and

$$R_{\text{Finsleroid Indicatrix}} \stackrel{g\to 0}{\Longrightarrow} R_{\text{Euclidean Sphere}} = 1.$$

Geometrically, the fact that the quantity (2.74) is independent of vectors R means that the indicatrix curvature is constant. Therefore, we have arrived at

Theorem 2.3. *The Finsleroid Indicatrix I_g^{PD} is a space of constant positive curvature*

Also, on comparing between the result (2.74) and equation (2.22)–(2.23), we obtain

Theorem 2.4. *The Finsleroid curvature relates to the discriminant of the input characteristic quadratic form* (2.22) *simply as*

$$R_{\text{Finsleroid Indicatrix}} = -\frac{1}{4}D_{\{B\}}. \tag{2.75}$$

Points of the indicatrix can be represented by means of the *unit vectors* $l = \{l^p\}$:

$$l^p = \frac{R^p}{K(g;R)}, \tag{2.76}$$

so that

$$K(g;l) \equiv 1. \tag{2.77}$$

The vectors can conveniently be parameterized as follows:

$$\begin{aligned} l^a &= n^a \frac{\sin f}{h} \exp\left(\frac{1}{2}G\left(f - \frac{\pi}{2}\right)\right), \\ l^N &= \left(\cos f - \frac{1}{2}G\sin f\right)\exp\left(\frac{1}{2}G\left(f - \frac{\pi}{2}\right)\right), \end{aligned} \tag{2.78}$$

where

$$f \in [0, \pi] \tag{2.79}$$

and n^a are the components that are taken to fulfill

$$r_{ab}n^a n^b = 1; \tag{2.80}$$

also,

$$J(g;l) = \exp\left(-\frac{1}{2}G\left(f-\frac{\pi}{2}\right)\right) \tag{2.81}$$

(cf. (2.31)). The reader is advised to verify that

$$A(g;l) = \frac{1}{J(g;l)}\cos f$$

and

$$\frac{hq}{A(g;l)} = \tan f.$$

Therefore, it is appropriate to take

$$f = \arctan\frac{hq}{A}, \tag{2.82}$$

in which case from (2.37) it follows that

$$\Phi(g;l) = \frac{\pi}{2} - f.$$

At the same time, for the function (2.22) we find

$$B(g;l) = \left(\frac{1}{J(g;l)}\right)^2 = \exp\left(G\left(f-\frac{\pi}{2}\right)\right).$$

This method can farther be extended for the whole space by taking the parameterizations

$$R^a = \frac{K}{hJ}n^a\sin f, \qquad R^N = \frac{K}{J}\left(\cos f - \frac{1}{2}G\sin f\right), \tag{2.83}$$

which entails

$$\frac{\partial R^p}{\partial K} = \frac{1}{K}R^p, \tag{2.84}$$

$$\frac{\partial R^a}{\partial f} = \frac{K}{hJ}n^a\left(\cos f + \frac{1}{2}G\sin f\right), \quad \frac{\partial R^N}{\partial f} = -\frac{K}{h^2J}\sin f, \tag{2.85}$$

$$\frac{\partial^2 R^a}{\partial f^2} = \frac{K}{hJ}n^a\left(G\cos f - \left(1-\frac{1}{4}G^2\right)\sin f\right), \tag{2.86}$$

and

$$\frac{\partial^2 R^N}{\partial f^2} = -\frac{K}{h^2 J}\left(\cos f + \frac{1}{2}G\sin f\right). \tag{2.87}$$

Last, we write down the explicit components of the relevant Finsleroid Cartan tensor

$$C_{pqr} := \frac{1}{2}\frac{\partial g_{pq}}{\partial R^r}:$$

$$R^N C_{NNN} = gw^3V^2Q^{-3}, \qquad R^N C_{aNN} = -gww_aV^2Q^{-3},$$

$$R^N C_{abN} = \frac{1}{2}gwV^2Q^{-2}r_{ab} + \frac{1}{2}g(1-gw-w^2)w_aw_bw^{-1}V^2Q^{-3},$$

$$R^N C_{abc} = -\frac{1}{2}gV^2Q^{-2}w^{-1}(r_{ab}w_c + r_{ac}w_b + r_{bc}w_a)$$
$$+ gw_aw_bw_cw^{-3}\left(\frac{1}{2}Q + gw + w^2\right)V^2Q^{-3};$$

and

$$R^N C_N{}^N{}_N = gw^3/Q^2, \qquad R^N C_a{}^N{}_N = -gww_a/Q^2,$$

$$R^N C_N{}^a{}_N = -gw(1+gw)w^a/Q^2,$$

$$R^N C_a{}^N{}_b = \frac{1}{2}gwr_{ab}/Q + \frac{1}{2}g(1-gw-w^2)w_aw_b/wQ^2,$$

$$R^N C_N{}^a{}_b = \frac{1}{2}gw\delta^a_b/Q + \frac{1}{2}g(1+gw-w^2)w^aw_b/wQ^2,$$

$$R^N C_a{}^b{}_c = -\frac{1}{2}g\left(\delta^b_a w_c + \delta^b_c w_a + (1+gw)r_{ac}w^b\right)/wQ$$
$$+ \frac{1}{2}g(gwQ + Q + 2w^2)w_aw^bw_c/w^3Q^2.$$

The components have been calculated by the help of the formulae (2.51)–(2.54).

The use of the contractions

$$R^N C_a{}^b{}_c r^{ac} = -g\frac{w^b}{w}\frac{1+gw}{Q}\left(\frac{N-2}{2} + \frac{1}{Q}\right)$$

and

$$R^N C_a{}^b{}_c w^a w^c = -g\frac{w}{Q^2}(1+gw)w^b$$

is convenient in many calculations.

Also

$$R^N C_N = \frac{N}{2}gwQ^{-1}, \qquad R^N C_a = -\frac{N}{2}g(w_a/w)Q^{-1},$$

$$R^N C^N = \frac{N}{2}gw/V^2, \qquad R^N C^a = -\frac{N}{2}gw^a(1+gw)/wV^2,$$

$$C^N = \frac{N}{2}gwR^N K^{-2}, \qquad C^a = -\frac{N}{2}gw^a(1+gw)w^{-1}R^N K^{-2},$$

$$C_p C^p = \frac{N^2}{4K^2}g^2.$$

The respective E_g^{PD}-*geodesic equation* reads

$$\frac{d^2R^p}{ds^2} + C_q{}^p{}_r(g;R)\frac{dR^q}{ds}\frac{dR^r}{ds} = 0, \tag{2.88}$$

where s is the arc-length parameter defined by

$$ds = \sqrt{g_{pq}(g;R)dR^p dR^q}. \tag{2.89}$$

3. Associated Finsleroid Hamiltonian function

Considering the co-vector space $\hat{V}_N$ dual to the vector space V_N used in the preceding Section 2, and denoting by $\hat{R}$ the respective co-vectors, so that $\hat{R} \in \hat{V}_N$, we may introduce the co-counterparts of the formulas (2.1)–(2.11), obtaining the topological product

$$\hat{V}_N = \hat{V}_{N-1} \times \hat{e}_N \tag{3.1}$$

and the separation $\hat{R} = \{\hat{\mathbf{R}}, R_N\}$, $R_N \in \hat{e}_N$ and $\hat{\mathbf{R}} \in \hat{V}_{N-1}$. Then we put $R_N = \hat{Z}$, $\hat{R} = \{\hat{\mathbf{R}}, \hat{Z}\}$, and introduce a metric $\hat{q} = \hat{q}(\hat{\mathbf{R}})$ over the $(N-1)$-dimensional co-vector space $\hat{V}_{N-1}$.

