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Minimizers of nonsmooth functionals on manifolds and nonlinear eigenvalue problems with constraints

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Abstract. In this paper first we prove a theorem on the relation between $C_0^1(\bar{Z})$ and $W_0^{1,p}(Z)$ local minimizers of a locally Lipschitz functional on a smooth submanifold of codimension one. Our result extends the semilinear, smooth works of Brezis–Nirenberg (CRAS, t.317 (1993), no constraints) and Tehrani (Non. Anal. 26 (1996), on manifolds). Then using this result and nonsmooth critical point theory, we prove a three critical points theorem for locally Lipschitz functionals on manifolds. Finally the results are used to establish three nontrivial solutions for eigenvalue problems with nonsmooth potential and constraints.

1. Introduction

Brezis–Nirenberg [2] proved an interesting result on the comparison of the H_0^1 versus the C_0^1 local minimizers of a C^1-functional. The result proved to be a useful tool in establishing multiplicity results for semilinear elliptic problems. Soon thereafter Tehrani [26] extended the result of Brezis–Nirenberg to C^1-functionals defined on suitable submanifolds of $H_0^1(Z)$. Tehrani used his extension to prove multiplicity results for certain semilinear eigenvalue problems. Recently the result of Brezis–Niremberg was extended by Kourogenis–Papageorgiou [15] to nonsmooth locally

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Lipschitz functionals defined on $W_0^{1,p}(Z)$ $(p \geq 2)$. Kourogenis–Papageorgiou used their result to study quasilinear elliptic problems driven by p-Laplacian differential operator with a nonsmooth potential. Such problems are known as "hemivariational inequalities" and are a new type of variational expressions which arise in Mechanics and Engineering, when one wants to consider more realistic laws of nonmonotone multivalued nature, which correspond to nonsmooth nonconvex energy functionals. For concrete applications, we refer to the book of Naniewicz–Panagiotopoulos [13].

In this paper we extend the work of Tehrani to nonsmooth locally Lipschitz functionals constrained on a smooth submanifold of $W_0^{1,p}(Z)$ $(2 \leq p < \infty)$. The result is then used to prove a multiplicity result (a three solutions theorem) for quasilinear eigenvalue problem for hemivariational inequalities with constraints. Semilinear hemivariational inequalities with constraints were examined using different methods by Bocea–Panagiotopoulos–Radulescu [5], Motreanu–Panagiotopoulos [21], Radulescu–Panagiotopoulos [24]. Also multiplicity results for eigenvalue problems were obtained by Gasinski–Papageorgiou [9], Goeleven–Motreanu–Panagiotopoulos [11], Radulescu [23] (semilinear problems) and Gasinski–Papageorgiou [10] (quasilinear problems).

Our approach uses the nonsmooth critical point theory as this was developed initially by Chang [4] and extended recently by Kourogenis–Papageorgiou [14] and Kyritsi–Papageorgiou [16]. For another approach with applications to problems with an area-type term, we refer to the paper of Degiovanni–Marzocchi–Radulescu [7] and the references therein.

2. Mathematical preliminaries

In this section for the convenience of the reader, we recall some basic definitions and facts from the critical point theory for nonsmooth locally Lipschitz functionals. This theory is based on the subdifferential theory of Clarke [6].

Let X be a Banach space and X^* its dual. By $(.,.)$ we denote the duality brackets for the pair (X, X^*). A map $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exists a neighborhood U of x and a constant $k_U > 0$ such that $|\varphi(z) - \varphi(y)| \leq k_U \|z - y\|$ for all $z, y \in U$. It is a well-known fact of convex analysis that a proper, convex and lower semicontinuous function $\psi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, is locally Lipschitz in the interior of its effective domain $\mathrm{dom}\psi = \{x \in X : \psi(x) < +\infty\}$. In analogy to the directional derivative of a convex function, for a locally Lipschitz function $\varphi : X \to \mathbb{R}$ we define the generalized directional derivative at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that the function $h \to \varphi^0(x; h)$ is sublinear continuous and so by the Hahn–Banach theorem, it is the support function of a nonempty, convex and w^*-compact set

$$\partial\varphi(x) = \{x^* \in X^* : (x^*, h) \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called the generalized (or Clarke) subdifferential of φ at $x \in X$. If $\varphi, \psi : X \to \mathbb{R}$ are two locally Lipschitz functions, then for all $x \in X$ and all $\lambda \in \mathbb{R}$, we have $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$. Moreover, if $\varphi : X \to \mathbb{R}$ is also convex, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis defined by $\partial\varphi(x) = \{x^* \in X^* : (x^*, y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}$. Also if $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

Let $C \subseteq X$ be a nonempty set. We know that the distance from C function $d_C(y) = \inf[\|y - x\| : x \in C]$, is Lipschitz. Moreover, if C is convex, then so is the function $x \to d_C(x)$. Given $x \in C$, let $T_C(x) = \{h \in X : d_C^0(x; h) = 0\}$. Clearly the set $T_C(x)$ is a closed, convex cone and is known as the tangent cone to C at $x \in C$. By definition, the normal cone to C at $x \in C$ is the set $N_C(x) = \{x^* \in X^* : (x^*, h) \leq 0$ for all $h \in T_C(x)\}$. So $N_C(x) = T_C^0(x) =$ the polar cone associated to $T_C(x)$. If C is convex, then $N_C(x)$ coincides with the normal cone of convex analysis, i.e. $N_C(x) = \{x^* \in X^* : (x^*, x - y) \geq 0 \text{ for all } y \in C\}$. If C is a C^1-manifold, then the tangent cone coincides with usual tangent space and the normal cone with the normal space. It is an easy consequence of this definition, that $N_C(x) = \overline{\bigcup_{\lambda \geq 0} \lambda\partial d_X(x)}^{w^*}$. If $\mathrm{int}C \neq \emptyset$ and $x \in \mathrm{int}\, C$, then

$T_C(x) = X$ and $N_C(x) = \{0\}$. Suppose that $\varphi : X \to \mathbb{R}$ is locally Lipschitz and attains a local minimum on C at $x \in C$. Then $0 \in \partial\varphi(x) + N_C(x)$.

