Publ. Math. Debrecen
67/3-4 (2005), 285–303

Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors

By TORU SASAHARA (Sapporo)

Abstract. We study Legendre surfaces whose mean curvature vectors are eigenvectors of the rough Laplacian. Some classification results on such Legendre surfaces are obtained. As an application, we classify nonminimal biharmonic Legendre surfaces in Sasakian space forms.

1. Introduction

Let $x : M^m \to N^n$ be an isometric immersion of an m-dimensional manifold M^m into an n-dimensional manifold N^n. Denote the rough Laplacian acting on the sections of the induced bundle x^*TN^n (resp. normal bundle $T^{\perp}M^m$) by Δ (resp. Δ^D).

During the last two decades, the class of submanifolds satisfying the following conditions has been investigated by many geometers:

$$\Delta^D H = \lambda H, \tag{1.1}$$

$$\Delta H = \lambda H, \tag{1.2}$$

where H is the mean curvature vector field and λ is a constant. (See, for instance, [5], [7]–[13], [18], [19], [22]–[24].)

Mathematics Subject Classification: 53C42, 53B25.
Key words and phrases: Sasakian manifolds, Legendre surfaces, rough Laplacian, biharmonic maps.

The first result on submanifolds with (1.1) was obtained by Barros and Garay ([5]). They classified Hopf cylinders with $\Delta^D H = 0$ in the 3-sphere. Inoguchi ([19]) generalized the classification results due to [5] to 3-dimensional Sasakian space forms. He also classified Legendre curves with (1.1) in 3-dimensional Sasakian space forms. In [23], the author investigated Legendre Chen surfaces in Sasakian space forms satisfying (1.1) under the condition that the mean curvature function is constant along a certain direction.

The first result on submanifolds satisfying (1.2) was obtained by Chen ([11], [12]). He proved that the surface in Euclidean 3-space with (1.2) is one of the following: (1) a minimal surface, (2) an open portion of an ordinary sphere, (3) an open portion of a circular cylinder. For more information about submanifolds satisfying (1.2) in Euclidean space, see [13].

In the ambient space with nonconstant sectional curvature, the study of submanifolds satisfying (1.2) has been initiated by the author. He ([22]) has studied Legendre surfaces with (1.2) in Sasakian manifolds of constant ϕ-sectional curvature -3. Also, in [24] he showed that Lagrangian surfaces satisfying (1.2) in Lorentzian complex space form are described by solutions of a certain system of partial differential equations of first order.

In this paper, first we establish the classification of Legendre Chen surfaces satisfying (1.1). This is a generalization of the classification theorem due to [23]. Secondly we classify Legendre surfaces with (1.1) under the condition that the mean curvature function is an eigenfunction of the Laplacian. Finally, we determine the intrinsic and the extrinsic structures of Legendre surfaces in Sasakian space forms satisfying (1.2), and moreover we apply this result to the theory of polyharmonic maps of order 2.

The notion of polyharmonic map of order k was introduced by Eells and Sampson ([17]). It is defined as a map which is a critical point of k-energy. Harmonic maps are polyharmonic maps of order 1. They are included in the class of polyharmonic maps of order k with $k \geq 2$. The polyharmonic maps of order 2 into Euclidean space are biharmonic in the sense of B.-Y. Chen ([13]), i.e., the components of the position vector are biharmonic functions of the Laplacian. On this reason, the polyharmonic map of order 2 is frequently called biharmonic map. Caddeo, Montaldo and Oniciuc ([7]) completely determined nonminimal biharmonic surface in $S^3(1)$. They ([8]) also stated that minimal submanifolds in $S^n\left(\frac{1}{\sqrt{2}}\right)$ are nonminimal biharmonic submanifolds in $S^{n+1}(1)$.

In [19] Inoguchi classified nonminimal biharmonic Legendre curves and Hopf cylinders in 3-dimensional Sasakian space forms.

In the last section, we classify nonminimal biharmonic Legendre surfaces in 5-dimensional Sasakian space forms. It is a 2-dimensional version of Inoguchi's result. Contrary to the 1-dimensional case, there exists a nonminimal biharmonic Legendre surface in the unit sphere. In fact, we obtain the explicit representation of such an immersion.

2. Preliminaries

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be an *almost contact manifold* if the structure group $\mathrm{GL}_{2n+1}\mathbf{R}$ of its linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to the existence of a tensor field ϕ of type (1,1), a vector field ξ and one-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

It follows that

$$\eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.2}$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{O}(2n+1)$, there exists a Riemannian metric g which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\xi, X) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM^{2n+1}$. The structure (ϕ, ξ, η, g) is called an *almost contact metric structure* and the manifold M^{2n+1} with an almost contact metric structure is said to be an *almost contact metric manifold.* If an almost contact metric manifold satisfies

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.4}$$

for all $X, Y \in TM^{2n+1}$, then M is said to be a *contact metric manifold.* On contact metric manifold, the vector field ξ is called the *characteristic vector field.*

A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on M^{2n+1}, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ.