With respect to a coordinate basis $\{\hat{e}_a\}$ dual to $\{e_a\}$, we obtain in $\hat{V}_{N-1}$ the coordinate representations

$$\hat{\mathbf{R}} = \{R_a\} = \{R_1, \ldots, R_{N-1}\} \tag{3.2}$$

and

$$\hat{R} = \{R_p\} = \{R_a, R_N\} \equiv \{R_a, \hat{Z}\} \tag{3.3}$$

together with

$$\hat{q}(\hat{\mathbf{R}}) = \sqrt{r^{ab} R_a R_b}, \tag{3.4}$$

where r^{ab} are the contravariant components of a symmetric positive-definte tensor defined over $\hat{V}_{N-1}$; the tensor is determined by the reciprocity $r_{ab} r^{bc} = \delta^c_a$. The co-version

$$r^{pq} = \{r^{NN} = 1,\ r^{Na} = 0,\ r^{ab}\} \tag{3.5}$$

of the input metric tensor (2.11) is meaningful over the space $\hat{V}_N$. The parameter g introduced in equations (2.12)–(2.13), as well as the explicated formulae (2.14)–(2.21), are applicable in the co-approach, too.

Under these conditions, the fundamental definition

$$H(g; \hat{R}) = K(g; R) \tag{3.6}$$

for the FHF is used, and the $\hat{E}^{PD}_g$*-space*

$$\hat{E}^{PD}_g := \{\hat{V}_N = \hat{V}_{N-1} \times \hat{e}_N;\ \hat{R} \in \hat{V}_N;\ H(g; \hat{R});\ g\} \tag{3.7}$$

is set forth.

A careful consideration on the basis of the formulae (3.6) and (2.59)–(2.63) leads to

Theorem 3.1. *The symmetry*

$$K(g; R) \stackrel{\left\{ \substack{g \longleftrightarrow -g, \\ R \longleftrightarrow \hat{R}} \right\}}{\Longleftrightarrow} H(g; \hat{R}) \tag{3.8}$$

holds fine.

Treating the FHF (3.6) as a function $\breve{H}$ of the arguments $\{g;\hat{q},\hat{Z}\}$, such that

$$\breve{H}(g;\hat{q},\hat{Z}) = H(g;\hat{R}), \tag{3.9}$$

the co-counterparts of equations (2.46)–(2.48) is appeared:

$$\breve{H}(g;\hat{q},-\hat{Z}) \neq \breve{H}(g;\hat{q},\hat{Z}), \quad \text{unless} \quad g=0; \tag{3.10}$$

the $g\hat{Z}$*-parity*:

$$\breve{H}(-g;\hat{q},-\hat{Z}) = \breve{H}(g;\hat{q},\hat{Z}); \tag{3.11}$$

and

$$H(g;R) \stackrel{R_a \leftrightarrow -R_a}{\Longleftrightarrow} H(g;R). \tag{3.12}$$

Definition. The body

$$\hat{F}_g^{PD} := \{\hat{R} \in \hat{V}_N : H(g;\hat{R}) \le 1\} \tag{3.13}$$

is called the *co-Finsleroid*.

Definition. The respective figuratrix defined by the equation

$$\hat{I}_g^{PD} := \{\hat{R} \in \hat{V}_N : H(g;\hat{R}) = 1\} \tag{3.14}$$

is called the *co-Finsleroid Indicatrix*.

We remain it to the reader to verify that Theorems 2.3–2.4 proven in the preceding Section 2 can well be re-formulated in the co-approach:

Theorem 3.2. *The co-Finsleroid Indicatrix $\hat{I}_g^{PD}$ is a constant-curvature space with the positive curvature value* (2.74)*:*

$$\mathcal{R}_{\text{co-Finsleroid Indicatrix}} = \mathcal{R}_{\text{Finsleroid Indicatrix}} \tag{3.15}$$

and

$$\mathcal{R}_{\text{co-Finsleroid Indicatrix}} = h^2 = 1 - \frac{1}{4}g^2, \qquad 0 < \mathcal{R}_{\text{co-Finsleroid Indicatrix}} \le 1. \tag{3.16}$$

The formula

$$\mathcal{R}_{\text{co-Finsleroid Indicatrix}} = -\frac{1}{4}D_{\{\hat{B}\}} \tag{3.17}$$

is valid.

4. Shape of Finsleroid and co-Finsleroid

The Finsleroid is not "uniform" in all directions and, therefore, does not permit general rotations. In terms of the function (2.45), the Finsleroid equation (see the definition (2.72)) reads

$$\breve{K}(g;q,Z) = 1. \tag{4.1}$$

From (2.30)–(2.35) it follows directly that the value

$$q^* := q\big|_{Z=0} \tag{4.2}$$

of the quantity q over the Finsleroid is given by

$$q^*(g) = \exp\left(-\frac{G}{2}\arctan\frac{G}{2}\right); \tag{4.3}$$

with the definitions

$$Z_1(g) = Z|_{q=0}, \quad \text{when} \quad Z < 0, \tag{4.4}$$

and

$$Z_2(g) = Z|_{q=0}, \quad \text{when} \quad Z > 0, \tag{4.5}$$

we obtain

$$Z_1(g) = -e^{G\pi/4} \quad \text{and} \quad Z_2(g) = e^{-G\pi/4}. \tag{4.6}$$

Thus at any given value g we obtain the simple and explicit value for the altitude of the Finsleroid:

Theorem 4.1. *We have*

$$\textit{The Altitude of Finsleroid} = Z_2(g) - Z_1(g) = 2\cosh\frac{G\pi}{4}.$$

The equation (4.1) cannot be resolved to find the function

$$Z = Z(g;q) \tag{4.7}$$

in an explicit form, because of a rather high complexity of the right-hand parts of equations (2.30)–(2.35). Nevertheless, differentiating the identity

$$\breve{K}(g;q,Z(g;q)) = 1 \tag{4.8}$$

(see (4.1)) yields, on using (2.51), the simple results for the first derivatives:

$$\frac{\partial Z(g;q)}{\partial q} = -\frac{q}{Z+gq} \tag{4.9}$$

and

$$\frac{\partial^2 Z(g;q)}{\partial q^2} = -\frac{B(g;R)}{(Z+gq)^3}. \tag{4.10}$$

We also get

$$\left.\frac{dZ(g;q)}{dq}\right|_{q=0} = 0 \quad \text{and} \quad \frac{dZ(g;q)}{dq} \underset{Z\to+0}{\Longrightarrow} -\frac{1}{g}. \tag{4.11}$$

Inversely, for the function

$$q = q(g;Z) \tag{4.12}$$

obeying (4.1) we obtain

$$\frac{\partial q}{\partial Z} = -g - \frac{Z}{q} \tag{4.13}$$

and

$$\frac{\partial^2 q}{\partial Z^2} = -\frac{B(g;R)}{q^3} < 0. \tag{4.14}$$

We have

$$\frac{\partial q(g;Z)}{\partial Z} > 0, \quad \text{if } Z < -gq; \qquad \frac{\partial q(g;Z)}{\partial Z} < 0, \quad \text{if } Z > -gq. \tag{4.15}$$

Also,

$$\frac{\partial q}{\partial Z} = 0, \quad \text{if } Z = Z^{**} \quad \text{with} \quad Z^{**} = -gq^{**}. \tag{4.16}$$

Inserting this Z^{**} in (2.30)–(2.35) yields

$$\Phi^{**} = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\frac{g^2-2}{2gh}, \quad \text{if } Z^{**} \ge 0 \sim g < 0, \tag{4.17}$$

$$\Phi^{**} = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\frac{g^2-2}{2gh}, \quad \text{if } Z^{**} \le 0 \sim g > 0, \tag{4.18}$$

and

$$q^{**}(g) = \mathrm{e}^{-\frac{1}{2}G\Phi^{**}} \tag{4.19}$$

together with

$$Z^{**}(g) = -g\,\mathrm{e}^{-\frac{1}{2}G\Phi^{**}}\,. \tag{4.20}$$

Therefore, the following assertion can be set up for the width of the Finsleroid.