It is well-known that the smooth critical point theory uses a compactness-type condition known as the Palais–Smale condition (PS-condition for short). In the present nonsmooth setting this condition takes the following form:

> "Let $C \subseteq X$ be nonempty and $\varphi : X \to \mathbb{R}$ locally Lipschitz. We say that φ satisfies the nonsmooth Palais–Smale condition at the level $c \in \mathbb{R}$ on C (nonsmooth PS_c-condition on C), if every sequence $\{x_n\}_{n\geq 1} \subseteq C$ such that $\varphi(x_n) \to c$ and $m_1(x_n) = \min[\|x^*\| : x^* \in \partial(\varphi|_C)(x_n)] \to 0$ has a strongly convergent subsequence."

3. $W^{1,p}$ versus C^1 minimizers on manifolds

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^{1,\alpha}$ boundary Γ $(0<\alpha<1)$. Let $f : Z \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function (i.e. f is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$). We know that such a function is jointly measurable (see for example Hu–Papageorgiou [12], p. 142). We assume that $f(z,x)$ satisfies the following growth condition. For almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|f(z,x)| \leq \alpha(z) + c|x|^{p-1} \quad \text{with } \alpha \in L^\infty(Z),\ c > 0.$$

We set $F(z,x) = \int_0^x f(z,r)dr$ (the potential function corresponding to f). We know that the functional $x \to \int_Z F(z,x(z))dz$ from $W_0^{1,p}(Z)$ into $\mathbb{R}$ is C^1 and its derivative at x is $N_f(x)$ is $N_f(x) =$ the Nemitsky operator corresponding to f, i.e. $N_f(x)(.) = f(.,x(.))$.

Also let $j : Z \times \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$ $z \to j(z,x)$ is measurable, for almost all $z \in Z$ $x \to j(z,x)$ is locally Lipschitz and for all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|j(z,x)| \leq \alpha_1(z) + c_1|x|^p \quad \text{with } \alpha_1 \in L^\infty(Z),\ c_1 > 0.$$

Let $J : W_0^{1,p}(Z) \to \mathbb{R}$ be the integral function defined by $J(x) = \int_Z j(z,x(z))dz$, for all $x \in W_0^{1,p}(Z)$. We know that J is locally Lipschitz (see Hu–Papageorgiou [13], p. 313).

Let $\psi : W_0^{1,p}(Z) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F(z, x(z))dz$$

and let $C = \{x \in W_0^{1,p}(Z) : \psi(x) = c\}$. Evidently C is a C^1-manifold in $W_0^{1,p}(Z)$ of codimension 1, provided that c is not a critical value of ψ. In what follows $\mathcal{F} : W_0^{1,p}(Z) \to W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ is the duality map for the Sobolev space $W_0^{1,p}(Z)$. Since $W_0^{1,p}(Z)$ and $W^{-1,q}(Z)$ are uniformly convex, $\mathcal{F}$ is a homeomorphism and $\mathcal{F}^{-1}$ is the duality map of $W^{-1,q}(Z)$ (see Hu–Papageorgiou [12], p. 313 or Zeidler [28], p. 861). In what follows by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $\big(W_0^{1,p}(Z), W^{-1,q}(Z)\big)$.

Theorem 1. *If $x_0 \in C_0^{1,\beta}(\bar{Z})$, $\beta \in (0,1)$ is a local minimizer of $J|_C$ for the $C_0^1(\bar{Z})$-topology and for all $x^* \in \partial J(x_0)$ we have $\big\langle x^*, \mathcal{F}^{-1}(\psi'(x_1))\big\rangle < 0$, then x_0 is a local minimizer of $J|_C$ for the $W_0^{1,p}(Z)$-topology.*

Proof. Suppose that the result is not true. For $\varepsilon > 0$, let $B_\varepsilon(x_0) = \{x \in W_0^{1,p}(Z) : \|x - x_0\| < \varepsilon\}$ (on $W_0^{1,p}(Z)$ we consider the norm $\|x\| = \|Dx\|_p$, by Poincaré's inequality). Then for every $n \geq 1$ we can find $0 < \varepsilon_n \leq \frac{1}{n}$ and $x_n \in \partial C \cap \bar{B}_{\varepsilon_n}(x_0)$ such that

$$J(x_n) = \inf\big[J(x) : x \in C \cap \bar{B}_{\varepsilon_n}(x_0)\big] < J(x_0), \quad n \geq 1.$$

Let $\vartheta \in C^1(W_0^{1,p}(Z))$ be defined by $\vartheta(x) = \frac{1}{p}\|Dx\|_p^p$. We have

$$J(x_n) = \inf\left[J(x) : \psi(x) = c, \quad \vartheta(x - x_0) \leq \frac{\varepsilon_n^p}{p}\right].$$

Invoking Theorem 1 and Proposition 13 of Clarke [5], we can find $x_n^* \in \partial J(x_n)$, $\mu_n \in \mathbb{R}$ and $\xi_n \leq 0$ such that

$$x_n^* = \mu_n \psi'(x_n) + \xi_n \vartheta'(x_n), \quad n \geq 1.$$

Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the continuous monotone (hence maximal monotone, see Hu–Papageorgiou [12], p. 309) operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

We have $\vartheta'(x_n) = A(x_n - x_0)$. So we can write that

$$x_n^* = \mu_n \psi'(x_n) + \xi_n A(x_n - x_0). \tag{1}$$

First suppose that $|\xi_n| \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$. Then from (1), for all $v \in W_0^{1,p}(Z)$ we have that

$$\langle x_n^*, v\rangle = \mu_n \left\langle \psi'(x_n), v\right\rangle + \xi_n \left\langle A(x_n - x_0), v\right\rangle . \tag{2}$$

Use as test function $v = \mathcal{F}^{-1}(\psi'(x_0))$. We have

$$\begin{aligned} \left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = \mu_n &\left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ &+ \xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle . \end{aligned} \tag{3}$$