The sectional curvature of a tangent plane which is invariant under ϕ is called ϕ-sectional curvature. If the sectional curvature is constant on all $p \in M^{2n+1}$ and all tangent planes in T_pM^{2n+1} which is invariant under ϕ, then M^{2n+1} is said to be of *constant ϕ-sectional curvature.* Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called *Sasakian space forms.* Denote Sasakian space forms of constant ϕ-sectional curvature ϵ by $M^{2n+1}(\epsilon)$. The curvature tensor $\bar{R}$ of $M(\epsilon)$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z = \frac{\epsilon+3}{4}\{g(Y,Z)X - g(Z,X)Y\} + \frac{\epsilon-1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\}. \end{aligned} \tag{2.5}$$

Let $x : N^m \to M^{2n+1}(\epsilon)$ be an isometric immersion. If η restricted to N^m vanishes, then N^m is an *integral submanifold*, in particular if $m = n$, it is called a *Legendre submanifold.*

Denote the Levi–Civita connection of $M^{2n+1}(\epsilon)$ (resp. N^m) by $\bar{\nabla}$ (resp. ∇). The formulas of Gauss and Weingarten are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

$$\bar{\nabla}_X V = -A_V X + D_X V, \tag{2.7}$$

where $X, Y \in TN^m$, $V \in T^{\perp}N^m$, h, A and D are the second fundamental form, the shape operator and the normal connection. The mean curvature vector H is given by $H = \frac{1}{m}\operatorname{trace} h$. Its length $\|H\|$ is called the *mean curvature function* of M^m.

If N^n is a Legendre submanifold, from [6] we have

$$A_{\phi Y}X = -\phi h(X,Y) = A_{\phi X}Y, \quad A_\xi = 0. \tag{2.8}$$

For more details see [6].

In the case that N^n is a Legendre submanifold, the equations of Gauss, Codazzi, Ricci are equivalent to

$$\langle R(X,Y)Z, W\rangle = \langle [A_{\phi Z}, A_{\phi W}]X, Y\rangle + \langle \bar{R}(X,Y)Z, W\rangle, \tag{2.9}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.10}$$

where $(\bar{\nabla}_X h)(Y,Z) := D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

The rough Laplacian acting on the sections of the induced bundle $x^*TM^{2n+1}(\epsilon)$ (resp. normal bundle) is defined by $\Delta = -\sum_{i=1}^m (\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i})$ (resp. $\Delta^D = -\sum_{i=1}^m (D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$), where $\{e_i\}$ is a local orthonormal frame of N^m.

One can obtain the following existence and uniqueness theorem by the similar way to those given in [14] and [15].

Theorem 1. *Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a symmetric bilinear TM^n-valued form on M^n satisfying*

(1) *$\langle\sigma(X,Y), Z\rangle$ is totally symmetric,*

(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$ is totally symmetric,*

(3) $R(X,Y)Z = \frac{\epsilon+3}{4}(g(Y,Z)X - g(X,Z)Y)$
$+ \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$.

Then there exists a Legendre immersion $x : (M^n, \langle\cdot,\cdot\rangle) \to N^{2n+1}(\epsilon)$ such that the second fundamental form h satisfies $h(X,Y) = \phi\sigma(X,Y)$.

Theorem 2. *Let $x^1, x^2 : M^n \to N^{2n+1}(\epsilon)$ be two Legendre immersions of a connected Riemannian n-manifold into a Sasakian manifold $N^{2n+1}(\epsilon)$ with second fundamental form h^1 and h^2. If*

$$\langle h^1(X,Y), \phi x^1_* Z\rangle = \langle h^2(X,Y), \phi x^2_* Z\rangle$$

for all vector fields X, Y, Z tangent to M^n, there exists an isometry A of $N^{2n+1}(\epsilon)$ such that $x^1 = A \circ x^2$.

3. Legendre surfaces whose mean curvature vectors are eigenvectors

Let $x : M^2 \to N^5(\epsilon)$ be a Legendre immersion. In the rest of paper we assume that the mean curvature function is nowhere zero. Let $\{e_i\}$ be an orthonormal frame along M^2 such that e_1, e_2 are tangent to M^2 and

$H = \frac{\alpha}{2}\phi e_1 = \frac{\alpha}{2}e_3$, with $\alpha > 0$. We shall work with such a frame in the remainder of this section. Then the second fundamental form takes the form:

$$\begin{aligned} h(e_1, e_1) &= (\alpha - c)\phi e_1 + b\phi e_2, \\ h(e_1, e_2) &= b\phi e_1 + c\phi e_2, \\ h(e_2, e_2) &= c\phi e_1 - b\phi e_2, \end{aligned} \tag{3.1}$$

for some functions b, c. From the equation of Codazzi we get

$$e_1 c + 3b\omega_1^2(e_1) = e_2 b + (\alpha - 3c)\omega_1^2(e_2), \tag{3.2}$$

$$-e_1 b + 3c\omega_1^2(e_1) = e_2 c + 3b\omega_1^2(e_2), \tag{3.3}$$

$$e_2(\alpha - c) - 3b\omega_1^2(e_2) = e_1 b + (\alpha - 3c)\omega_1^2(e_1), \tag{3.4}$$

where $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle$. Combining (3.3) and (3.4) yields

$$e_2\alpha = \alpha\omega_1^2(e_1). \tag{3.5}$$

Suppose that M^2 satisfies $\Delta^D H = \lambda H$. Then by comparing the components of ϕe_1, ϕe_2 and ξ of that condition, we obtain