Theorem 4.2. *With any given g,*

$$\textit{The Width of Finsleroid} = 2q^{**}(g) = 2\,\mathrm{e}^{-\frac{1}{2}G\Phi^{**}}\,.$$

The formulas (4.19) and (4.20) may also be interpreted by saying that The *Equatorial Section* of the Finsleroid is of the radius

$$r_{\text{Equatorial}} = q^{**} \tag{4.21}$$

and cuts the Z-axis at

$$Z_{\text{Equatorial}} = Z^{**}. \tag{4.22}$$

With the parameter value $|g|$ being increasing, the Finsleroid is stretching in wide and altitude:

$$q^{**} \underset{|g|\to 2}{\Longrightarrow} \infty \tag{4.23}$$

and

$$|Z^{**}| \underset{|g|\to 2}{\Longrightarrow} \infty, \tag{4.24}$$

tending in its shape to a cone:

$$\frac{q^{**}}{|Z^{**}|} \underset{|g|\to 2}{\Longrightarrow} \frac{1}{2}, \tag{4.25}$$

such that the vertex of the Finsleroid tends to approach the origin point "O". From (4.23) one can infer

Theorem 4.3. *We have:*

$$\textit{The Limiting Vertex Angle} = 2\arctan\frac{1}{2}.$$

The above formulae, particularly the negative sign of the second derivative (4.10), can be used directly to verify the following

Theorem 4.4. *The Finsleroid Indicatrix I_g^{PD} is closed, regular, and strongly convex.*

The co-Finsleroid equation

$$\breve{H}(g;\hat{q},\hat{Z}) = 1 \tag{4.26}$$

(cf. equations (3.9) and (3.14)) can be studied in a similar way, leading to the relations obtainable from equations (4.3)–(4.20) by means of the formal replacement $\{g \to -g, R \to \hat{R}\}$, owing to the fundamental symmetry (3.8).

Therefore, we can state the following:

Theorem 4.5. *The co-Finsleroid Indicatrix $\hat{I}_g^{PD}$ is closed, regular, and also strongly convex.*

Theorem 4.6. *At any given parameter g, the Finsleroid and the co-Finsleroid mirror one another under the g-reflection:*

$$F_g^{PD} \overset{g \longleftrightarrow -g}{\Longleftrightarrow} \hat{F}_{-g}^{PD}. \tag{4.27}$$

All figures shown below have been prepared by means of a precise use of Maple9.

In Figures 2–7 bold lines serve to draw the Finsleroids, while unit circles simulate the ordinary Euclidean spheres. Figure 2 may be used as a convenient demonstration example (the *trainer*) for the Finsleroid by showing various structure details, including the equatorial section and the characteristic tangents, in a distinct way. We remain it to the reader to evaluate the angles that are depicted in the example and find among them equal cases.

Figures 2–7 clearly support the validity of Theorem 4.4 about regularity and convexity and make an idea of existence a diffeomorphic spherical map (see equation (5.1) below) a quite trustworthy one.

If one compares between Figure 2 and Figure 3 between Figure 4 and Figure 5 or between Figure 6 and Figure 7, one observes immediately that the change of sign of the characteristic Finslerian parameter g does *turn up* the figures and, therefore, verifies the fundamental Finslerian Z-parity property (as given by equation (2.47)) in a due visual way. In a narrow sense, Figures 2–7 show the geometry of the generatrix for the Finsleroids, the latter being (hyper)surfaces of revolution over the Z-axis.

It can be traced also how the parameter g effects the shape of Finsleroid. A positive value g (deforms and) shifts the unit sphere in the down-wise manner, respectively a negative value in the up-wise manner.

Figure 8 and Figure 9 model the important functions (4.3) and (4.20), respectively.

Figure 10 visualizes the $\{|g| \to 2\}$-limiting case.

There is no need to picture co-Finsleroids, for they mirror Finsleroids with respect to the $(R^N = 0)$-plane (according to our Theorems 4.5 and 4.6 and corresponding equations (3.8) and (4.27)). In particular, at $g = -0.4$, the co-Finsleroid looks like the demonstration example given by Figure 1 at $g = 0.4$ (with interchanging respectively the coordinate axes: R^N with P_N and $\mathbf{R}$ with $\mathbf{P}$).

5. Quasi-Euclidean map of Finsleroid

Theorem 4.4 can be continued farther by indicating the diffeomorphism

$$F_g^{PD} \stackrel{i_g}{\Longrightarrow} B^{PD} \tag{5.1}$$

of the Finsleroid $F_g^{PD} \subset V_N$ to the unit ball $B^{PD} \subset V_N$:

$$B^{PD} := \{R \in B^{PD} : S(R) \leq 1\}, \tag{5.2}$$

where

$$S(R) = \sqrt{r_{pq}R^pR^q} \equiv \sqrt{(R^N)^2 + r_{ab}R^aR^b} \tag{5.3}$$

is the input Euclidean metric function (see (2.11)). Indeed, the diffeomorphism (5.1) can always be extended to get the diffeomorphic map

$$V_N \stackrel{\sigma_g}{\Longrightarrow} V_N \tag{5.4}$$

of the whole vector space V_N by means of the homogeneity:

$$\sigma_g \cdot (bR) = b\sigma_g \cdot R, \quad b > 0. \tag{5.5}$$

To this end it is sufficient to take merely

$$\sigma_g \cdot R = \|R\| i_g \cdot \left(\frac{R}{\|R\|}\right), \tag{5.6}$$

where

$$\|R\| = K(g; R). \tag{5.7}$$

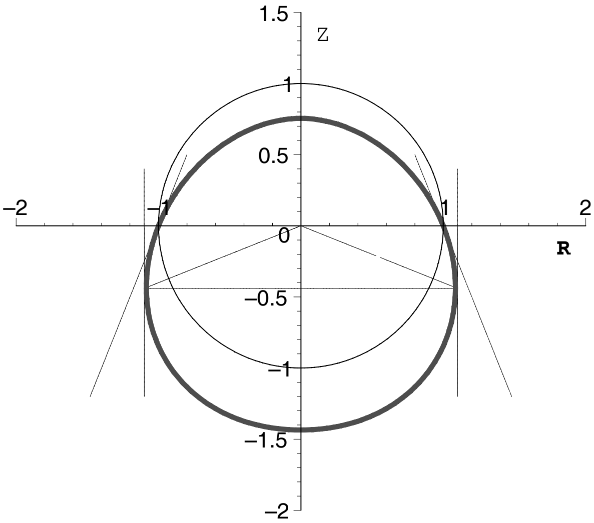

Fig 1. $[g = 0.4]$

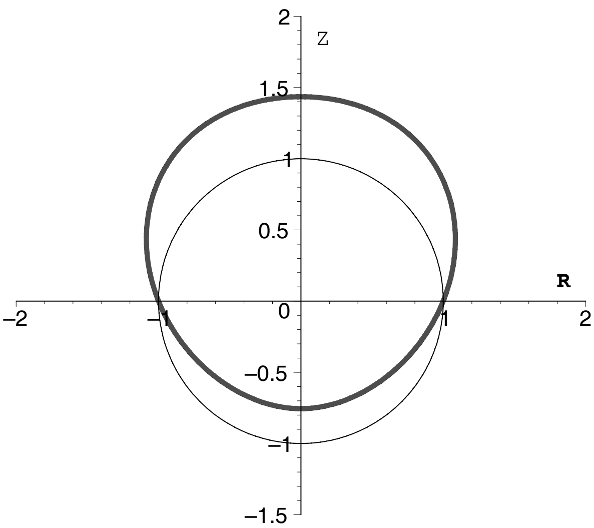

Fig 2. $[g = -0.4]$

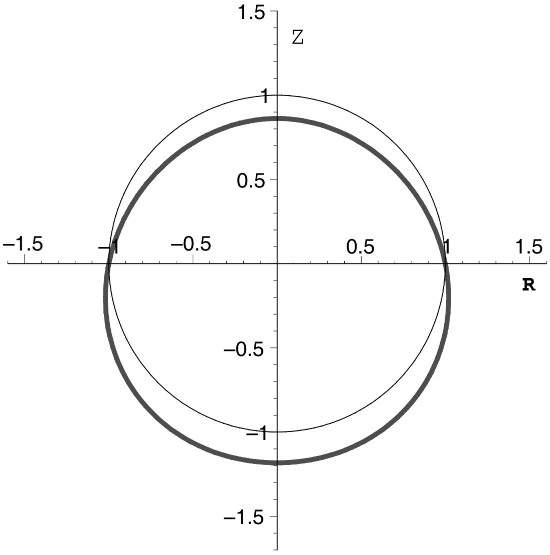

Fig 3. $[g = 0.2]$

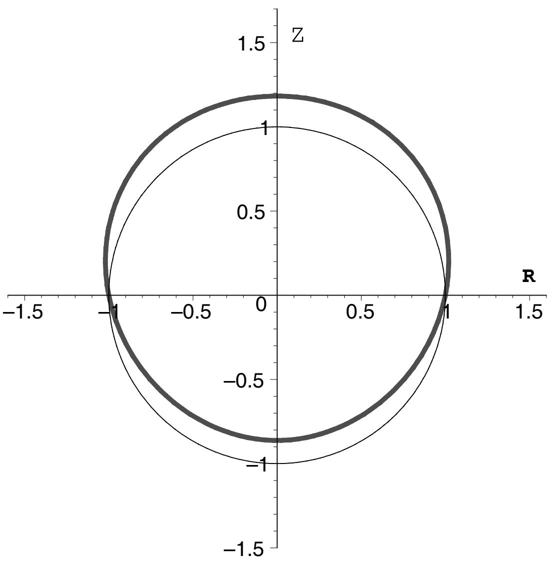

Fig 4. $[g = -0.2]$

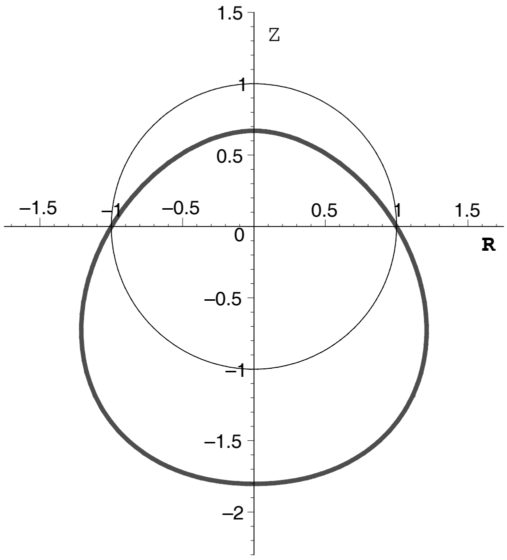

Fig 5. $[g = 0.6]$

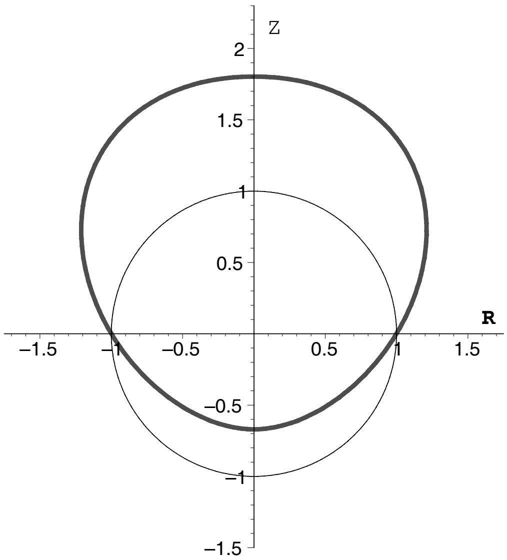

Fig 6. $[g = -0.6]$

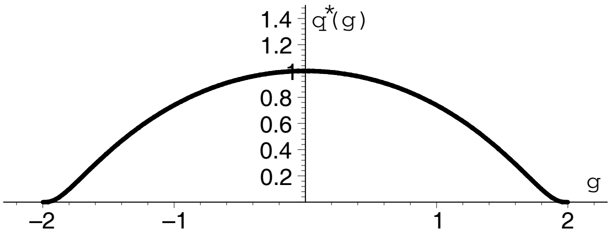

Fig 7. [Equation (4.3)]

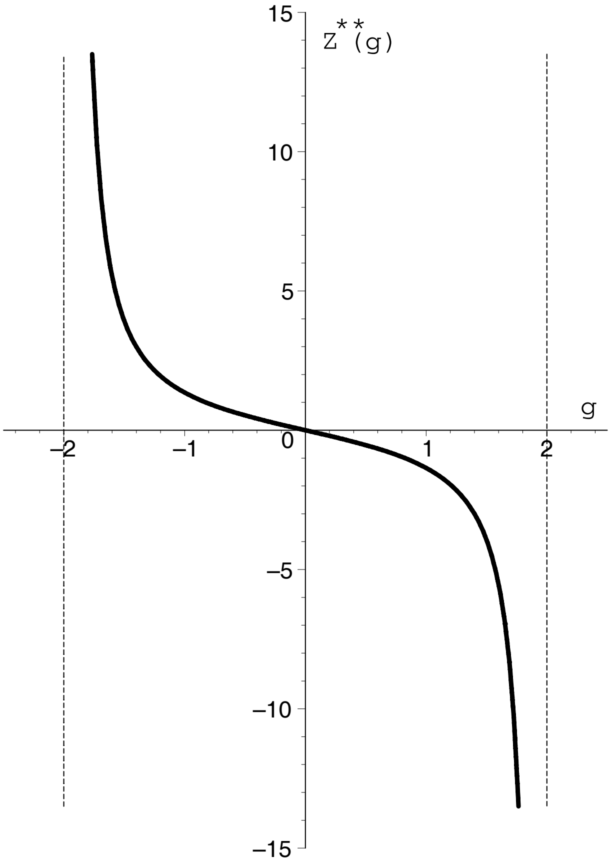

Fig 8. [Equation (4.20)]

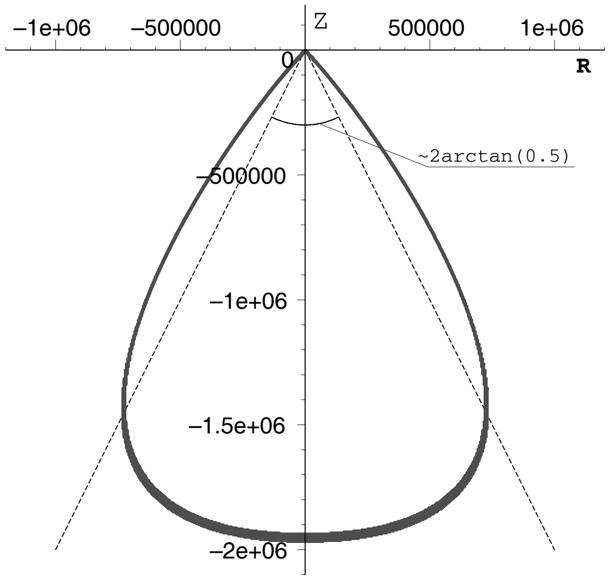

Fig 9. $[g = 1.96]$

Equations (5.1)–(5.7) entail

$$K(g; R) = S(\sigma_g \cdot R). \tag{5.8}$$

At the same time, the identity (2.57) suggests taking the map

$$\bar{R} = \sigma_g \cdot R \tag{5.9}$$

by means of the components

$$\bar{R}^p = \sigma^p(g; R) \tag{5.10}$$

with

$$\sigma^a = R^a h J(g; R), \qquad \sigma^N = A(g; R) J(g; R), \tag{5.11}$$

where $J(g; R)$ and $A(g; R)$ are the functions (2.31) and (2.38). Indeed, inserting (5.11) in (5.3) and taking into account equations (2.30) and (2.57), we get the identity

$$S(\bar{R}) = K(g; R) \tag{5.12}$$

which is tantamount to the implied relation (5.8).

