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On the spectrum of the difference equations of second order

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Abstract. In this paper we investigate the discrete spectrum of the boundary value problem

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} + f_n = \lambda y_n,\ n \in \mathbb{N} = \{1, 2, \dots\}, \qquad y_0 = 0$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are complex sequences.

1. Introduction

Let T be a non-selfadjoint, closed linear operator in a Hilbert space H. We will denote the continuous spectrum and the set of all eigenvalues of T by $\sigma_c(T)$ and $\sigma_d(T)$, respectively. In this paper we consider some singular points which are the poles of the kernel of the resolvent and are also embedded in the continuous spectrum, but they are not eigenvalues. Hereafter, we will call such singular points as spectral singularities of T and denote the set of all spectral singularities of T by $\sigma_{ss}(T)$.

Eigenvalue problems of selfadjoint difference equations have been treated by various authors (for the relevant references one may consult Agarwal [3] or Agarwal and Wong [4]). But spectral theory of non-selfadjoint difference equations with spectral singularities has not been investigated extensively. In [7] it has been shown by some examples that

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the non-selfadjoint difference equations have spectral singularities. Afterwards spectral analysis of difference equations with spectral singularities became interesting subject in this field. Some problems of spectral theory of non-selfadjoint difference equations with spectral singularities were studied in [1], [7], [8], [10]. Note that spectral analysis of Sturm–Liouville, Schrödinger and Klein–Gordon differential equations with spectral singularities has been investigated in detail in [2], [5], [6], [9], [11], [12], [14], [16], [17].

Let us consider the boundary value problem (BVP)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} + f_n = \lambda y_n,\ a_0 = 1, \quad n \in \mathbb{N}, \tag{1.1}$$

$$y_0 = 0, \tag{1.2}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are complex sequences, $a_n \neq 0$ for all $n \in \mathbb{N}$ and λ is a complex parameter. Note that we can write equation (1.1) in the following Sturm–Liouville form:

$$\Delta\left(a_{n-1}\Delta y_{n-1}\right) + q_n y_n + f_n = \lambda y_n, \quad n \in \mathbb{N},$$

where $q_n = a_{n-1} + a_n + b_n$ and Δ is the forward difference operator, i.e., $\Delta y_n = y_{n+1} - y_n$.

In this paper we investigate spectral properties of the BVP (1.1), (1.2) using the boundary behavior of analytic functions. In particular, we prove that under the conditions

$$\sup_{n\in\mathbb{N}}\left\{\exp\left(\varepsilon\sqrt{n}\right)\left(|1-a_n| + |b_n|\right)\right\} < \infty,$$

$$\sup_{n\in\mathbb{N}}\left\{\exp\left(\varepsilon n^{1+\beta}\right)|f_n|\right\} < \infty$$

for some $\varepsilon > 0$ and $\beta > 0$, the BVP (1.1), (1.2) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

2. The solution of (1.1), (1.2)

Related with equation (1.1) we will consider the following difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{2.1}$$

i.e. the case $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Suppose that the complex sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy

$$\sum_{n=1}^{\infty} n\left(|1-a_n| + |b_n|\right) < \infty. \tag{2.2}$$

Under the condition (2.2) equation (2.1) has the following solution

$$e_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{n,m} e^{imz}\right), \quad n \in \mathbb{N}\cup\{0\}, \tag{2.3}$$

for $\lambda = 2\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$, and $A_{n,m}$, α_n are expressed in terms of $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. Moreover α_n and $A_{n,m}$ satisfy

$$\alpha_n = \left\{\prod_{k=n}^{\infty} a_k\right\}^{-1}, \quad |A_{n,m}| \leq c \sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} \left(|1-a_k| + |b_k|\right), \tag{2.4}$$

where $[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$ and $c > 0$ is a constant ([15], see also [7]). Therefore the solution $e(z) = \{e_n(z)\}_{n=0}^{\infty}$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and

$$e_n(z + 2\pi) = e_n(z), \qquad z \in \overline{\mathbb{C}}_+,\ n \in \mathbb{N}\cup\{0\},$$

$$e_n(z) = e^{inz}[1 + o(1)], \qquad z \in \overline{\mathbb{C}}_+,\ n \to \infty, \tag{2.5}$$

$$e_n(z) = \alpha_n e^{inz}[1 + o(1)], \qquad n \in \mathbb{N},\ z \in \mathbb{C}_+,\ \operatorname{Im} z \to \infty, \tag{2.6}$$

hold ([15]).