Since $x_n^* \in \partial J(x_n)$ and $\{x_n\}_{n\geq 1} \in C \cap \partial B_{\varepsilon_n}(x_0) \subseteq W_0^{1,p}(Z)$, hence bounded, because the Clarke subdifferential is a multifunction mapping bounded sets to bounded sets, it follows that $\{x_n^*\}_{n\geq 1} \subseteq W^{-1,q}(Z)$ is bounded. So by passing to a subsequence if necessary, we may assume that $x_n^* \xrightarrow{w} x^*$ in $W^{-1,q}(Z)$. For every $n \geq 1$ and for every $v \in W_0^{1,p}(Z)$, we have that

$$\langle x_n^*, v\rangle \leq J^0(x_n; v).$$

Because $x_n \to x_0$ in $W_0^{1,p}(Z)$ and $(x, v) \to J^0(x; v)$ is upper semicontinuous (see Clarke [6], p. 25), we obtain

$$\begin{aligned} &\langle x^*, v\rangle \leq J^0(x_0; v) \quad \text{for all } v \in W_0^{1,p}(Z) \\ \Rightarrow \quad & x^* \in \partial J(x_0). \end{aligned}$$

If we pass to the limit in (3) and recalling that we have assumed that $|\xi_n| \leq M_1$ for all $n \geq 1$, because $x_n^* \xrightarrow{w} x^*$, $\psi'(x_n) \to \psi'(x_0)$ (since $\psi \in C^1(W_0^{1,p}(Z))$) and $A(x_n - x_0) \to 0$, we obtain

$$\begin{aligned} 0 > \left\langle x^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle &= (\lim \mu_n) \left\langle \psi'(x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ &= (\lim \mu_n) \|\mathcal{F}^{-1}(\psi'(x_0))\|^2. \end{aligned} \tag{4}$$

Here we have used the fact that $\mathcal{F}^{-1}$ is the duality map of $W^{-1,q}(Z)$. From (4) it follows that $\lim \mu_n < 0$ and so we can find $n_0 \geq 1$ such that for all $n \geq n_0$, we have $\mu_n < 0$.

Next suppose that $|\xi_n| \to \infty$ as $n \to \infty$.

We shall show that there exist $\varepsilon_0 > 0$ and $n_0 \geq 1$ such that for all $n \geq n_0$, we have $|\mu_n| \geq \varepsilon_0$. Suppose that this is not true. Then by passing to a subsequence if necessary, we may assume that $\mu_n \to 0$. Let $y_n = \frac{x_n - x_0}{\|x_n - x_0\|}$, $n \geq 1$. We may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$. In (2) we use as test function $v = y_n$. We have

$$\begin{aligned}\langle x_n^*, y_n\rangle &= \mu_n \left\langle \psi'(x_n), y_n\right\rangle + \xi_n \left\langle A(x_n - x_0), y_n\right\rangle \\ &= \mu_n \left\langle \psi'(x_n), y_n\right\rangle + \xi_n \|Dx_n - Dx_0\|_p^{p-1}.\end{aligned} \tag{5}$$

Because $\{\langle \psi'(x_n), y_n\rangle\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and by hypothesis $\mu_n \to 0$, it follows that $\mu_n \langle \psi'(x_n), y_n\rangle \to 0$. Also since $x_n^* \in \partial J(x_n) \subseteq L^q(Z)$ (see Chang [4] or Clarke [6], p. 47), we have that $\{x_n^*\}_{n\geq 1} \subseteq L^q(Z)$ is bounded and so we may assume that $x_n^* \xrightarrow{w} x^*$ in $L^q(Z)$ and $x^* \in \partial J(x_0)$ (recall that the graph of ∂J is sequentially closed in $L^p(Z) \times L^q(Z)_w$, see Clarke [6], p. 29). Also from the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $y_n \to y$ in $L^p(Z)$ and so $\langle x_n^*, y_n\rangle = (x_n^*, y_n)_{pq} \to (x^*, y)_{pq} = \langle x^*, y\rangle$ (by $(.,)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$). Hence from (5) in the limit as $n \to \infty$, we obtain

$$\langle x^*, y\rangle = \lim \xi_n \|Dx_n - Dx_0\|_p^{p-1}. \tag{6}$$

From the mean value theorem, we have

$$\begin{aligned}&0 = \psi(x_n) - \psi(x_0) = \left\langle \psi'(x_0 + t_n(x_n - x_0)), x_n - x\right\rangle \quad \text{with } t_n \in (0,1)\\ &\Rightarrow 0 = \left\langle \psi'(x_0 + t_n(x_n - x_0)), y_n\right\rangle \quad \text{for all } n \geq 1.\end{aligned}$$

Because $x_n \to x_0$ in $W_0^{1,p}(Z)$, it follows that $t_n \to 0$ and since $\psi \in C^1(W_0^{1,p}(Z))$, we have

$$0 = \lim_{n\to\infty} \left\langle \psi'(x_0 + t_n(x_n - x_0)), y_n\right\rangle = \left\langle \psi'(x_0), y\right\rangle.$$

From Lusternik's theorem (see for example Zeidler [29], p. 276), we know that

$$T_C(x_0) = \{h \in W_0^{1,p}(Z) : \left\langle \psi'(x_0), h\right\rangle = 0\}.$$

Therefore $y \in T_C(x_0)$.

Since x_0 is a local minimizer of $J\big|_C$ in $C_0^1(\bar{Z})$, the latter space is dense in $W_0^{1,p}(Z)$ and $h \to J^0(x_0;h)$ is continuous on $W_0^{1,p}(Z)$, we infer that

$$0 \le J^0(x_0;h) \quad \text{for all } h \in T_C(x_0).$$

Because $T_C(x_0)$ is a linear subspace of $W_0^{1,p}(Z)$ of codimension 1, we have

$$0 \le J^0(x_0;-h) \quad \text{for all } h \in T_C(x_0).$$

We know that $J^0(x_0;-h) = (-J)^0(x_0;h)$ (see Clarke [6]). Remark that x_0 is a local maximizer in $C_0^1(\bar{Z})$ of $(-J)\big|_C$. So as above, we deduce that

$$\begin{aligned} & (-J)^0(x_0;h) \le 0 && \text{for all } h \in T_C(x_0), \\ \Rightarrow \quad & J^0(x_0;h) \le 0 && \text{for all } h \in T_C(x_0) \\ \Rightarrow \quad & J^0(x_0;h) = 0 && \text{for all } h \in T_C(x_0). \end{aligned}$$

Moreover, from the nonsmooth Lagrange multiplier rule (see Clarke [5], [6]), we can find $\widehat{x}^* \in \partial J(x_0)$ such that

$$\langle \widehat{x}^*, h \rangle = J^0(x_0;h) \quad \text{for all } h \in T_C(x_0).$$