Thus we have arrived at

Theorem 5.1. *The map given explicitly by equations* (5.9)–(5.11) *assigns the diffeomorphism between the Finsleroid and the unit ball according to equations* (5.1)–(5.8).

The inverse

$$R = \mu_g \cdot \bar{R}, \qquad \mu_g = (\sigma_g)^{-1} \tag{5.13}$$

of the transformation (5.9)–(5.11) can be presented by the components

$$R^p = \mu^p(g; \bar{R}) \tag{5.14}$$

with

$$\mu^a = \bar{R}^a / hk(g; \bar{R}), \qquad \mu^N = I(g; \bar{R})/k(g; \bar{R}), \tag{5.15}$$

where

$$k(g; \bar{R}) := J(g; \mu(g; \bar{R})) \tag{5.16}$$

and

$$I(g; \bar{R}) = \bar{R}^N - \frac{1}{2} G \sqrt{r_{ab} \bar{R}^a \bar{R}^b}. \tag{5.17}$$

The identity

$$\mu^p(g; \sigma(g; R)) \equiv R^p \tag{5.18}$$

can readily be verified. Notice that

$$\frac{\bar{R}^a}{S(\bar{R})} = \frac{hR^a}{\sqrt{B(g;R)}}, \qquad \frac{\bar{R}^N}{S(\bar{R})} = \frac{A(g;R)}{\sqrt{B(g;R)}},$$

$$\frac{\sqrt{r_{ab}\bar{R}^a \bar{R}^b}}{\bar{R}^N} = \frac{hq}{A(g;R)}, \qquad w^a = \frac{R^a}{R^N} = \frac{\bar{R}^a}{hI(g;\bar{R})}, \tag{5.19}$$

and

$$\sqrt{B}/R^N = S/I, \qquad \sqrt{Q} = S/I. \tag{5.20}$$

The σ_g-image

$$\phi(g; \bar{R}) := \Phi(g; R)\big|_{R=\mu(g;\bar{R})} \tag{5.21}$$

of the function Φ described by equations (2.32)–(2.42) is of a clear meaning of angle:

$$\phi(g;\bar{R}) = \arctan \frac{\bar{R}^N}{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}} \tag{5.22}$$

(equation (5.19) has been used) which ranges over

$$-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}. \tag{5.23}$$

We have

$$\begin{aligned} \phi &= \frac{\pi}{2}, \quad \text{if } \bar{R}^a = 0 \text{ and } \bar{R}^N > 0; \\ \phi &= -\frac{\pi}{2}, \quad \text{if } \bar{R}^a = 0 \text{ and } \bar{R}^N < 0, \end{aligned} \tag{5.24}$$

and also

$$\phi|_{\bar{R}^N=0} = 0. \tag{5.25}$$

Comparing equations (5.16) and (2.31) shows that

$$k = \mathrm{e}^{\frac{1}{2}G\phi}. \tag{5.26}$$

The right-hand parts in (5.11) are homogeneous functions of degree 1:

$$\sigma^p(g;bR) = b\sigma^p(g;R), \quad b > 0. \tag{5.27}$$

Therefore, the identity

$$\sigma^p_s(g;R)R^s = \bar{R}^p \tag{5.28}$$

should be valid for the derivatives

$$\sigma^q_p(g;R) := \frac{\partial \sigma^q(g;R)}{\partial R^p}. \tag{5.29}$$

The simple representations

$$\begin{aligned} \sigma^N_N(g;R) &= \left(B + \frac{1}{2}gqA\right)\frac{J}{B}, \\ \sigma^N_a(g;R) &= -\frac{g(ZA-B)}{2q}\frac{Jr_{ab}R^b}{B}, \end{aligned} \tag{5.30}$$

$$\sigma^a_N(g;R) = \frac{1}{2}gq\frac{JR^a h}{B},$$
$$\sigma^a_b(g;R) = \left(B\delta^a_b - \frac{gr_{bc}R^cR^aZ}{2q}\right)\frac{Jh}{B}, \tag{5.31}$$

and also the determinant

$$\det(\sigma^q_p) = h^{N-1}J^N \tag{5.32}$$

are obtained.

Henceforth, to simplify notation, we shall use the substitution

$$t^p = \bar{R}^p. \tag{5.33}$$

Again, we can note the homogeneity

$$\mu^p(g;bt) = b\mu^p(g;t), \quad b > 0, \tag{5.34}$$

for the functions (5.15), which entails the identity

$$\mu^p_s(g;t)t^s = R^p \tag{5.35}$$

for the derivatives

$$\mu^p_q(g;t) := \frac{\partial \mu^p(g;t)}{\partial t^q}. \tag{5.36}$$

We find

$$\mu^N_N = 1/k(g;t) - \frac{1}{2}g\frac{m(t)I(g;t)}{hk(g;t)(S(t))^2},$$
$$\mu^N_a = -\frac{1}{2}g\frac{r_{ac}t^cI^*(g;t)}{h^2k(g;t)(S(t))^2}, \tag{5.37}$$

$$\mu^a_N = -\frac{1}{2}g\frac{m(t)\,t^a}{h^2k(g;t)(S(t))^2},$$
$$\mu^a_b = \frac{1}{hk(g;t)}\delta^a_b + \frac{1}{2}g\frac{t^Nt^ar_{bc}t^c}{m(t)\,h^2k(g;t)(S(t))^2}, \tag{5.38}$$

where

$$m(t) = \sqrt{r_{ab}t^at^b}, \tag{5.39}$$

$$I^*(g;t) = hm(t) + \frac{1}{2}gt^N, \tag{5.40}$$

and

$$S(t) = \sqrt{r_{rs}t^r t^s} \equiv \sqrt{(t^N)^2 + r_{ab}t^a t^b}. \tag{5.41}$$

The relations

$$\frac{\partial(1/k(g;t))}{\partial t^N} = -\frac{1}{2}g\frac{m(t)}{hk(g;t)(S(t))^2},$$

$$\frac{\partial(1/k(g;t))}{\partial t^a} = \frac{1}{2}g\frac{t^N r_{ab}t^b}{m(t)hk(g;t)(S(t))^2}$$

are obtained.

Also,

$$R_p\mu^p_q = t_q, \qquad t_p\sigma^p_q = R_q. \tag{5.42}$$

The unit vectors

$$L^p := \frac{t^p}{S(t)}, \qquad L_p := r_{pq}L^q \tag{5.43}$$

fulfill the relations

$$L^q = l^p\sigma^q_p, \qquad l^p = \mu^p_q L^q, \qquad l_p = \sigma^q_p L_q, \qquad L_p = \mu^q_p l_q, \tag{5.44}$$

where $l^p = R^p/K(g;R)$ and $l_p = R_p/H(g;R)$ are the initial Finslerian unit vectors.