Let $\hat{e}(z) = \{\hat{e}_n(z)\}_{n=0}^{\infty}$ denote the solution of (2.1) for $\lambda = 2\cos z$, subject to the condition

$$\lim_{n\to\infty} \hat{e}_n(z) e^{inz} = 1,\ z \in \overline{\mathbb{C}}_+. \tag{2.7}$$

The solution $\hat{e}(z)$ is analytic in $\mathbb{C}_+$ and continuous on the real axis and

$$\hat{e}_n(z + 2\pi) = \hat{e}(z), \quad n \in \mathbb{N} \cup \{0\}.$$

Note that $e(z)$ is a bounded solution of (2.1) for $\lambda = 2\cos z$, but $\hat{e}(z)$ is unbounded. From (2.5) and (2.7) we get that Wronskian of solutions $e(z)$ and $\hat{e}(z)$ is obtained as

$$\begin{aligned} W\left[e(z), \hat{e}(z)\right] &= a_n \left[e_n(z)\hat{e}_{n+1}(z) - e_{n+1}(z)\hat{e}_n(z)\right] \\ &= -2i \sin z, \quad z \in \overline{\mathbb{C}}_+. \end{aligned}$$

Let us define

$$\begin{aligned} E_n(z) = \frac{1}{2i\sin z} \Bigg\{ &\sum_{k=n}^{\infty} f_{k+1}\hat{e}_{k+1}(z) e_n(z) - \sum_{k=n}^{\infty} f_{k+1} e_{k+1}(z)\hat{e}_n(z) \\ &+ \hat{A}(z) e_n(z) - A(z)\hat{e}_n(z) \Bigg\}, \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \hat{A}(z) &= \sum_{k=1}^{\infty} f_k \hat{e}_k(z) + \hat{e}_0(z), \\ A(z) &= \sum_{k=1}^{\infty} f_k e_k(z) + e_0(z). \end{aligned} \tag{2.9}$$

It is obvious that $E(z) := \{E_n(z)\}_{n=0}^{\infty}$ is the solution of the BVP (1.1), (1.2) for $\lambda = 2\cos z$ and

$$E_n(z + 2\pi) = E_n(z), \quad z \in \overline{\mathbb{C}}_+, \ n \in \mathbb{N} \cup \{0\}.$$

3. Discrete spectrum of (1.1), (1.2)

Let us define the semi-strips

$$P_0 = \{z : z \in \mathbb{C}, \ 0 \le \operatorname{Re} z < 2\pi, \ \operatorname{Im} z > 0\},$$

$$P = \{z : z \in \mathbb{C}, \ 0 \le \operatorname{Re} z < 2\pi, \ \operatorname{Im} z \ge 0\}.$$

We also denote the set of eigenvalues and the set of spectral singularities of BVP (1.1), (1.2) by σ_d and σ_{ss}, respectively.

Lemma 3.1. *If for some* $\varepsilon > 0$ *and* $\beta > 0$ (2.2) *and*

$$\sup_{n\in\mathbb{N}} \left\{ \exp(\varepsilon n^{1+\beta})|f_n| \right\} < \infty, \tag{3.1}$$

hold, then

$$\sigma_d = \{\lambda : \lambda = 2\cos z,\ z \in P_0,\ A(z) = 0\}, \tag{3.2}$$

where the function A *is defined by* (2.9).

Proof. Let $\lambda_0 = 2\cos z_0$ and $z_0 \in P_0$. Using (2.5) and (2.7), we get that $e(z_0) \in \ell_2(\mathbb{N})$ and $\hat{e}(z_0) \notin \ell_2(\mathbb{N})$. Since

$$e_n(z_0) \sum_{k=n}^{\infty} f_{k+1} \hat{e}_{k+1}(z_0) = O\big(e^{-\frac{\in}{2} n^{1+\beta}}\big), \quad n \to \infty,$$

and

$$\hat{e}_n(z_0) \sum_{k=n}^{\infty} f_{k+1} e_{k+1}(z_0) = O\big(e^{-\frac{\in}{2} n^{1+\beta}}\big), \quad n \to \infty,$$

it follows from (2.8) that $E(z_0)$ belongs to $\ell_2(\mathbb{N})$ if and only if $A(z_0) = 0$. □

Analogously to Sturm–Liouville difference equation, we have

$$\sigma_{ss} = \{\lambda : \lambda = 2\cos z,\ z \in (0, 2\pi),\ z \neq \pi,\ A(z) = 0\}, \tag{3.3}$$

(see [1], [7], [8]).