Recall that $x^* \in \partial J(x_0)$. So we have

$$\begin{aligned} & \langle \widehat{x}^*, h \rangle \ge \langle x^*, h \rangle, \\ \Rightarrow \quad & \langle \widehat{x}^* - x^*, h \rangle \ge 0 \quad \text{for all } h \in T_C(x_0) \\ \Rightarrow \quad & \widehat{x}^* = x^*. \end{aligned}$$

Since $y \in T_C(x_0)$ it follows that

$$\begin{aligned} & \langle x^*, y \rangle = 0 \\ \Rightarrow \quad & \lim_{n\to\infty} \xi_n \|Dx_n - Dx_0\|_p^{p-1} = 0. \end{aligned}$$

We return to (2) and use $v = \mathcal{F}^{-1}(\psi'(x_0)) \in W_0^{1,p}(Z)$. We obtain

$$\begin{aligned} \left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0)) \right\rangle = {} & \mu_n \left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0)) \right\rangle \\ & + \xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0)) \right\rangle \end{aligned}$$

$$\Rightarrow \quad 0 > \left\langle x^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = (\lim \mu_n)\|\psi'(x_0)\|^2 \\ + \lim \xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ = (\lim \mu_n)\|\psi'(x_0\|^2 = 0,$$

a contradiction (note that $|\xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle| \leq |\xi_n|\|Dx_n - Dx_0\|_p^{p-1}\|\mathcal{F}^{-1}(\psi'(x_0))\|_p$). So indeed we can find $\varepsilon_0 > 0$ and $n_0 \geq 1$ such that for all $n \geq n_0$ we have $|\mu_n| \geq \varepsilon_0$.

Moreover, from (3) we have that

$$\left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle - \xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ = \mu_n \left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle,$$

$$\Rightarrow \quad \lim_{n\to\infty} \mu_n \left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = \left\langle x^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle < 0 \qquad \text{(see (6))},$$

$$\Rightarrow \quad (\lim_{n\to\infty} \mu_n) \left\langle \psi'(x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = (\lim_{n\to\infty} \mu_n)\|\psi'(x_0)\|^2 < 0,$$

$$\Rightarrow \quad \mu_n < 0 \quad \text{for all } n \geq n_0.$$

Once again we return to (2) and as above we employ the test function $v = \mathcal{F}^{-1}(\psi'(x_0))$. We obtain

$$\left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = \mu_n \left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ + \xi_n \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle.$$

Divide by ξ_n (recall that we are assuming that $|\xi_n| \to +\infty$). So we have

$$\frac{1}{\xi_n} \left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle = \frac{\mu_n}{\xi_n} \left\langle \psi'(x_n), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \\ + \left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle.$$

Remark that $\frac{1}{\xi_n} \left\langle x_n^*, \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \to 0$ while from the previous arguments we have that $\left\langle A(x_n - x_0), \mathcal{F}^{-1}(\psi'(x_0))\right\rangle \to 0$. So in the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{\mu_n}{\xi_n}\|\psi'(x_0)\|^2 = 0 \\ \Rightarrow \quad \lim_{n\to\infty} \frac{\mu_n}{\xi_n} = 0. \tag{7}$$

Since $x_n^* = \mu_n \psi'(x_n) + \xi_n A(x_n - x_0)$ (see (1)) and $\psi'(x_n) = A(x_n) - N_f(x_n)$ we have

$$A(x_n) = \frac{1}{\mu_n} x_n^* + N_f(x_n) - \frac{\xi_n}{\mu_n} A(x_n - x_0)$$

$$\Rightarrow \quad \langle A(x_n), \vartheta \rangle = \left\langle \frac{1}{\mu_n} x_n^* + N_f(x_n) - \frac{\xi_n}{\mu_n} A(x_n - x_0), \vartheta \right\rangle$$

for all $\vartheta \in C_0^\infty(Z)$.

From the representation theorem for the elements of $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ (see ADAMS [1], p. 51) we have $-\operatorname{div}(\|Dx_n\|^{p-2} Dx_n)$ and $\operatorname{div}(\|D(x_n - x_0)\|^{p-2} D(x_n - x_0)) \in W^{-1,q}(Z)$. So after integration by parts and since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we obtain

$$\begin{aligned}-\operatorname{div}(\|Dx_n(z)\|^{p-2} Dx_n(z)) - \frac{\xi_n}{\mu_n} \operatorname{div}(\|D(x_n - x_0)(z)\|^{p-2} D(x_n - x_0)(z))\\ = \frac{1}{\mu_n} x_n^*(z) + f(z, x_n(z)) \quad \text{a.e. on } Z.\end{aligned}$$

From the previous considerations we have that either $\mu_n < 0$ for $n \geq 1$ large or (7) is valid, hence $\frac{\xi_n}{\mu_n} \to +\infty$ as $n \to \infty$.

Also because by hypothesis x_0 is a local minimizer of $J\big|_C$, for some $\lambda < 0$, we have

$$-\operatorname{div}(\|Dx_0(z)\|^{p-2} Dx_0(z)) - f(z, x_0(z)) = \frac{1}{\lambda} u_0(z) \quad \text{a.e. on } Z$$

with $u_0 \in L^q(Z)$, $u_0(z) \in \partial j(z, x_0(z))$ a.e. on Z. So we have

$$\begin{aligned}&-\operatorname{div}\Big(\|Dx_n\|^{p-2} Dx_n + \frac{\xi_n}{\mu_n} \|D(x_n - x_0)\|^{p-2} (Dx_n - Dx_0)\\ &\qquad + \frac{\xi_n}{\mu_n} \|Dx_0\|^{p-2} Dx_0\Big)(z)\\ &\quad = \frac{1}{\mu_n} x_n^*(z) + f(z, x_n(z)) - \frac{\xi_n}{\mu_n} f(z, x_0(z)) - \frac{1}{\lambda} \frac{\xi_n}{\mu_n} u_0(z) \quad \text{a.e. on } Z.\end{aligned}$$

Let $\eta(z) = Dx_0(z)$. Then $\eta \in C^\beta(\bar{Z})$. We consider the vector field

$$A_n(z, \zeta) = \|\zeta\|^{p-2} \zeta + \frac{\xi_n}{\mu_n} \|\zeta - \eta(z)\|^{p-2} (\zeta - \eta(z)) + \frac{\xi_n}{\mu_n} \|\eta(z)\|^{p-2} \eta(z).$$