Now we use the explicit formulae (2.62)–(2.63) and (5.30)–(5.31) to find the transform

$$n^{rs}(g;t) := \sigma^r_p\sigma^s_q g^{pq} \tag{5.45}$$

of the FMT under the F^{PD}_g-induced map (5.9)–(5.11), which, after rather lengthy direct calculations, results in

Theorem 5.2. *One obtains the simple representation*

$$n^{rs} = h^2 r^{rs} + \frac{1}{4}g^2 L^r L^s. \tag{5.46}$$

The covariant version reads

$$n_{rs} = \frac{1}{h^2}r_{rs} - \frac{1}{4}G^2 L_r L_s. \tag{5.47}$$

The determinant of this tensor is a constant:

$$\det(n_{rs}) = h^{2(1-N)} \det(r_{ab}). \tag{5.48}$$

Notice that

$$L^p L_p = 1, \quad n_{pq} L^q = L_p, \quad n^{pq} L_q = L^p, \quad n_{pq} L^p L^q = 1,$$

$$n_{pq} t^p t^q = (S(t))^2.$$

Equation (5.47) obviously entails

$$g_{pq} = n_{rs}(g;t)\sigma^r_p \sigma^s_q. \tag{5.49}$$

Let us introduce

Definition. The metric tensor $\{n_{pq}(g;t)\}$ of the form (5.47) is called *quasi-Euclidean.*

Definition. The *quasi-Euclidean space*

$$Q_N := \{V_N; n_{pq}(g;t); g\} \tag{5.50}$$

is an extension of the Euclidean space $\{V_N; r_{pq}\}$ to the case $g \neq 0$.

This motivates the following

Definition. Under these conditions, the maps (5.1) and (5.4) are called *quasi-Euclidean.*

Now we state that the following theorem is valid.

Theorem 5.3. *The quasi-Euclidean metric tensor is conformal to the Euclidean metric tensor.*

Indeed, if we consider the map

$$\bar{R}^p \to \tilde{R}: \qquad \tilde{R}^p = \xi(g;\bar{R})\bar{R}^p/h \tag{5.51}$$

with

$$\xi(g;\bar{R}) = a\left(g;\ \frac{1}{2}S^2(\bar{R})\right) \tag{5.52}$$

and use the coefficients

$$k^p_q := \frac{\partial \tilde{R}^p}{\partial \bar{R}^q} = (\xi \delta^p_q + a' \bar{R}^p \bar{R}_q)/h \tag{5.53}$$

to define the tensor

$$c^{pq}(g;\tilde{R}) := k^p_r k^q_s n^{rs}(g;\bar{R}), \tag{5.54}$$

we find that

$$c^{pq} = \xi^2 r^{pq} \tag{5.55}$$

whenever

$$\xi = \left[\frac{1}{2}S^2(\bar{R})\right]^{(h-1)/2}. \tag{5.56}$$

The proof of Theorem 5.3 is complete.

6. Scalar product, angle and geodesics

Given two vectors $R_1 \in V_N$ and $R_2 \in V_N$. Let us define the E^{PD}_g-*scalar product*

$$\begin{aligned} \langle R_1, R_2 \rangle := {} & K(g;R_1)K(g;R_2) \\ & \times \cos\left[\frac{1}{h}\arccos\frac{A(g;R_1)A(g;R_2) + h^2 r_{be} R^b_1 R^e_2}{\sqrt{B(g;R_1)}\sqrt{B(g;R_2)}}\right] \end{aligned} \tag{6.1}$$

so that the E^{PD}_g-*angle*

$$\alpha(R_1, R_2) \ := \frac{1}{h}\arccos\frac{A(g;R_1)A(g;R_2) + h^2 r_{be} R^b_1 R^e_2}{\sqrt{B(g;R_1)}\sqrt{B(g;R_2)}} \tag{6.2}$$

is appeared between the vectors R_1 and R_2; the functions B, K, as well as A can be found in Section 2.

The general solution

$$R^p = R^p(s) \tag{6.3}$$

to the E^{PD}_g-space geodesic equations (presented by equations (2.88)–(2.89)) proves to be given explicitly by means of the components

$$R^N(s) = \left(t^N(s) - \frac{1}{2}Gm(s)\right)/k(s), \qquad R^a(s) = \frac{1}{h}t^a(s)/k(s) \tag{6.4}$$

with

$$t^N(s) = \frac{K_s}{\sin(h\alpha)}\left[\frac{A(g;R_1)}{\sqrt{B(g;R_1)}}\sin(h(\alpha-\nu)) + \frac{A(g;R_2)}{\sqrt{B(g;R_2)}}\sin(h\nu)\right], \tag{6.5}$$

$$t^a(s) = h\frac{K_s}{\sin(h\alpha)}\left[\frac{R_1^a}{\sqrt{B(g;R_1)}}\sin(h(\alpha-\nu)) + \frac{R_2^a}{\sqrt{B(g;R_2)}}\sin(h\nu)\right], \tag{6.6}$$

where

$$K_s = \sqrt{(K(g;R_1))^2 + 2bs + s^2}, \tag{6.7}$$

$$b = K(g;R_1)\sqrt{1 - \left(\frac{K(g;R_2)\sin\alpha}{\Delta s}\right)^2}, \tag{6.8}$$

and

$$k(s) = \exp\left(\frac{1}{2}G\arctan\frac{t^N(s)}{m(s)}\right), \qquad m(s) = \sqrt{r_{ab}t^a(s)t^b(s)}. \tag{6.9}$$

The intermediate angle ν is equal to

$$\nu = \arctan\frac{sK(g;R_2)\sin\alpha}{K(g;R_1)\Delta s + [K(g;R_2)\cos\alpha - K(g;R_1)]s} \tag{6.10}$$

and is showing the property

$$\nu_{|s=0} = \alpha.$$

Along the geodesics,

$$K(g;R(s)) = K_s \tag{6.11}$$

so that the behaviour law for the squared FMF K^2 is quadratic with respect to the parameter s.

The picture symbolizes the role which the angles (6.2) and (6.10) are playing in featuring the geodesic line C which joins two points P_1 and P_2.

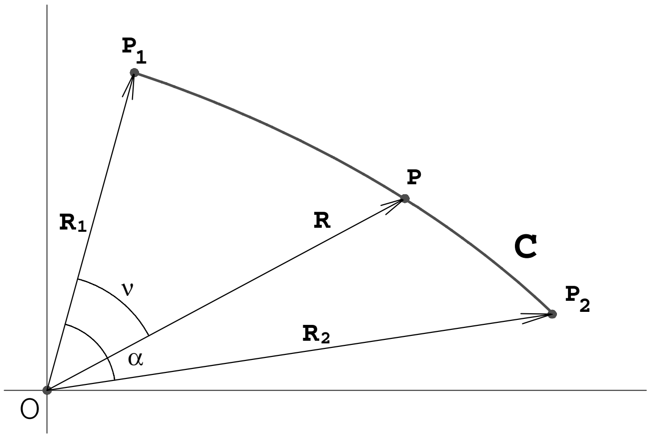

Fig 10. The geodesic C and the angles $\alpha = \angle P_1OP_2$ and $\nu = \angle P_1OP$

On this way the following substantive items can be arrived at.
The E_g^{PD}-Case Two-Point Distance Δs:

$$(\Delta s)^2 = (K(g;R_1))^2 + (K(g;R_2))^2 - 2K(g;R_1)K(g;R_2)\cos\alpha. \quad (6.12)$$

The E_g^{PD}-Case Scalar Product

$$\langle R_1, R_2\rangle = K(g;R_1)K(g;R_2)\cos\alpha. \quad (6.13)$$

At equal vectors, the reduction

$$\langle R, R\rangle = (K(g;R))^2 \quad (6.14)$$

takes place, that is, the two-vector scalar product (6.1) reduces exactly to the squared FMF.
The E_g^{PD}-Case Perpendicularity

$$\langle R, R^{\perp}\rangle = 0, \quad (6.15)$$

in which case $\alpha = \pi/2$.

Under the identification

$$|R_1 \ominus R_2| = \Delta s \quad (6.16)$$

the formula (6.12) can be read as

The E_g^{PD}-Case Cosine Theorem

$$|R_1 \ominus R_2|^2 = (K(g;R_1))^2 + (K(g;R_2))^2 - 2\langle R_1, R_2\rangle. \tag{6.17}$$

From this we can also conclude that

The E_g^{PD}-Case Pythagoras Theorem

$$|R \ominus R^{\perp}|^2 = (K(g;R))^2 + (K(g;R^{\perp}))^2 \tag{6.18}$$

holds fine.