In order to investigate the structure of the eigenvalues and the spectral singularities of the BVP (1.1), (1.2), by (3.2) and (3.3), we need to discuss the quantitative properties of the zeros of A in P. In order to do so, write

$$M_1 = \{z : z \in P_0,\ A(z) = 0\},$$

$$M_2 = \{z : z \in [0, 2\pi],\ A(z) = 0\}.$$

From (3.2) and (3.3), we see that

$$\sigma_d = \{\lambda : \lambda = 2\cos z,\ z \in M_1\}, \tag{3.4}$$

$$\sigma_{ss} = \big\{\lambda : \lambda = 2\cos z,\ z \in M_2 \backslash \{0, \pi, 2\pi\}\big\}. \tag{3.5}$$

Lemma 3.2. *If* (2.2) *and* (3.1) *hold, then*

i. *The set* M_1 *is bounded and has at most a countable number of elements, and its limit points can lie only in* $[0, 2\pi]$.

ii. *The set* M_2 *is compact and* $\mu(M_2) = 0$, *where* μ *denotes the Lebesgue measure in the real axis.*

PROOF. Using (2.3) and (2.4) we get that the function A is analytic in P_0, continuous in P and

$$A(z) = \alpha_0 + \sum_{k=1}^{\infty} \varphi_k e^{ikz}, \tag{3.6}$$

where

$$\begin{aligned} \varphi_1 &= f_1\alpha_1 + \alpha_0 A_{0,1}, \\ \varphi_k &= f_k\alpha_k + \alpha_0 A_{0,k} + \sum_{m=1}^{k-1} f_m \alpha_m A_{m,k-m}, \quad k \geq 2. \end{aligned} \tag{3.7}$$

It follows from (2.4), (2.6) and (3.1) that

$$A(z) = \alpha_0 + o(1), \quad z \in P_0, \ |z| \to \infty. \tag{3.8}$$

Since $\alpha_0 \neq 0$, equation (3.8) shows the boundedness of M_1. From the analyticity of A in P_0 we find that M_1 has at most a countable number of elements and its limit points can lie only in $[0, 2\pi]$. By the boundary value uniqueness theorem for analytic functions we obtain that the set M_2 is closed and $\mu(M_2) = 0$ ([13]). □

From (3.4), (3.5) and Lemma 3.2 we get the following.

Theorem 3.3. *Under the conditions* (2.2) *and* (3.1)

i. *The set of eigenvalues of the BVP* (1.1), (1.2) *is bounded, is no more than countable and its limit points can lie only in* $[-2, 2]$.

ii. $\sigma_{ss} \subset [-2, 2]$ *and* $\mu(\sigma_{ss}) = 0$.

Theorem 3.4. *If for some* $\varepsilon > 0$, (3.1) *and*

$$\sup_{n \in \mathbb{N}} \left\{ e^{\varepsilon n} \left(|1 - a_n| + |b_n| \right) \right\} < \infty, \tag{3.9}$$

hold, then the BVP (1.1), (1.2) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

PROOF. From (2.4) and (3.9) we obtain that

$$|A_{n,m}| \le ce^{-\frac{\varepsilon}{2}m}, \quad n,m \in \mathbb{N},$$

$$\left|\sum_{m=1}^{k-1} f_m \alpha_m A_{m,k-m}\right| \le ce^{-\frac{\varepsilon}{2}k}, \quad k \in \mathbb{N},$$

and consequently it follows from (3.7) that

$$|\varphi_k| \le ce^{-\frac{\varepsilon}{2}k}, \quad k \in \mathbb{N}, \tag{3.10}$$

where $c > 0$ is a constant. By (3.6) and (3.10) we observe that the function A has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{2}$. Since A is a 2π periodic function, the limit points of M_1 and M_2 cannot lie in $[0, 2\pi]$. Using Theorem 3.3 we get that the bounded sets σ_d and σ_{ss} have no limit points, i.e., the sets σ_d and σ_{ss} have a finite number of elements. From analyticity of A in $\operatorname{Im} z > -\frac{\varepsilon}{2}$ we get that all zeros of A in P have a finite multiplicity. Consequently, all eigenvalues and spectral singularities of the BVP (1.1), (1.2) have a finite multiplicity. □

Briefly, (3.1) and (3.9) guarantee the analytic continuation of the function A from the real axis to lower half-plane and finiteness of eigenvalues and spectral singularities of the BVP (1.1), (1.2) can be obtained as a result of this analytic continuation.

Now let us suppose that for some $\varepsilon > 0$, and $\frac{1}{2} \le \delta < 1$,

$$\sup_{n\in\mathbb{N}} \left\{ e^{\varepsilon n^\delta} \left(|1-a_n| + |b_n| \right) \right\} < \infty, \tag{3.11}$$

holds, which is weaker than (3.9). It is evident that under conditions (3.1) and (3.11), the function A is analytic in $\mathbb{C}_+$ and infinitely many times differentiable on the real axis. But A does not necessarily have an analytic continuation from the real axis to lower half-plane. Therefore, under the conditions (3.1) and (3.11), the finiteness of eigenvalues and spectral singularities of the BVP (1.1), (1.2) cannot be proved by the same technique used in Theorem 3.4.