Evidently $A_n(z,0)=0$ and $\langle A_n(z,\zeta),\zeta\rangle \geq \|\zeta\|^p$. Moreover, if $\frac{\xi_n}{\mu_n}\in[0,1]$, straightforward calculations can verify that

$$\begin{aligned}
&\|A_n(z,\zeta)\| \leq \beta_1+\beta_2\|\zeta\|^{p-1},\\
&\big(A_n(z,\zeta)-A_n(z,\zeta'),\zeta-\zeta'\big)_{\mathbb{R}^N} \geq \beta_3\|\zeta-\zeta'\|^p,\\
&\|A_n(z,\zeta)-A_n(z',\zeta)\| \leq \beta_4\|z-z'\|(1+\|\zeta\|^{p-1}),\\
&\big(D_\zeta A_n(z,\zeta)\vartheta,\vartheta\big)_{\mathbb{R}^N} \geq \beta_5\left(\|\zeta\|^{p-2}+\frac{\xi_n}{\mu_n}\|\eta(z)-\zeta\|^{p-2}\right)\|\vartheta\|^2,\\
&\big(D_\zeta A_n(z,\zeta)\vartheta,\vartheta\big)_{\mathbb{R}^N} \leq \beta_6\left(\|\zeta\|^{p-2}+\frac{\xi_n}{\mu_n}\|\eta(z)-\zeta\|^{p-2}\right)\|\vartheta\|^2
\end{aligned}$$

with $\beta_k>0$ $k=1,\dots,6$ independent of $\frac{\xi_n}{\mu_n}\in[0,1]$. Also

$$\frac{1}{\beta_7}\left(\left(1+\frac{\xi_n}{\mu_n}\right)\|\zeta\|^{p-2}+\frac{\xi_n}{\mu_n}\|\eta(z)\|^{p-2}\right)$$

$$\leq\left(\|\zeta\|^{p-2}+\frac{\xi_n}{\mu_n}\|\eta(z)-\zeta\|^{p-2}\right)\leq\beta_7\left(\left(1+\frac{\xi_n}{\mu_n}\right)\|\zeta\|^{p-2}+\frac{\xi_n}{\mu_n}\|\eta(z)\|^{p-2}\right)$$

with $\beta_7>1$ depending only on $p\geq 2$.

Then we have that

$$\begin{aligned}
-\operatorname{div} A_n\big(z,Dx_n(z)\big) &= \frac{1}{\mu_n}x^*(z)+f(z,x_n(z))-\frac{\xi_n}{\mu_n}f(z,x_0(z))\\
&\quad-\frac{1}{\lambda}\frac{\xi_n}{\mu_n}u_0^*(z)\quad\text{a.e. on } Z.
\end{aligned}$$

First invoking Theorem 7.1, p. 286 of Ladyzhenskaya–Uraltseva [18], we obtain $M_1>0$

$$\|x_n\|_\infty \leq M_1 \quad\text{for all } n\geq 1.$$

Then the properties of the vector field $A_n(z,\zeta)$, permit the use of Theorem 1 of Lieberman [20] (see also Di Benedetto [8], Chapter IX) and obtain $M_2>0$, $\alpha\in(0,1)$, such that

$$\|x_n\|_{C_0^{1,\alpha}(\bar Z)} \leq M_2 \quad\text{for all } n\geq 1.$$

Now if $\frac{\xi_n}{\mu_n}>1$. Then if $w_n=x_n-x_0$, we have

$$-\operatorname{div}\left(\|Dw_n\|^{p-2}Dw_n+\frac{\xi_n}{\mu_n}\|D(w_n+x_n)\|^{p-2}D(w_n+x_0)\right.$$

$$- \frac{\xi_n}{\mu_n} \|Dx_0\|^{p-2} Dx_0\Big)(z)$$
$$= \frac{1}{\xi_n} x_n^*(z) + \frac{\xi_n}{\mu_n} f\big(z, w_n(z) + x_0(z)\big) + \frac{\xi_n}{\mu_n} f\big(z, x_0(z)\big)$$
$$+ \frac{1}{\lambda} \frac{\xi_n}{\mu_n} u_0(z) \quad \text{a.e. on } Z.$$

Introduce the vector field

$$\widehat{A}_n(z,\zeta) = \|\zeta\|^{p-2}\zeta + \frac{\xi_n}{\mu_n}\|\zeta + \eta(z)\|^{p-2}\big(\zeta + \eta(z)\big) - \frac{\xi_n}{\mu_n}\|\eta(z)\|^{p-2}\eta(z).$$

This satisfies the same properties as $A_n(z,\zeta)$ and

$$-\operatorname{div} \widehat{A}_n(z, Dw_n(z)) = \frac{1}{\xi_n} x_n^*(z) + \frac{\xi_n}{\mu_n} f\big(z, w_n(z) + x_0(z)\big) + \frac{\xi_n}{\mu_n} f(z, x_0(z)) + \frac{1}{\lambda}\frac{\xi_n}{\mu_n} u_0(z) \text{ a.e.}$$

So we are back in the previous situation and again

$$\|x_n\|_{C_0^{1,\alpha}(\bar{Z})} \le M_2 \quad \text{for all } n \ge 1.$$

Since the embedding of $C_0^{1,\alpha}(\bar{Z})$ into $C_0^1(\bar{Z})$ is compact (see Kufner–John–Fučik [16], p. 38) and $x_n \to x_0$ in $W_0^{1,p}(Z)$, it follows that $x_n \to x_0$ in $C_0^1(\bar{Z})$. Recall that

$$J(x_n) < J(x_0) \quad \text{and} \quad \psi(x_n) = c \quad \text{for all } n \ge 1.$$

This contradicts the hypothesis that x_0 is a local $C_0^1(\bar{Z})$-minimizer of J on C. $\square$

4. Eigenvalue problems with constraints

In this section, we prove a three solutions theorem for the following constrained eigenvalue problem with nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) \in \lambda \partial j(z, x(z)) \text{ a.e on } Z \\ x|_\Gamma = 0, \ \|Dx\|_p = 1, \ 2 \le p < \infty, \ \lambda \in \mathbb{R}. \end{array} \right\} \tag{8}$$

Our hypotheses on the nonsmooth nonlinearity $j(z,x)$, are the following:

$\mathbf{H}(j) = j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $\int_Z j(z,0)dz = 0$ and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have $|u| \leq \alpha_1(z) + c_1|x|^{p-1}$, with $\alpha_1 \in L^\infty(Z)$, $c_1 > 0$;

(iv) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have $ux \geq 0$ and $0 \in \partial j(z,x)$ a.e. on Z if and only if $x = 0$;

(v) $\lim_{x\to 0} \frac{u}{|x|^{p-2}x} = 0$ uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$.