The symmetry

$$|R_1 \ominus R_2| = |R_2 \ominus R_1| \tag{6.19}$$

is obvious.

Note. One can easily execute the formula (6.12) from the representation (6.7) if one inserts (6.8) in (6.7), takes the case $s = \Delta s$, uses the equality $K(g;R_2) = K_{\Delta s}$ (see (6.11)), and resolves the resultant equation to find $(\Delta s)^2$.

Particularly, from (6.2) it directly ensues that the value of the angle α formed by a vector R with the Finsleroid R^N-axis is given by

$$\alpha = \frac{1}{h} \arccos \frac{A(g;R)}{\sqrt{B(g;R)}}, \tag{6.20}$$

where A is the function (2.38), and with $(N-1)$-dimensional equatorial $\{\mathbf{R}\}$-plane of Finsleroid is prescribed as

$$\alpha = \frac{1}{h} \arccos \frac{L(g;R)}{\sqrt{B(g;R)}}, \tag{6.21}$$

where L is the function (2.36).

Comparing (6.2) with (2.82) leads to the equality

$$\alpha = \frac{1}{h} f. \tag{6.22}$$

Therefore, *the Finsleroid angle α between two vectors ranges over*

$$0 \le \alpha \le \alpha_{\max}$$

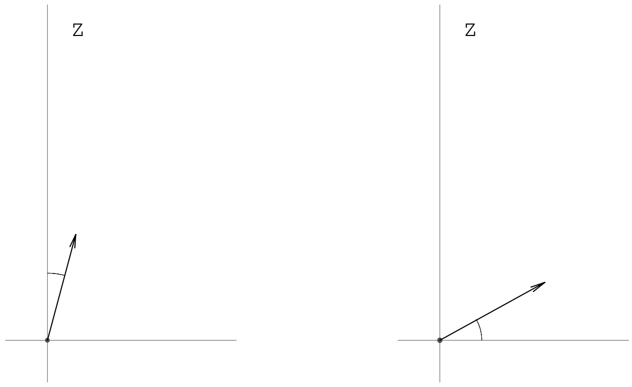

Fig 11. The angle cases (6.20) and (6.21), respectively

(notice (2.79)), where

$$\alpha_{max} = \frac{1}{h}\pi \ge \pi \text{ with equality if and only if } g = 0,$$

so that

$$\alpha_{max} \underset{|g|\to 2}{\Longrightarrow} \infty.$$

Note. In the Euclidean limit proper, the angle (6.2) is reduced to read merely

$$\alpha(R_1, R_2)\Big|_{g=0} = \arccos \frac{R_1^N R_2^N + r_{be} R_1^b R_2^e}{\sqrt{(R_1^N)^2 + r_{be} R_1^b R_1^e}\sqrt{(R_2^N)^2 + r_{be} R_2^b R_2^e}}.$$

Using (6.5) and (6.6) in (6.4) yields

$$R^p(s) = \frac{K_s}{k(s)\sqrt{B(g;R_1)}}\frac{\sin(h(\alpha-\nu))}{\sin(h\alpha)}R_1^p + \frac{K_s}{k(s)\sqrt{B(g;R_2)}}\frac{\sin(h\nu)}{\sin(h\alpha)}R_2^p + X(s)\delta_N^p \tag{6.23}$$

with

$$X(s) = \frac{1}{2}g\frac{K_s}{k(s)}\left[\frac{\sin(h(\alpha-\nu))}{\sin(h\alpha)}\frac{q_1}{\sqrt{B(g;R_1)}}\right.$$

$$+\frac{\sin(h\nu)}{\sin(h\alpha)}\frac{q_2}{\sqrt{B(g;R_2)}}-\frac{m(s)}{hK_s}\Bigg]. \quad (6.24)$$

Since the additional term $X(s)\delta^p_N$ has appeared in the right-hand part of (6.23), and the right-hand part in (6.24) does not vanish identically, we are to conclude that in general the vector $R^p(s)$ is not spanned by two end vectors R^p_1 and R^p_2. Therefore, *in general the E^{PD}_g-geodesic curves obtained are not plane curves.*

The velocity components

$$U^p(s) \ := \frac{dR^p}{ds} \quad (6.25)$$

can conveniently be deduced from the equalities

$$U^p(s)=\mu^p_q(g;\mathbf{t}(s))\frac{dt^q}{ds}, \quad (6.26)$$

where μ^p_q are the functions that are given by the list (5.37)–(5.38). Calculations show that

$$U^p(s)=\frac{b+s}{(K_s)^2}R^p(s)+\frac{hK(g;R_1)K(g;R_2)\sin\alpha}{k_sK_s\sin(h\alpha)\,\Delta s}T^p(s) \quad (6.27)$$

with

$$T^N(s)=\frac{A(g;R_2)}{h^2\sqrt{B(g;R_2)}}\cos(h\nu)-\frac{A(g;R_1)}{h^2\sqrt{B(g;R_1)}}\cos(h(\alpha-\nu)) \quad (6.28)$$

and

$$\begin{gathered}T^a(s)=\frac{1}{\sqrt{B(g;R_2)}}\left[R^a_2-\frac{1}{2}g\frac{A(g;R_2)}{h^2q}R^a(s)\right]\cos(h\nu)\\ -\frac{1}{\sqrt{B(g;R_1)}}\left[R^a_1-\frac{1}{2}g\frac{A(g;R_1)}{h^2q}R^a(s)\right]\cos(h(\alpha-\nu)).\end{gathered} \quad (6.29)$$

It follows that

$$U^p(s)\Big|_{g=0}=\frac{R^p_2-R^p_1}{\Delta s}, \quad (6.30)$$

and the contraction

$$R_p(s)U^p(s)=b+s \quad (6.31)$$

is valid (where R_p are the covariant vector components (2.59)). Also,

$$g_{pq}(g;R(s))U^p(s)U^q(s)=1. \tag{6.32}$$

The *initial-data solution*

$$R_2^p=R_2^p(g;R_1,U_1,\Delta s) \tag{6.33}$$

can also be explicitly found, namely we get

$$R_2^p=\mu^p(g;t_2) \tag{6.34}$$

with the functions

$$t_2^p=z(\Delta s)\sigma^p(g;R_1)+n(\Delta s)\sigma^p_q(g;R_1)U_1^q, \tag{6.35}$$

$$z(\Delta s)=\frac{1}{h}\frac{\sin(h\alpha)}{\sin\alpha}+\frac{K(g;R_2)}{K(g;R_1)}\left[\cos(h\alpha)-\frac{1}{h}\frac{\sin(h\alpha)}{\sin\alpha}\cos\alpha\right], \tag{6.36}$$

$$n(\Delta s)=\frac{1}{h}\frac{\sin(h\alpha)}{\sin\alpha}\Delta s, \tag{6.37}$$

$$K(g;R_2)=\sqrt{(K(g;R_1))^2+2b\Delta s+(\Delta s)^2}, \tag{6.38}$$

$$b=R_{1q}U_1^q, \tag{6.39}$$

and the angle value α can be taken as

$$\alpha=\arccos\frac{(K(g;R_1))^2+b\Delta s}{K(g;R_1)K(g;R_2)}. \tag{6.40}$$

The functions σ^p and σ^p_q entering equation (6.34) can be found in the lists (5.11) and (5.29)–(5.30).