Let us denote the set of all limit points of M_1 and M_2 by M_3 and M_4, respectively, and the set of all zeros of A with infinite multiplicity in P by M_5. From the boundary uniqueness theorem of analytic functions

$$M_1 \cap M_5 = \emptyset, \quad M_3 \subset M_2, \quad M_4 \subset M_2, \quad M_5 \subset M_2,$$

and

$$\mu\left(M_3\right)=\mu(M_4)=\mu(M_5)=0.$$

Using the continuity of all derivatives of A on the real axis, we have

$$M_3\subset M_5,\quad M_4\subset M_5. \tag{3.12}$$

To prove the next result, we will use the following uniqueness theorem for the analytic functions on the upper half-plane.

Theorem 3.5 ([7])**.** *Let us assume that the 2π periodic function f is analytic in the open upper half-plane, all of its derivatives are continuous in the closed upper half-plane and*

$$\sup_{z\in P}\left|f^{(k)}(z)\right|\leq B_k,\quad k\in\mathbb{N}\cup\{0\}.$$

If the set G with Lebesgue measure zero is the set of all zeros of the function f with infinite multiplicity in P and

$$\int_0^\omega \ln F(s)d\mu(G_s)=-\infty,$$

where

$$F(s)=\inf_{k\in\mathbb{N}\cup\{0\}}\frac{B_k s^k}{k!}$$

and $\mu\left(G_s\right)$ is the Lebesgue measure of s-neighborhood of G and $\omega\in(0,2\pi)$ is an arbitrary constant, then $f\equiv 0$ in $\overline{\mathbb{C}}_+$.

Lemma 3.6. *If* (3.1) *and* (3.11) *hold, then $M_5=\emptyset$.*

Proof. It follows from (3.1), (3.6) and (3.11) that

$$|A^{(n)}(z)|\leq Q_n,$$

where

$$Q_n=c2^n\sum_{m=1}^{\infty}m^n e^{-\frac{\varepsilon}{2}m^\delta},\quad n=0,1,2,\dots .$$

Now we obtain the estimate

$$
\begin{aligned}
Q_n &\le 2c\,2^n \int_0^\infty t^n e^{-\frac{\varepsilon}{2}t^\delta} dt \\
&= c\,2^{n+1}\frac{1}{\delta\varepsilon}\left(\frac{2}{\varepsilon}\right)^{\frac{2}{\delta}-1}\int_0^\infty t^{\frac{k+1}{\delta}-1}e^{-t}dt \qquad (3.13)\\
&\le cq^n n!\, n^{n\left(\frac{1}{\delta}-1\right)},
\end{aligned}
$$

c and q are positive constants depending on ε, β and δ. Since the function A is not equal to zero identically, then by Theorem 3.5, M_5 satisfies

$$
\int_0^\omega \ln F(s) d\mu\left(M_{5,s}\right) > -\infty, \qquad (3.14)
$$

where

$$
F(s) = \inf_{n\in\mathbb{N}\cup\{0\}} \frac{Q_n s^n}{n!}
$$

and Q_n is constant defined by (3.13). Substituting (3.13) in the definition of $F(s)$, we arrive at

$$
F(s) \le c\exp\left\{-\frac{1-\delta}{\delta}e^{-\frac{1}{1-\delta}}q^{-\frac{1}{1-\delta}}s^{-\frac{\delta}{1-\delta}}\right\},
$$

by (3.14), we get that

$$
\int_0^\omega s^{-\frac{\delta}{1-\delta}} d\mu(M_{5,s}) < \infty. \qquad (3.15)
$$

Since $\frac{\delta}{1-\delta} \ge 1$, (3.15) holds for arbitrary s if and only if $\mu\left(M_{5,s}\right) = 0$ or $M_5 = \emptyset$. $\square$

Theorem 3.7. *Under the conditions* (3.1) *and* (3.11) *the BVP* (1.1), (1.2) *has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem, we have to show that the function A has a finite number of zeros with finite multiplicities in P. From (3.12) and Lemma 3.6 we get that $M_3 = M_4 = \emptyset$. So the bounded sets M_1 and M_2 have no limit points, i.e., the function A has only finite number of zeros in P. Since $M_5 = \emptyset$, these zeros are of finite multiplicity. $\square$

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