*Publ. Math. Debrecen*
**67/3-4** (2005), 315–331

# Hilbert's inequalities and their reverses

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**Abstract.** In this paper, we make some further generalizations of Hilbert's well known inequality and its equivalent form in both the integral and the discrete case. A reverse of Hilbert's inequality is also given in the integral case. Several other results of this type obtained in recent years, follow as special cases of our results.

## 1. Introduction

Let us first repeat the Hilbert's well known inequality and its equivalent in both the integral and the discrete case.

**Theorem A.** *If $f$ and $g \in L^2[0,\infty)$, then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy \le \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right)^{\frac{1}{2}},$$

*and*

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^2 dy \le \pi^2 \int_0^\infty f^2(x)dx,$$

*where $\pi$ and $\pi^2$ are the best constants.*

*Mathematics Subject Classification:* 26D15.
*Key words and phrases:* inequalities, Hilbert's inequality, Hölder's inequality, reverse inequality, equivalent inequalities, gamma function, beta function.

**Theorem B.** *The following inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi\left(\sum_{n=1}^{\infty}{a_n}^2\sum_{m=1}^{\infty}{b_m}^2\right)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 \le \pi^2\left(\sum_{m=1}^{\infty}{a_n}^2\right),$$

*where $\pi$ and $\pi^2$ are the best constants.*

In recent years there were lots of generalizations of this theorem. Let us mention some authors who gave a number of results: Yang, Hong Yong, Gavrea, Peachey.

Brnetić and Pečarić ([1], [2]) gave some further generalizations of Hilbert's inequality. We shall state their main theorems that will attract our attention. Let us note that in all theorems and corollaries that follow, we suppose that all integrals and series converge, so we shall omit of conditions of this type.

**Theorem C.** *If $n \ge 2$ is an integer, $\lambda > n-2$ and $\sum_{i=1}^{n}\frac{1}{p_i} = 1$, $p_i > 0$, $i = 1, 2, \ldots, n$ then*

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1+x_2+\cdots+x_n)^{\lambda}}dx_1dx_2\ldots dx_n < K\left(\int_0^{\infty}x^{n-1-\lambda+p_i(A_i-A_{i+1})}f_i^{p_i}(x)dx\right)^{\frac{1}{p_i}} \quad (1)$$

*where*

$$K = \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\left(\Gamma(1-A_{i+1}p_i)\Gamma(\lambda-n+1+A_{i+1}p_i)\right)^{\frac{1}{p_i}},$$

$$A_i \in \left(\frac{n-\lambda-1}{p_{i-1}}, \frac{1}{p_{i-1}}\right),$$

*$i = 1, 2, \ldots, n$, while $\Gamma$ is the gamma function. We use the conventions $p_0 = p_n$ and $A_{n+1} = A_1$.*

The two authors considered the special case $n = 2$ ([2]), and they obtained equivalent forms in both the integral and the discrete case.

**Theorem D.** *If $\lambda > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\, dy$$
$$< L\Big(\int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_0^\infty x^{1-\lambda+q(A_2-A_1)} g^q(x)dx\Big)^{\frac{1}{q}}, \tag{2}$$

*and*

$$\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)} \Big(\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx\Big)^p dy$$
$$< L^p \Big(\int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x)dx\Big), \tag{3}$$

*where $L = (B(1 - A_2 p, \lambda - 1 + A_2 p))^{\frac{1}{p}} (B(1 - A_1 q, \lambda - 1 + A_1 q))^{\frac{1}{q}}$, $A_1 \in \big(\frac{1-\lambda}{q}, \frac{1}{q}\big)$ and $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$, where $B$ is the beta function.*

**Theorem E.** *If $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, $\lambda > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then the following inequalities hold and are equivalent:*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda}$$
$$< L\Big(\sum_{m=1}^\infty m^{1-\lambda+p(A_1-A_2)} {a_m}^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^\infty n^{1-\lambda+q(A_2-A_1)} {b_n}^q\Big)^{\frac{1}{q}},$$

*and*

$$\sum_{n=1}^\infty n^{(\lambda-1)(p-1)+p(A_1-A_2)} \Big(\sum_{m=1}^\infty \frac{a_m}{(m+n)^\lambda}\Big)^p < L^p \sum_{m=1}^\infty m^{1-\lambda+p(A_1-A_2)} {a_m}^p$$

*where $L$ is defined as in the previous theorem, $A_1 \in \big(\frac{1-\lambda}{q}, \frac{1}{q}\big)$, $A_2 \in \big(\frac{1-\lambda}{p}, \frac{1}{p}\big)$, and $A_1, A_2 > 0$.*

In the proofs of these main theorems Hölder's inequality was used. In this paper we shall use the reverse of Hölder's inequality ([4]) to obtain appropriate reverses of the inequalities of the authors mentioned. We shall also make some further generalizations in both the integral and the discrete case.