7. Two-dimensional case

When turning to the dimension $N=2$, the consideration is restricted to the *Finsleroid–Minkowski plane*

$$P=P^-\cup P^+, \tag{7.1}$$

where

$$P^- \;:= \{R \in P^- : \; R^1 \le 0\}, \quad P^+ \;:= \{R \in P^- : \; R^1 \ge 0\} \tag{7.2}$$

are respectively the left semi-plane and the right semi-plane. Let us introduce the *indicator*

$$\epsilon \;:= \{\epsilon = -1, \text{ if } R^1 < 0; \; \epsilon = 0, \text{ if } R^1 = 0; \; \epsilon = 1, \text{ if } R^1 > 0\} \tag{7.3}$$

and take also the orthogonalized form for the input Euclidean metric tensor $\{r_{pq}\}$, so that $r_{pq} = \delta_{pq}$. Then the Finsleroid-adapted vector components (cf. the representation (2.83)) take on the form

$$R^1 = \epsilon \frac{K}{hJ} \sin f, \quad R^2 = \frac{K}{J}\left(\cos f - \frac{1}{2} G \sin f\right), \tag{7.4}$$

from which it follows that

$$R^2 \frac{\partial R^1}{\partial f} - R^1 \frac{\partial R^2}{\partial f} = \frac{1}{hJ^2} K^2 \epsilon. \tag{7.5}$$

Since also $\sqrt{\det(g_{pq})} = J^2$ (cf. equation (2.64)), from (7.5) we conclude

$$d\alpha_{\text{Landsberg}} = \frac{1}{h} df$$

(see, e.g., p. 85 of [8] for the definition of the Landsberg angle), where h is the constant (2.13).

Thus, restricting the Finsleroid geometry to the Minkowski plane, the quantity f in the representation (7.4) is the factor h of the Landsberg angle. Noting also (6.22), we arrive at the following theorem.

Theorem 7.1. *The equality*

$$\alpha_{\text{Finsleroid}} = \alpha_{\text{Landsberg}}$$

holds.

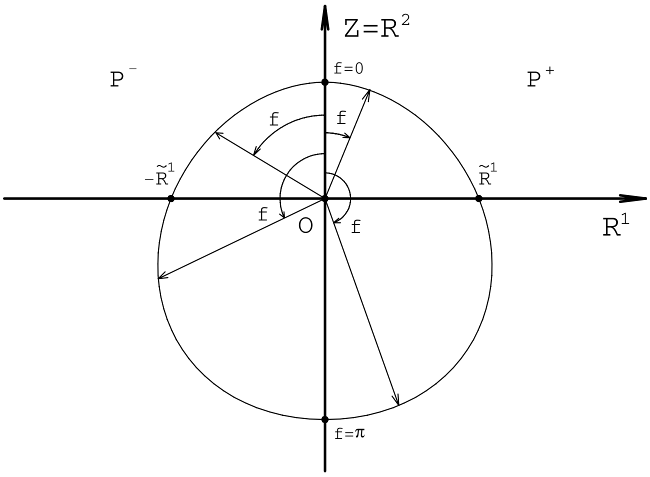

Fig 12. The angle f

It is also possible to draw

Theorem 7.2. *The Finsleroid Indicatrix on the Minkowski plane is strongly convex.*

Proof. Let us verify the relevant criterion formulated on p. 88 of [8]. In terms of our notation, we calculate accordingly:

$$\frac{\frac{\partial^2 R^2}{\partial f^2}\frac{\partial R^1}{\partial f}-\frac{\partial R^2}{\partial f}\frac{\partial^2 R^1}{\partial f^2}}{\frac{\partial R^2}{\partial f}R^1-R^2\frac{\partial R^1}{\partial f}}=\frac{-\frac{1}{h^3}}{-\frac{1}{h}}=\frac{1}{h^2}.$$

Since the right-hand side here is always positive, the criterion works fine and, therefore, Theorem 7.2 is valid. □

Since that $ds := \sqrt{g_{pq}(g;R)dR^p dR^q} = \frac{1}{h}df$, we can also conclude that

$$ds = \frac{1}{h}df. \tag{7.6}$$

In particular, the latter equality entails

Theorem 7.3. *The length $L_I := \int ds$ of the Finsleroid Indicatrix is*

$$L_I = \frac{2\pi}{h} \geq 2\pi, \tag{7.7}$$

showing the properties

$$L_I = 2\pi \quad \textit{if and only if } g = 0 \quad \textit{(the Euclidean case)} \tag{7.8}$$

and

$$L_I \to \infty \quad \textit{when} \quad |g| \to 2. \tag{7.9}$$

From (7.4) and (7.5) it can readily be explicated that the *Rund equation*

$$\frac{d^2R^p}{ds^2} + I\frac{dR^p}{ds} + R^p = 0 \tag{7.10}$$

holds fine with

$$I = -g. \tag{7.11}$$

If the meaning of the Cartan scalar is acquired to the quantity I thus appeared in (7.10) (cf. [8]), one may state the following:

Theorem 7.4. *The Cartan scalar for the Finsleroid–Minkowski plane is the constant which equals the negative of the characteristic Finsleroid parameter g.*

Equations (7.4) suggest naturally to propose the following E_g^{PD}-*Generalized Trigonometric Functions*:

$$\operatorname{Cos} f \;:= \frac{1}{J}\left(\cos f - \frac{G}{2}\sin f\right), \qquad \operatorname{Sin} f := \frac{1}{hJ}\sin f, \tag{7.12}$$

and

$$\operatorname{Cos}^* f \;:= \frac{1}{J}\left(\cos f + \frac{G}{2}\sin f\right). \tag{7.13}$$

They reveal the properties

$$R^1 = K \operatorname{Sin} f, \qquad R^2 = K \operatorname{Cos} f, \tag{7.14}$$

and

$$(\operatorname{Cos} f)' = -\frac{1}{h}\operatorname{Sin} f, \qquad (\operatorname{Sin} f)' = \frac{1}{h}\operatorname{Cos}^* f, \tag{7.15}$$

together with

$$(\operatorname{Cos}^* f)' = -\frac{1}{h}\operatorname{Sin} f + G\operatorname{Cos}^* f, \tag{7.16}$$

where the prime stands for the derivative with respect to f.

References

[1] E. Cartan, Les espaces de Finsler, Actualites 79, *Hermann*, *Paris*, 1934.

[2] H. Busemann, Angular Measure and Integral Curvature, *Canad. J. Math.* **1** (1949), 279.

[3] H. Rund, The Differential Geometry of Finsler spaces, *Springer-Verlag*, *Berlin*, 1959.

[4] G. S. Asanov, Finsler Geometry, Relativity and Gauge Theories, *D. Reidel Publ. Comp.*, *Dordrecht*, 1985.

[5] R. S. Ingarden, Differential Geometry and Physics, *Tensor* **30** (1976), 201.

[6] R. S. Ingarden and L. Tamássy, On Parabolic Geometry and Irreversible macroscopic Time, *Rep. Math. Phys.* **32** (1993), 11.

[7] I. Kozma and L. Tamássy, Finsler Geometry without Line Element Faced to Application, *Rep. Math. Phys.* **42** (2003), 77.

[8] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann–Finsler Geometry, *Springer*, *New York, Berlin*, 2000.

[9] A. C. Thompson, Minkowski Geometry, Encycl. of Math. and its Appl., Vol. 63, *Cambridge Univ. Press*, *Cambridge*, 1996.

[10] G. S. Asanov, Finsler Cases of GF-Spaces, *Aeq. Math.* **49** (1995), 234.

[11] G. S. Asanov, Finslerian Metric Functions over the Product $R \times M$ and their Potential Applications, *Rep. Math. Phys.* **41** (1998), 117.

[12] G. S. Asanov, Finsleroid-Space, arXiv:math-ph/0310019, 2003.

G. S. ASANOV
DIVISION OF THEORETICAL PHYSICS
MOSCOW STATE UNIVERSITY
119992 MOSCOW
RUSSIA

E-mail: asanov@newmail.ru

(Received February 13, 2004; revised June 2, 2004)