$$\Delta_M \alpha + \alpha\big\{1 - \lambda + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\big\} = 0, \tag{3.6}$$

$$2e_1\alpha\omega_1^2(e_1) + 2e_2\alpha\omega_1^2(e_2) + \alpha\big\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\big\} = 0, \tag{3.7}$$

$$e_1\alpha + \alpha\omega_1^2(e_2) = 0, \tag{3.8}$$

where Δ_M is the Laplace operator acting on $C^\infty(M)$. Using (3.5) and (3.8), a straightforward computation shows

$$\left[\frac{1}{\alpha}e_1, \frac{1}{\alpha}e_2\right] = 0. \tag{3.9}$$

Consequently, we obtain that there exists local coordinates x, y such that

$$e_1 = \alpha\partial_x, \quad e_2 = \alpha\partial_y. \tag{3.10}$$

Here $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$. It follows from (3.10) that the metric tensor is given by

$$g = \frac{1}{\alpha^2}dx^2 + \frac{1}{\alpha^2}dy^2. \tag{3.11}$$

Thus we have

$$\omega_1^2(e_1) = \alpha_y, \quad \omega_1^2(e_2) = -\alpha_x, \tag{3.12}$$

where $f_x = \partial_x f$ and $f_x = \partial_x f$.

By (3.12) we obtain that (3.7) is satisfied automatically. The relation (3.6) can be rewritten as

$$-\alpha\alpha_{yy} - \alpha\alpha_{xx} + 1 - \lambda + (\alpha_x)^2 + (\alpha_y)^2 = 0. \tag{3.13}$$

From the Gauss equation, (3.12) and (3.13) we have

$$\begin{aligned} \alpha c - 2c^2 - 2b^2 + \frac{\epsilon+3}{4} &= \langle R(e_1,e_2)e_2, e_1\rangle \\ &= -(\omega_1^2(e_1))^2 - (\omega_1^2(e_2))^2 + e_2(\omega_1^2(e_1)) - e_1(\omega_1^2(e_2)) \\ &= -(\alpha_y)^2 - (\alpha_x)^2 + \alpha\alpha_{yy} + \alpha\alpha_{xx} = 1 - \lambda. \end{aligned} \tag{3.14}$$

Therefore we obtain the following.

Proposition 3. *Let M^2 be a Legendre surface of $N^5(\epsilon)$. If M^2 satisfies $\Delta^D H = \lambda H$ for a constant λ, then the Gauss curvature $G = 1 - \lambda$.*

The allied mean curvature vector $a(H)$ of M^2 in $N^5(\epsilon)$ is defined by

$$\sum_{r=4}^{5} (\mathrm{trace}\, A_H A_{e_r}) e_r.$$

If $a(H)$ vanishes identically on M^2, it is called a *Chen* surface. In [23] the author has classified Legendre Chen surfaces of $N^5(\epsilon)$ with $\Delta^D H = \lambda H$ under the condition $\phi H \|H\| = 0$ or $(\phi H)^\perp \|H\| = 0$, where $(\phi H)^\perp$ is the unit vector field perpendicular to ϕH in TM^2. More precisely, the author obtained the following theorem.

Theorem 4 ([23])**.** *Let M^2 be a Legendre Chen surface of $N^5(\epsilon)$ satisfying $\Delta^D H = \lambda H$ with non-constant mean curvature function. If $(\phi H)^\perp \|H\| = 0$ (resp. $\phi H \|H\| = 0$), then $1 - 4\lambda - \epsilon = 0$ and there exists a coordinate system $\{x, y\}$ defined in a neighborhood $V \subset I \times \mathbf{R}$ of $p \in M^2$ such that the metric tensor of M^2 is given by*

$$g = dx^2 + F(x)^2 dy^2, \tag{3.15}$$

and the second fundamental form is given by

$$\begin{aligned} h(\partial_x,\partial_x) &= \frac{1}{2F}\phi\partial_x,\\ h(\partial_y,\partial_y) &= \frac{F}{2}\phi\partial_x, \qquad (3.16)\\ h(\partial_x,\partial_y) &= \frac{1}{2F}\phi\partial_y, \end{aligned}$$

$$(\textit{resp. } h(\partial_x,\partial_x) = h(\partial_x,\partial_y) = 0,\ h(\partial_y,\partial_y) = \phi\partial_y)\,, \qquad (3.17)$$

where $F : I \to \mathbf{R} : x \to F(x)$ *is one of the following functions which are positive on* I*:*

$$c_1 \cos\frac{\sqrt{\epsilon+3}}{2}x + c_2 \sin\frac{\sqrt{\epsilon+3}}{2}x \quad (\epsilon > -3), \qquad (3.18)$$

$$c_1 x + c_2 \quad (\epsilon = -3), \qquad (3.19)$$

$$c_1 \exp\left(\frac{\sqrt{-\epsilon-3}}{2}x\right) + c_2 \exp\left(-\frac{\sqrt{-\epsilon-3}}{2}x\right) \quad (\epsilon < -3), \qquad (3.20)$$

where c_1, c_2 *are some constants.*

Conversely, suppose that c_1, c_2, $\epsilon = 1 - 4\lambda$ *are constants and* $F(x)$ *is a function which is positive on an open interval* I *satisfying one of* (3.18)–(3.20)*. Let* g *be the metric tensor on a simply-connected domain* $V \subset I \times \mathbf{R}$ *defined by* (3.15)*. Then, up to rigid motions of* $N^5(\epsilon)$*, there exists a unique Legendre Chen immersion of* (V, g) *into* $N^5(\epsilon)$ *whose second fundamental form is given by* (3.16) *(resp.* (3.17)*). Moreover such a surface satisfies* $\Delta^D H = \lambda H$ *and* $(\phi H)^{\perp}\|H\| = 0$ *(resp.* $\phi H\|H\| = 0$*).*

Remark 5. The mean curvature function $\|H\|$ of surfaces obtained in Theorem 4 is given by $\frac{1}{2F(x)}$.