In what follows $S_1 = \{x \in W_0^{1,p}(Z) : \|Dx\|_p = 1\}$, the constraint set.

We start with a general result which is actually of independent interest.

Proposition 2. *If X is a reflexive Banach space, $S \subseteq X$ is a C^1-manifold of codimension 1, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, $y_1, y_2 \in S$ are two distinct local minimizers of φ on S and φ satisfies the nonsmooth* PS_β*-condition on S for every $\beta > \max\{\varphi(y_1), \varphi(y_2)\}$, then φ has a third critical point on S.*

Proof. Without any loss of generality, we assume that $\varphi(y_1) \leq \varphi(y_2)$. We also assume that y_2 is an isolated local minimizer of φ on S, or otherwise there is nothing to prove. So let $r \in (0, \|y_1 - y_2\|)$ such that

$$\varphi(y_1) \leq \varphi(y_2) < \xi = \inf[\varphi(y) : y \in \partial B_r(y_2) \cap S].$$

Let g be a continuous curve on S with $g(0) = y_1$ and $g(1) = y_2$. We introduce the sets

$$E = \partial B_r(y_2) \cap S \quad \text{and} \quad F = g([0,1]).$$

It is clear that E and $\partial F = \{y_1, y_2\}$ link in S (see Struwe [25], p. 116). We define

$$\Gamma = \{\gamma \in C(S,S) : \gamma|_{\partial F} = \text{identity}\} \quad \text{and} \quad \beta = \inf_{\gamma\in\Gamma} \sup_{u\in F} \varphi(\gamma(u)).$$

Evidently $\varphi(y_1) \leq \varphi(y_2) < \xi \leq \beta$. We shall show that β is a critical value of φ on S. To this end let $K_\beta = \{x \in S : \varphi(x) = \beta$ and $0 \in$

$\partial(\varphi|_S)(x)\}$. If β is not a critical value of φ on S, then $K_\beta = \emptyset$. Let $\varepsilon_0 = \xi - \varphi(y_2) > 0$ and $U = \emptyset$ and apply the deformation theorem of Chang [4] (see Remark 3 in the paper) with $0 < \varepsilon < \varepsilon_0$. We obtain a continuous one-parameter family of homeomorphisms $\eta : [0,1] \times S \to S$ with the properties postulated by the result of Chang. From the choice of $\varepsilon_0 > 0$, we see that for all $t \in [0,1]$, $\eta(t,.)|_{\partial F} =$ identity. Also we choose $\gamma \in \Gamma$ such that $\varphi(\gamma(u)) < \beta + \varepsilon$ for all $u \in F$ (recall the definition of β). Set $\gamma_1 = \eta(1,.) \circ \gamma$. From the deformation theorem, we have that $\gamma_1 \in \Gamma$ and also $\sup[\varphi(\gamma_1(u)) : u \in F] < \beta - \varepsilon$, which contradicts the definition of β. This proves that $K_\beta \neq \emptyset$ and so we have a critical point $y_3 \in S$ of φ on S such that $\beta \leq \varphi(y_3)$. This means that y_3 is distinct from y_1, y_2. $\square$

Now we return to the analysis of problem (8) and consider the integral functional $J : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$J(x) = -\int_Z j(z, x(z))dz.$$

We know that J is locally Lipschitz (see Hu–Papageorgiou [13], p. 313).

Proposition 3. *If hypotheses* H*(j) hold, then* J *satisfies the non-smooth* PS_c*-condition on* S_1 *for every* $c \neq 0$.

Proof. Let $\{x_n\}_{n\geq 1} \subseteq S_1$ be a sequence such that

$$J(x_n) \to c \neq 0 \quad \text{and} \quad m_1(x_n) \to 0.$$

By virtue of the Poincaré inequality $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$ (from the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$). Hence $J(x_n) \to J(x) = c \neq 0$. By virtue of hypothesis $\mathrm{H}(j)$(iv) and the Lebourg mean value theorem (see Lebourg [9] or Clarke [6], p. 41), we see that for almost all $z \in Z$ and all $u \neq 0$, we have $j(z,u) > 0$ and so $J(x) \neq 0$ implies $x \neq 0$. Because the norm in a Banach space is weakly lower semicontinuous and since the set $\partial(J|_{S_1})(x_n) \subseteq W^{-1,q}(Z)$ is weakly compact, we can find $x_n^* \in \partial J(x_n)$ such that

$$m_1(x_n) = \|x_n^* - \langle x_n^*, x_n\rangle \mathcal{F}(x_n)\| \quad \text{for all } n \geq 1.$$

From Clarke [6], p. 47 (see also Chang [4], Theorem 2.2), we have that $\{x_n^*(\cdot)\}_{n\geq 1} \subseteq L^q(Z)$ and because $x_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z, from hypothesis H(j)(iii) it follows that $\{x_n^*\}_{n\geq 1} \subseteq L^q(Z)$ is bounded. So we may assume that $x_n^* \xrightarrow{w} x^*$ in $L^q(Z)$, hence $x_n^* \to x^*$ in $W^{-1,q}(Z)$ (from the compact embedding of $L^q(Z)$ into $W^{-1,q}(Z)$). Exploiting the lower semicontinuity of $J^0(\cdot;\cdot)$ and since $\langle x_n^*, h\rangle \leq J^0(x_n; h)$ for almost all $h \in W_0^{1,p}(Z)$ and all $n \geq 1$, in the limit we have $\langle x^*, h\rangle \leq J^0(x; h)$ for all $h \in W_0^{1,p}(Z)$ and so $x^* \in \partial J(x) \subseteq L^q(Z)$ and $x^*(z) \in \partial j(z, x(z))$ a.e. on Z (see Clarke [6], p. 76). Therefore, since $x \neq 0$, from hypothesis H(j)(iv) we have that $x^* \neq 0$. Also we have $\langle x_n^*, x_n\rangle = (x_n^*, x_n)_{pq} \to (x^*, x)_{pq} = \langle x^*, x\rangle$. Moreover, since by the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq S_1$, we have $\|x_n^* - \langle x_n^*, x_n\rangle \mathcal{F}(x_n)\| \to 0$ and $x_n^* \to x^*$ in $W^{-1,q}(Z)$, we infer that

$$\langle x_n^*, x_n\rangle \mathcal{F}(x_n) \to x^* \quad \text{in } W^{-1,q}(Z).$$