We shall investigate Legendre Chen surfaces with $\Delta^D H = \lambda H$ without the additional condition $\phi H\|H\| = 0$ or $(\phi H)^{\perp}\|H\| = 0$.

If M^2 is a Chen surface, we have $b = 0$ by using (3.1). Then the relations (3.2)–(3.4) reduce to the following.

$$\alpha c_x = -(\alpha - 3c)\alpha_x, \qquad (3.21)$$

$$3c\alpha_y = \alpha c_y. \qquad (3.22)$$

On the other hand, differentiating (3.14) by y, we have

$$\alpha_y c + (\alpha - 4c)c_y = 0. \tag{3.23}$$

Combining (3.22) and (3.23) implies

$$\alpha_y c(\alpha - 3c) = 0. \tag{3.24}$$

If $\alpha = 3c$ on an open set, it follows from (3.22) that $\alpha_y = 0$ on that open set. Hence by continuity we have that (3.24) can be replaced by

$$\alpha_y c = 0. \tag{3.25}$$

If $c = 0$ on an open set, we have from (3.21) that $\alpha_x = 0$ on that open set. Therefore by continuity we obtain $\alpha_x \alpha_y = 0$. Consequently, we get the following.

Proposition 6. *Let M^2 be a Legendre surface of $N^5(\epsilon)$ satisfying $\Delta^D H = \lambda H$. Assume that the mean curvature function vanishes nowhere. If M^2 is a Chen surface, then $\phi H\|H\| = 0$ or $(\phi H)^\perp \|H\| = 0$.*

Remark 7. Theorem 4 and Proposition 6 determine Legendre Chen surfaces in Sasakian space forms satisfying (1.1).

Next we shall classify Legendre surfaces with (1.1) and $\Delta_M \|H\| = \mu \|H\|$.

Proposition 8. *Let M^2 be a Legendre surface of $N^5(\epsilon)$ satisfying $\Delta^D H = \lambda H$ and $\Delta_M \alpha = \mu \alpha$ for constants λ and μ. Suppose that α is not constant and nowhere zero. Then $\mu = 0$. If $a(H) \equiv 0$, then $1 - 4\lambda - \epsilon = 0$ and there exists a suitable coordinate system $\{x, y\}$ on a neighborhood U of $p \in M^2$ such that the metric tensor and the second fundamental form take the form (3.15) and (3.16) with $F(x) = C \exp\left(\pm\frac{\sqrt{-\epsilon-3}}{2}x\right)$ for some constant C. Conversely, such a surface is locally obtained by $F(x)$. If $a(H) \neq 0$ at each point, there exists a suitable coordinate system $\{x, y\}$ on a neighborhood U of $p \in M^2$ such that*

(1) *α is written as*

$$\alpha = \sqrt{\lambda - 1}\cos\theta x + \sqrt{\lambda - 1}\sin\theta y + E \quad (> 0), \tag{3.26}$$

and the metric tensor of M is given by

$$g = \frac{1}{\alpha^2}(dx^2 + dy^2), \tag{3.27}$$

where θ and $E\,(>0)$ are constant.

(2) *the second fundamental form is given by* (3.1), *where* $e_1 = \alpha\partial_x$, $e_2 = \alpha\partial_y$,

$$b = \frac{\sqrt{8\alpha c - 16c^2 + 2\epsilon - 2 + 8\lambda}}{4}, \tag{3.28}$$

and c is a solution of the following system of differential equations of first order.

$$\begin{aligned} &\alpha c_x \pm 3b\sqrt{\lambda - 1}\sin\theta \\ &\quad = \pm\alpha\left(\frac{(\alpha - 4c)c_y + \sqrt{\lambda - 1}\sin\theta c}{4b}\right) - (\alpha - 3c)\sqrt{\lambda - 1}\cos\theta, \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\alpha c_y \mp 3b\sqrt{\lambda - 1}\cos\theta \\ &\quad = \mp\alpha\left(\frac{(\alpha - 4c)c_x + \sqrt{\lambda - 1}\cos\theta c}{4b}\right) + 3c\sqrt{\lambda - 1}\sin\theta, \end{aligned} \tag{3.30}$$

(double signs in same order)

where θ is constant.

Conversely, suppose that $\alpha(x, y) = \sqrt{\lambda - 1}\cos\theta x + \sqrt{\lambda - 1}\sin\theta y + E$, *with $E > 0$, is a function which is positive on a simply-connected domain $U \subset \mathbf{R}^2$ and c is a solution of the system of differential equations* (3.29) *and* (3.30). *Let g be the metric tensor on U defined by* (3.27). *Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre immersion of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by* (3.1). *Moreover such a surface satisfies $\Delta^D H = \lambda H$ and $\Delta_M \alpha = 0$.*

Proof. Let M^2 be a Legendre surface of $N^5(\epsilon)$ which satisfies $\Delta^D H = \lambda H$. As mentioned before, there exits a coordinate system $\{x, y\}$ such that the metric tensor is given by (3.27). We put $e_1 = \alpha\partial_x$ and $e_2 = \alpha\partial_y$. We shall use this local orthonormal frame field.

Suppose that α satisfies $\Delta_M\alpha = \mu\alpha$ for a constant μ. We put $f = \omega_1^2(e_1)$ and $g = \omega_1^2(e_2)$. Then we get

$$e_2f - e_1g = -\mu. \tag{3.31}$$

Furthermore it follows from (3.6) that

$$f^2 + g^2 = \lambda - 1 - \mu. \tag{3.32}$$

Differentiating (3.32) by e_1 and e_2, we obtain

$$fe_1f + ge_1g = 0, \quad fe_2f + ge_2g = 0. \tag{3.33}$$

It is clear that $e_1f + e_2g = 0$ by (3.12). Combining this relation, (3.31) and (3.33) yields

$$fe_1f + ge_2f = -\mu g, \quad -ge_1f + fe_2f = 0. \tag{3.34}$$

Since α is not constant, we have $f^2 + g^2 \neq 0$ from (3.5) and (3.8). By solving (3.34) with respect to e_1f and e_2f, we obtain

$$e_1f = \frac{-\mu fg}{\lambda - 1 - \mu}, \quad e_2f = \frac{-\mu g^2}{\lambda - 1 - \mu}. \tag{3.35}$$

We replace (3.31) and (3.35) by the derivatives with respect to x and y. Since $e_1f + e_2g = 0$ holds, we have

$$f_x = \frac{-\mu fg}{\alpha(\lambda - 1 - \mu)}, \qquad f_y = \frac{-\mu g^2}{\alpha(\lambda - 1 - \mu)}, \tag{3.36}$$

$$g_x = \frac{-\mu g^2}{\alpha(\lambda - 1 - \mu)} + \frac{\mu}{\alpha}, \qquad g_y = \frac{\mu fg}{\alpha(\lambda - 1 - \mu)}. \tag{3.37}$$

We shall consider the case of $\mu \neq 0$. Then it follows from the integrability conditions $g_{xy} = g_{yx}$ and (3.37) that

$$f_xg + fg_x = -2gg_y. \tag{3.38}$$

By substituting (3.36) and (3.37) into (3.38), we have $f\mu = 0$ and hence $f = 0$. Then (3.31) and (3.32) give us $\mu = 0$. It is a contradiction. Thus we obtain that μ must be 0. Then (3.36) and (3.37) imply that f and g

are constant. Therefore it follows from (3.32) and (3.12) that α can be written as

$$\alpha = \sqrt{\lambda - 1}\,\cos\theta x + \sqrt{\lambda - 1}\,\sin\theta y + E, \tag{3.39}$$

where θ and $E\,(>0)$ are constant.

When $a(H) \equiv 0$, we have $\epsilon - 1 + 4\lambda = 0$ and moreover $\cos\theta = 1$ or $\sin\theta = 1$ in (3.39) from Theorem 4 and Proposition 6. Using the suitable coordinate change, we prove the assertion.

When $a(H) \neq 0$ at each point, the relations (3.2)–(3.4) and (3.14) imply (3.28)–(3.30). The converse is proved by applying Theorems 1 and 2. □

Remark 9. The system of equations (3.29) and (3.30) is complicated. However, in case that $\epsilon - 1 + 4\lambda = 0$ and $\cos\theta = \pm\sin\theta$, we can find a solution $c = \frac{\alpha}{4}$ of the differential system. Here the double signs are in same order.

Finally we shall determine Legendre surfaces satisfying $\Delta H = \lambda H$ in $N^5(\epsilon)$. We obtain the following.

Theorem 10. *Let M^2 be a nonminimal Legendre surface in $N^5(\epsilon)$ satisfying $\Delta H = \lambda H$. Then at each point $p \in M^2$ there exists a suitable local coordinate system $\{x, y\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:*

(1) $g = dx^2 + dy^2$,

(2) *the second fundamental form takes the following form:*

$$h(\partial_x, \partial_x) = \sqrt{\lambda - 1}\,\cos\theta\phi\partial_x,$$

$$h(\partial_x, \partial_y) = \sqrt{\lambda - 1}\,\sin\theta\phi\partial_y,$$

$$h(\partial_y, \partial_y) = \sqrt{\lambda - 1}\,\sin\theta\phi\partial_x,$$

where θ is a constant which satisfies

$$\sin\theta(\cos\theta - \sin\theta) = \frac{\epsilon + 3}{4(1 - \lambda)}. \tag{3.40}$$

Conversely, suppose that θ, $\lambda\,(>1)$ and ϵ are constants satisfying (3.40). Let $g = dx^2 + dy^2$ be the metric tensor on a simply-connected *domain $V \subset \mathbf{R}^2$. Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre immersion of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by (2). Moreover such a surface satisfies $\Delta H = \lambda H$.*