Remark that $\{\mathcal{F}(x_n)\}_{n\geq 1} \subseteq W^{-1,q}(Z)$ is bounded. So we may assume that $\mathcal{F}(x_n) \xrightarrow{w} w$ in $W^{-1,q}(Z)$. Hence $\langle x^*, x\rangle\, w = x^* \neq 0$ and so $\langle x^*, x\rangle \neq 0$. Therefore $\mathcal{F}(x_n) \to \frac{x^*}{\langle x^*, x\rangle}$ in $W^{-1,q}(Z)$ and since $\mathcal{F}$ is a homeomorphism, we conclude $x_n \to \frac{\mathcal{F}^{-1}(x^*)}{|\langle x^*, x\rangle|}$ in $W_0^{1,p}(Z)$, which completes the proof. □

Now we are ready for the "three solutions theorem" for problem

Theorem 4. *If hypotheses H(j) hold, then problem* (8) *has at least three distinct nontrivial solutions* $(\lambda_k, x_k) \in \mathbb{R} \times W_0^{1,p}(Z)$, $k = 1, 2, 3$.

Proof. Let $\tau_+ : \mathbb{R} \to \mathbb{R}_+$ be the truncation map defined by

$$\tau_+(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Set $j_+(z, x) = j(z, \tau_+(x))$. From Clarke [6], p. 42, we know that for almost all $z \in Z, j_+(z, .)$ is locally Lipschitz and

$$\partial j_+(z, x) = \begin{cases} \partial j(z, x) & \text{if } x > 0, \\ \{0\} & \text{if } x \leq 0. \end{cases}$$

Let $J_+ : W_0^{1,p}(Z) \to \mathbb{R}$ be the integral functional defined by

$$J_+(x) = -\int_Z j_+(z, x(z))dz.$$

Also let $\psi : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by $\psi(x) = \|x\|^p = \|Dx\|_p^p$ (by Poincaré's inequality, on $W_0^{1,p}(Z)$ we consider the norm $\|x\| = \|Dx\|_p$). Evidently $\psi \in C^1(W_0^{1,p}(Z))$ and $S_1 = \{x \in W_0^{1,p}(Z) : \psi(x) = 1\}$. Hence S_1 is a manifold of codimension 1. We consider the following optimization problem:

$$\xi_+ = \inf[J_+(x) : x \in S_1]. \tag{9}$$

If $x \in W_0^{1,p}(Z)$, $x(z) \geq 0$ a.e. on Z, then from the Lebourg mean value theorem we know that we can find $u_t \in \partial J_+(tx)$ with $t \in (0,1)$ such that

$$J_+(x) - J_+(0) = (u_t, x)_{pq}.$$

But by hypothesis $J_+(0) = 0$ and so

$$J_+(x) = (u_t, x)_{pq} = \int_Z u_t(z)x(z)dz$$

with $-u_t(z) \in \partial j_+(z, x(z)) \subseteq \partial j(z, x(z))$ a.e. on Z. Then by hypothesis H(j)(iv) we have that

$$\int_Z u_t(z)x(z)dz < 0 \quad \text{for all } x \in W_0^{1,p}(Z),\ x(z) \geq 0 \text{ a.e. on } Z,\ x \neq 0,$$
$$\Rightarrow \quad J_+(x) < 0 \quad \text{for all } x \in W_0^{1,p}(Z),\ x(z) \geq 0 \text{ a.e. on } Z,\ x \neq 0.$$

It follows that $\xi_+ < 0$. Let $\{x_n\}_{n \geq 1} \subseteq S_1$ be a minimizing sequence for the optimization problem (9), i.e. $J_+(x_n) \downarrow \xi_+$. As before, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. So $J_+(x_n) \to J_+(x) = \xi_+ < 0$, hence $x \geq 0$, $x \neq 0$. We claim that $\|Dx\|_p = 1$ (i.e. $x \in S_1$). From the weak lower semicontinuity of the norm in a Banach space, we have $\|Dx\|_p \leq 1$. If $\|Dx\|_p < 1$, then we can find $\hat{\vartheta} > 1$ such that $\|D(\hat{\vartheta}x)\|_p = 1$. Consider the map $k : (0, +\infty) \to \mathbb{R}$ defined by $k(\xi) = -\int_Z j_+(z, \xi x(z))dz$. Evidently k is locally Lipschitz and so it is differentiable almost everywhere. Moreover, from the chain rule of Clarke [6], p. 42, for almost all $\xi > 0$, we have that

$$k'(\xi) = -\int_Z x(z)u(z)dz \text{ with } u \in L^q(Z),\ u(z) \in \partial j_+(z, \xi x(z)) \text{ a.e. on } Z,$$
$$\Rightarrow k'(\xi) < 0 \quad (\text{by hypothesis H}(j)(\text{iv})).$$

So the function k is strictly decreasing on $(0,+\infty)$. Note that $k(1) = J_+(x)$. Therefore $J_+(\hat{\vartheta}x) = k(\hat{\vartheta}) < k(1) = J_+(x) = \xi_+$, a contradiction to the definition of ξ_+, since $\hat{\vartheta}x \in S_1$. This means that $\|Dx\|_p = 1$, hence $x \in S_1$ and $J_+(x) = \xi_+$. From the Lagrange multiplier rule of Clarke [5], we can find $\lambda_1 > 0$ such that $A(x) + \lambda_1 \partial J_+(x) \ni 0$. As in the proof of Theorem 1, we obtain that

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2} Dx(z)\right) \in \lambda_1\ \partial j_+(z, x(z)) \quad \text{a.e. on } Z. \tag{10}$$