Proof. By the Gauss and Weingarten formulae we have

$$\Delta H = \mathrm{tr}(\bar{\nabla} A_H) + \Delta^D H + (\mathrm{tr} A^2_{\phi e_1})H + a(H), \tag{3.41}$$

where $\mathrm{tr}(\bar{\nabla} A_H) = \sum_{i=1}^2 (A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i)$. Suppose that $\Delta H = \lambda H$ for a constant λ. By comparing of the component of ϕe_2,

$$\begin{aligned} &2e_1 \alpha \omega_1^2(e_1) + 2e_2 \alpha \omega_1^2(e_2) \\ &\qquad + \alpha\big\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\big\} + \langle a(H), \phi e_2 \rangle = 0. \end{aligned} \tag{3.42}$$

From (3.12) and (3.42) we have $a(H) = 0$, i.e. M^2 is a Chen surface. Since $\mathrm{tr}\bar{\nabla} A_H = 0$, we get

$$2(\alpha - c)e_1\alpha + \alpha\{(\alpha - c)\omega_1^2(e_2) + e_1(\alpha - c)\} = 0, \tag{3.43}$$

$$2ce_2\alpha + \alpha\{(\alpha - c)\omega_1^2(e_1) + e_2(\alpha - c)\} = 0. \tag{3.44}$$

By replacing (3.2), (3.3), (3.43) and (3.44) by the derivatives with respect to x and y, we have

$$\alpha c_x = -(\alpha - 3c)\alpha_x, \tag{3.45}$$

$$3c\alpha_y = \alpha c_y, \tag{3.46}$$

$$2(\alpha - c)\alpha_x - (\alpha - c)\alpha_x + \alpha(\alpha - c)_x = 0, \tag{3.47}$$

$$2c\alpha_y + (\alpha - c)\alpha_y + \alpha(\alpha - c)_y = 0. \tag{3.48}$$

Solving the system (3.45)–(3.48), we obtain that α and c are constant.

Thus the equation of Gauss yields

$$\alpha c - 2c^2 + \frac{\epsilon + 3}{4} = 0. \tag{3.49}$$

Also since the relation $\langle \Delta^D H + (\mathrm{tr} A^2_{\phi e_1} - \lambda)H, \phi e_1 \rangle = 0$ holds, we have

$$(\alpha - c)^2 + c^2 = \lambda - 1. \tag{3.50}$$

Consequently, we obtain that with respect to a suitable coordinate system $\{x, y\}$, the metric tensor and the second fundamental form are given by (1) and (2) respectively. The proof of converse follows from applying Theorems 1 and 2. $\square$

The mean curvature vector H is said to be *C-parallel* if $DH \| \xi$. Baikoussis and Blair ([2]) classified Legendre surfaces in $N^5(\epsilon)$ whose mean curvature vector is C-parallel. From Theorem 10 we see that the mean curvature vector of Legendre surfaces with (1.2) in $N^5(\epsilon)$ is C-parallel.

Remark 11. Theorem 10 is a generalization of the classification result on Legendre Chen surfaces in a Sasakian space form $N^5(-3)$ due to [22].

Remark 12. In case $\epsilon = -3$ (resp. 1), we obtain the explicit representation of the position vectors of Legendre surfaces with $\Delta H = \lambda H$ in $\mathbf{R}^5$ (resp. $\mathbf{R}^6$) by virtue of [22] (resp. [1]).

4. Biharmonic Legendre surfaces

In this section, by applying Theorem 10 we determine nonminimal biharmonic Legendre surfaces in Sasakian space forms.

Let (M^m, g) and (N^n, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map. We denote by ∇ and $\bar{\nabla}$ the Levi–Civita connections on M and N respectively. Then the *tension field* $\tau(\phi)$ is a section of the vector bundle ϕ^*TN defined by

$$\tau(\phi) := \mathrm{tr}(\nabla^{\phi} d\phi) = \sum_{i=1}^{m} \{\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)\}.$$

Here ∇^{ϕ} and $\{e_i\}$ denote the induced connection by ϕ on the bundle ϕ^*TN, which is the pull-back of $\bar{\nabla}$, and a local orthonormal frame field of M, respectively.

A smooth map ϕ is said to be a *harmonic map* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \int_{\Omega} \sum_{i=1}^{m} h(d\phi(e_i), d\phi(e_i)) dv_g$$

over every compact supported region Ω of M.

EELLS and SAMPSON ([17]) suggested to study *polyharmonic maps of order k* which are critical points of k-energy E_k:

$$E_k(\phi) = \int_\Omega |(d + d^*)^k \phi|^2 dv_g.$$

Here d^* is the codifferential operator and $|\cdot|$ denotes the Hilbert–Schmidt norm. They are frequently called *k-harmonic maps*. The 1-harmonic map coincides with the harmonic map. In case of $k = 2$, we have

$$E_2(\phi) = \int_\Omega h(\tau(\phi), \tau(\phi)) dv_g.$$

The Euler–Lagrange equation of the functional E_2 was computed by JIANG ([20], [21]) as follows.

$$\tau_2(\phi) := -\mathcal{J}_\phi(\tau(\phi)) = 0. \tag{4.1}$$

Here the operator $\mathcal{J}_\phi$ is the *Jacobi operator* defined by

$$\mathcal{J}_\phi(V) := \bar{\Delta}_\phi V - \mathcal{R}_\phi(V), \quad V \in \Gamma(\phi^* TN), \tag{4.2}$$

$$\bar{\Delta}_\phi := -\sum_{i=1}^{m} \left(\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i} \right), \tag{4.3}$$

$$\mathcal{R}_\phi(V) := \sum_{i=1}^{m} R^N(V, d\phi(e_i)) d\phi(e_i), \tag{4.4}$$

where R^N is the curvature tensor of N.