From Ladyzhenskaya–Uraltseva [18], p. 286, we have that $x \in L^\infty(Z)$ and so the regularity result of Lieberman [19], implies that $x \in C^{1,\alpha}(\bar{Z})$, for some $0 < \alpha < 1$. Moreover, from hypotheses H(j)(iii) and (v), it follows that for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j_+(z,x)$, we have

$$|u| \leq \alpha_2(z)|x|^{p-1} \quad \text{with} \quad \alpha_2 \in L^\infty(Z).$$

Therefore we can apply Theorem 5 of Vazquez [27] and infer that

$$x(z) > 0 \text{ for all } z \in Z \quad \text{and} \quad \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \Gamma.$$

Here n denotes the outward unit normal on the boundary Γ of Z. From this it follows that if $\varepsilon > 0$ is small, for all $y \in \{y \in C_0^1(\bar{Z}) : y \in S_1, \|x-y\|_{C_0^1(\bar{Z})} < \varepsilon\}$, we have that $J_+(x) \leq J_+(y)$, in other words x is a local $C_0^1(\bar{Z})$-minimizer of J_+ on S_1. Recall that by definition $\psi(x) = \|Dx\|_p^p$. So if $\psi_1 : W_0^{1,p}(Z) \to L^p(Z,\mathbb{R}^N)$ is given by $\psi_1(x) = Dx$ and $\psi_2 : L^p(Z,\mathbb{R}^N) \to \mathbb{R}$ is given by $\psi_2(v) = \frac{1}{p}\|v\|_p^p$, then $\psi = \psi_2 \circ \psi_1$ and from the chain rule we have that $\psi'(x) = \|Dx\|_p^{p-1} \frac{\mathcal{F}_p(Dx)}{\|Dx\|_p} \circ \psi_1(x) = \|Dx\|_p^{p-2} \mathcal{F}_p(Dx) \circ \psi_1(x)$ with $\mathcal{F}_p$ being the duality map of the Banach space $L^p(Z,\mathbb{R}^N)$. It is well known (see for example Hu–Papageorgiou [12], p. 317), that $\mathcal{F}_p(v) = \frac{\|v(.)\|^{p-2}v(.)}{\|v\|_p^{p-2}}$. So for all $\vartheta \in C_0^\infty(Z)$, $\langle \psi'(x), \vartheta\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\vartheta(z))_{\mathbb{R}^N}\, dz$, hence $\psi'(x) = A(x)$ with $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ being the strongly monotone, demicontinuous, coercive operator (thus surjective, see Hu–Papageorgiou [12], p. 322) introduced earlier. Recall that if $\mathcal{F}$ is the duality map of $W_0^{1,p}(Z)$, $\mathcal{F}^{-1}$ is the duality map of $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ and for all $u^* \in W^{-1,q}(Z)$, $\mathcal{F}^{-1}(u^*) = \|D(A^{-1}u^*)\|_p^{2-p} A^{-1}(u^*)$. So if $x^* \in \partial J_+(x)$, we have $x^* \in L^q(Z)$ and

$-x^*(z) \in \partial j_+(z, x(z)) = \partial j(z, x(z))$ a.e. on Z (since $x(z) > 0$ for all $z \in Z$) and it follows that

$$\begin{aligned} \left\langle x^*, \mathcal{F}^{-1}(\psi'(x))\right\rangle &= \|Dx\|_p^{2-p} \langle x^*, x\rangle = \|Dx\|_p^{2-p} (x^*, x)_{pq} \\ &= \|Dx\|_p^{2-p} \int_Z x^*(z)x(z)dz < 0 \text{ (see hypothesis H}(j)\text{(iv)).} \end{aligned}$$

So we have satisfied the hypotheses of Theorem 1 and we infer that x is a local $W_0^{1,p}(Z)$-minimizer of J on S_1 (recall that $J_+ = J$ on the nonnegative elements of $W_0^{1,p}(Z)$). Moreover, from (10) we see that $(\lambda_1, x_1 = x) \in (\mathbb{R}_+ \setminus \{0\}) \times W_0^{1,p}(Z)_+$ is a solution of problem (8).

Next let $\tau_- : \mathbb{R} \to \mathbb{R}_-$ be the truncation function defined by $\tau_-(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ and set $j_-(z, x) = j(z, \tau_-(x))$. Arguing as above, we obtain $y \in C^1(\bar{Z})$, $y(z) < 0$ for all $z \in Z$ which is a local minimizer of J on S_1 and solves (8) with $\lambda_2 < 0$. Hence $(\lambda_2, x_2 = y) \in (\mathbb{R}_- \setminus \{0\}) \times W_0^{1,p}(Z)_-$ is a solution of problem (8).

Let $D = \{x \in S_1 : J(x) < 0\}$. Evidently $x_1, x_2 \in D$. First assume that D is path-connected. Then we can find $g \in C([0,1], S_1)$ such that $g(0) = x_1$, $g(1) = x_2$ and $g(t) \in D$ for all $t \in [0,1]$. Then with this choice of g, if E, F and β are as in the proof of Proposition 2, we have that $\beta < 0$. By virtue of Proposition 3, J satisfies the nonsmooth PS$_\beta$-condition on S_1. From Proposition 2 (check the proof), we deduce that $J|_{S_1}$ has another critical point x_3 distinct from x_1 and x_2 such that $\beta \leq J(x_3)$. Hence $x_3 \neq 0$. As before we can find $\lambda_3 \in \mathbb{R} \setminus \{0\}$ such that $(\lambda_3, x_3) \in (\mathbb{R} \setminus \{0\}) \times W_0^{1,p}(Z)$ is a third nontrivial solution of problem (8).

If D is not path-connected and x_1, x_2 belong to different path-connected components, we can find $g \in C([0,1], S_1)$ such that $g(t) \in G = \{x \in S_1 : J(x) > 0\}$ for all $t \in (t_1, t_2)$ with $t_1, t_2 \in (0,1)$, $t_1 < t_2$. Again with this g, let E, F and β be as in the proof of Proposition 2. Then $\beta > 0$ and so from Proposition 3, J satisfies the PS$_\beta$-condition on S_1. From Proposition 2, we obtain a third critical point x_3 of J on S_1, with $\beta \leq J(x_3)$, hence $x_3 \neq 0$. Again there exists $\lambda_3 \in \mathbb{R} \setminus \{0\}$ such that $(\lambda_3, x_3) \in (\mathbb{R} \setminus \{0\}) \times W_0^{1,p}(Z)$ is a nontrivial solution of problem (8). $\square$

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