Remark 13. Let $\phi : M \to N$ be an isometric immersion. Then its tension field is mH. Thus the functional E_2 is given by

$$E_2(\phi) = m^2 \int_\Omega h(H, H) dv_g.$$

In case that M is 2-dimensional, $E_2(\phi)$ is the total mean curvature of Ω up to constant multiple. (See [10], Section 5.3.)

In particular, if N is the Euclidean n-space $\mathbf{E}^n$ and ϕ is an isometric immersion, then

$$\tau_2(\phi) = \Delta_M \Delta_M \phi,$$

since $\Delta_M \phi = -mH$. Thus the 2-harmonicity for an isometric immersion into Euclidean space is equivalent to the biharmonicity in the sense of Chen. (See [13].) 2-harmonic maps are frequently called *biharmonic maps.*

Now consider the case that $m = 2$ and N is a Sasakian space form $N^5(\epsilon)$. Then from (4.1)–(4.4) and (2.5) we see that ϕ is an isometric biharmonic Legendre immersion if and only if

$$\Delta H - \left(\frac{5\epsilon + 3}{4}\right) H = 0, \tag{4.5}$$

where Δ is the Laplace operator described in Section 2.

We put $\lambda = \frac{5\epsilon+3}{4}$ in Theorem 10. Since α and c in (3.49) and (3.50) are real numbers, ϵ must satisfy $\epsilon \geq \frac{-11+32\sqrt{2}}{41}$. Solving (3.49) and (3.50) with respect to α and c implies the following.

Corollary 14. *Let M^2 be a nonminimal biharmonic Legendre surface in $N^5(\epsilon)$. Then $\epsilon \geq \frac{-11+32\sqrt{2}}{41}$ and at each point $p \in M^3$ there exists a suitable local coordinate system $\{x, y\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:*

(1) $$g = dx^2 + dy^2,$$

(2) $$h(\partial_x, \partial_x) = \frac{\epsilon - 1}{\alpha}\phi\partial_x$$
$$h(\partial_y, \partial_y) = \left(\alpha - \frac{\epsilon - 1}{\alpha}\right)\phi\partial_x,$$
$$h(\partial_x, \partial_y) = \left(\alpha - \frac{\epsilon - 1}{\alpha}\right)\phi\partial_y$$

where

$$\alpha = \begin{cases} \sqrt{\frac{13\epsilon - 9 \pm \sqrt{41\epsilon^2 + 22\epsilon - 47}}{8}} & (\epsilon \neq 1), \\ 1 & (\epsilon = 1). \end{cases}$$

Conversely, suppose that ϵ is a constant satisfying $\epsilon \geq \frac{-11+32\sqrt{2}}{41}$ and let g be the metric tensor on a simply-connected domain $V \subset \mathbf{R}^2$ defined by (1). Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendre immersion of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by (2). Moreover such an immersion is nonminimal biharmonic.

We consider the complex Euclidean $(n+1)$-space $\mathbf{C}^{n+1}$ and identify $z = (x_1 + iy_1, \ldots, x_{n+1} + iy_{n+1}) \in \mathbf{C}^{n+1}$ with $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \in \mathbf{E}^{2n+2}$. Let J be its usual almost complex structure. It is well-known that a Sasakian space form $N^{2n+1}(1)$ is isomorphic to $S^{2n+1}(1)$ endowed with the Sasakian structure induced by J of $\mathbf{C}^{n+1}$. (For example, see [6].)

By Corollary 14 and the same computations as in [1], we can explicitly represent nonminimal biharmonic Legendre immersions into $S^5(1)$ in $\mathbf{C}^3$ as follows.

Corollary 15. *Let $f_L : M^2 \to S^5(1) \subset \mathbf{C}^3$ be a nonminimal biharmonic Legendre immersion. Then the position vector $f_L = f_L(x,y)$ of M^2 in $\mathbf{C}^3$ is given by*

$$f_L(x,y) = \frac{1}{\sqrt{2}}\big(e^{ix}, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y\big). \tag{4.6}$$

We see that $f_L(x,y)$ is doubly periodic. More precisely, $f_L(x,y)$ is periodic with period 2π w.r.t. x and $\sqrt{2}\pi$ w.r.t. y. Thus, f_L is a nonminimal biharmonic Legendre immersion from a torus into $S^5(1)$.

Let $f : M \to \mathbf{E}^n$ be an isometric immersion. If the position vector f can be written as

$$f = f_1 + f_2, \quad \Delta_M f_1 = \lambda_1 f_1, \quad \Delta_M f_2 = \lambda_2 f_2,$$

for two different constants λ_1 and λ_2, then f is said to be of 2-*type*. Now, we put

$$f_1(x,y) := \frac{1}{\sqrt{2}}(e^{ix}, 0, 0),$$

$$f_2(x,y) := \frac{1}{\sqrt{2}}(0, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y).$$

Then we have $f_L = f_1 + f_2$, $\Delta_M f_1 = f_1$ and $\Delta_M f_2 = 3f_2$. Thus (4.6) is of 2-type.

In case that the ambient space is $S^5(1)$, by [1]–[3], Theorem 10 and Corollary 14 we obtain the following inclusions between the different conditions for nonminimal Legendre surfaces:

$$\text{Biharmonic} \subset \Delta H = \lambda H \subset \text{2-type in } \mathbf{E}^6 \quad \subset \quad DH \parallel \xi. \tag{4.7}$$

Now, put

$$g_1(x,y) := (\cos x, \sin x),$$

$$g_2(x,y) := \frac{1}{\sqrt{2}}(1, \sin\sqrt{2}y, \cos\sqrt{2}y).$$

Then we see that $f_L(x,y)$ can be written as $f_L(x,y) = g_1 \otimes g_2$ (see [16]). g_2 is a biharmonic curve in $S^2(1)$ ([8]) and hence f_L is a tensor product of two biharmonic curves in the unit sphere.

Remark 16. In case that N^n is a real space form or a Sasakian space form or a complex space form, the geometry of submanifolds with $\Delta H = \lambda H$ is closely related to that of biharmonic submanifolds. (See [24] for Lagrangian surfaces satisfying $\Delta H = \lambda H$.)

Acknowledgments. The author would like to express his hearty gratitude to Professor J. Inoguchi for his helpful advice and constant encouragement. The author also thank the referee whose comments have helped to improve this paper.

References

[1] C. Baikoussis and D. E. Blair, 2-type integral surfaces in $S^5(1)$, *Tokyo J. Math.* **14** (1991), 345–356.

[2] C. Baikoussis and D. E. Blair, Integral surfaces in Sasakian space forms, *J. Geometry* **43** (1992), 30–40.

[3] C. Baikoussis, T. Koufogiorgos and F. Defever, Coordinate finite type integral surfaces in S^5, *Algebras, Groups and Geometries* **10** (1993), 227–239.

[4] P. Baird and D. Kamissoko, On constructing biharmonic maps and mertics, *Annals of Global Analysis and Geometry* **23** (2003), 65–75.

[5] M. Barros and O. J. Garay, On submanifolds with harmonic mean curvature, *Proc. Amer. Math. Soc* **123** (1995), 2545–2549.

[6] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, Vol. 203, *Birkhauser Boston, Inc., Boston, MA*, 2002.

[7] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds of S^3, *Internat. J. Math* **12** (2001), 867–876.

[8] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds in spheres, *Israel J. Math* **130** (2002), 109–123.

[9] R. Caddeo, C. Oniciuc and P. Piu, Explicit formulas for biharmonic non-geodesic curves of the Heisenberg group, *preprint*.

[10] B. Y. Chen, Total Mean Curvature and Submanifold of Finite Type, *World Scientific Publ., Singapore*, 1984.

[11] B. Y. Chen, Null 2-type surfaces in E^3 are circular cylinders, *Kodai Math. J.* **11** (1988), 295–299.

[12] B. Y. Chen, Null 2-type surfaces in Euclidean space, *Algebra, Analysis and Geometry* (1988), 1–18.

[13] B. Y. Chen, A report on submanifolds of finite type, *Soochow J. Math* **22** (1996), 117–337.

[14] B. Y. Chen and L. Vrancken, Addendum to: Existence and uniqueness theorem for slant immersions and its applications, *Results in Math* **39** (2001), 18–22.

[15] B. Y. Chen and L. Vrancken, Existence and uniqueness theorem for slant immersions and its applications, *Results in Math* **31** (1997), 28–39.

[16] F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken, The semiring of immersions of manifolds, *Beiträge Algebra Geometry* **5** (1994), 209–215.

[17] J. Eells and J. H. Sampson, Variational theory in fiber bundles, Proc. U.S.-Japan Seminar in Differential Geometry, *Kyoto*, 1965, 22–33.

[18] A. Ferrández, P. Lucas and M. A. Meroño, Biharmonic Hopf cylinders, *Rocky Mountain J. Math* **28** (1998), 957–975.

[19] J. Inoguchi, Submanifolds with harmonic mean curvature vector field in contact 3-manifolds, *Colloq. Math.* **100** (2004), 163–179.

[20] G. Y. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. A* **7** (1986), 130–144.

[21] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, *Chinese Ann. Math. A* **7** (1986), 389–402.

[22] T. Sasahara, Spectral decomposition of the mean curvature vector field of surfaces in a Sasakian manifold $\mathbf{R}^{2n+1}(-3)$, *Results in Math.* **43** (2003), 168–180.

[23] T. Sasahara, Legendre surfaces whose mean curvature vectors are eigenvectors of the Laplace operator, *Note di Mat.* **22** (2003), 49–58.

[24] T. Sasahara, Quasi-minimal Lagrangian surfaces whose mean curvature vectors are eigenvectors (*to appear in* Demonstratio Mat.).

TORU SASAHARA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO 060-0810
JAPAN

E-mail: t-sasa@math.sci.hokudai.ac.jp

(Received October 13, 2003; revised September 3, 2004)