## 2. Integral case

First we shall generalize Theorem C and obtain the reverse inequality. More precisely, we have the following

**Theorem 1.** *If* $n \geq 2$ *is an integer,* $\lambda > n-2$ *and* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, *with* $0 < p_1 < 1$, $p_i < 0$, $i = 2,3,\ldots,n$, *and* $A_2 \in \left(\frac{n-\lambda-1}{p_1}, \frac{1}{p_1}\right)$, $A_i \in \left(\frac{1}{p_{i-1}}, \frac{n-\lambda-1}{p_{i-1}}\right)$, $i \neq 2$, $p_0 = p_n$, $A_{n+1} = A_1$, *then the reverse inequality in* (1) *is valid.*

Proof. We start with the identity

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1+x_2+\cdots+x_n)^\lambda} dx_1 dx_2 \ldots dx_n$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\frac{{x_1}^{A_1}}{{x_2}^{A_2}} f_2(x_2)\frac{{x_2}^{A_2}}{{x_3}^{A_3}} \cdots f_n(x_n)\frac{{x_n}^{A_n}}{{x_1}^{A_1}}}{(x_1+x_2+\cdots+x_n)^\lambda} dx_1 dx_2 \ldots dx_n$$

Now, if we apply the reverse of Hölder's inequality ([4]) we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1+x_2+\cdots+x_n)^\lambda} dx_1 dx_2 \ldots dx_n$$

$$\geq \prod_{i=1}^{n} \left( \int_0^\infty \cdots \int_0^\infty \left( \frac{{x_i}^{A_i}}{{x_{i+1}}^{A_{i+1}}} \right)^{p_i} \frac{f_i^{p_i}(x_i)}{(x_1+x_2+\cdots+x_n)^\lambda} dx_1 dx_2 \ldots dx_n \right)^{\frac{1}{p_i}},$$

with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $0 < p_1 < 1$ and $p_i < 0$, $i = 2,3,\ldots,n$. Observe that we use the convention $x_{n+1} = x_1$.

It can easily be seen that this inequality is strict. Namely, equality holds if the numbers

$$\left( f_i(x_i) \frac{{x_i}^{A_i}}{{x_{i+1}}^{A_{i+1}}} \right)^{p_i} \qquad i = 1,2,\ldots,n$$

are proportional. In that case our integral diverges, which is a contradiction.

Now, recall that for the gamma function and the beta function $B(a,b)$ we have the well known formula

$$\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

So, by integrating

$$\int_0^\infty \frac{1}{(1+u_2+\cdots+u_n)^\lambda}du_n = \frac{(1+u_2+\cdots+u_{n-1})^{-\lambda+1}}{\lambda-1}$$

we obtain

$$\begin{aligned}\int_0^\infty \cdots \int_0^\infty & \frac{{u_2}^{-\alpha}}{(1+u_2+\cdots+u_n)^\lambda}du_2\,du_3\ldots du_n \\ &= \frac{1}{(\lambda-1)(\lambda-2)\cdots(\lambda-n+2)}\int_0^\infty \frac{{u_2}^{-\alpha}}{(1+u_2)^{\lambda-n+2}}du_2 \\ &= \frac{1}{\Gamma(\lambda)}\Gamma(1-\alpha)\Gamma(\lambda-n+1+\alpha).\end{aligned} \tag{4}$$

The previous equality holds for $n-\lambda-1<\alpha<1$, since the gamma function is defined for positive reals.

Now, if we put $x_k = x_1u_k$, $k=2,3,\ldots,n$, then we get

$$\begin{aligned}&{x_1}^{A_1p_1}\int_0^\infty \cdots \int_0^\infty \frac{1}{{x_2}^{A_2p_1}(x_1+x_2+\cdots+x_n)^\lambda}dx_2\,dx_3\ldots dx_n \\ &= {x_1}^{n-1-\lambda+p_1(A_1-A_2)}\int_0^\infty \cdots \int_0^\infty \frac{{u_2}^{-A_2p_1}}{(1+u_2+\cdots+u_n)^\lambda}du_2\,du_3\ldots du_n,\end{aligned} \tag{5}$$

and the result easily follows from (4) and (5).

Taking into account that the gamma function is defined for positive reals, we obtain the conditions $A_2 \in \left(\frac{n-\lambda-1}{p_1}, \frac{1}{p_1}\right)$ and $A_i \in \left(\frac{1}{p_{i-1}}, \frac{n-\lambda-1}{p_{i-1}}\right)$ for $i \neq 2$. Also, note that from this we have $\lambda > n-2$. □

It is interesting to consider the special case of Theorem 1, when $n=2$. We shall also give an equivalent form in that case. This is the following

**Theorem 2.** *If $\lambda > 0$ and $\frac{1}{p}+\frac{1}{q}=1$ with $0<p<1$, $q<0$, and $A_1 \in \left(\frac{1}{q}, \frac{1-\lambda}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$, then the reverse inequalities in* (2) *and* (3) *are valid, as well as the inequality*

$$\begin{aligned}\int_0^\infty x^{(\lambda-1)(q-1)+q(A_2-A_1)}&\left(\int_0^\infty \frac{g(y)}{(x+y)^\lambda}dy\right)^q dx \\ &< L^q\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)}g^q(y)dy\right).\end{aligned} \tag{6}$$

*In particular, these inequalities are equivalent.*

Proof. Let us show that the reverses in (2) and (3) are equivalent. Suppose that the reverse in (2) is valid. If we put

$$g(y) = y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^{p-1},$$

then taking into account $\frac{1}{p}+\frac{1}{q}=1$ and using the reverse in (2) we have

$$\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy = \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy$$

$$> L\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda+q(A_2-A_1)}g^q(x)dx\right)^{\frac{1}{q}}$$

$$= L\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy\right)^{\frac{1}{q}},$$

whence we have the reverse in (3).

Now let us suppose that the reverse inequality in (3) is valid. By applying the reverse of Hölder's inequality ([4]) and the reverse in (2) we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy$$

$$= \int_0^\infty \left(y^{-\frac{(1-\lambda)+q(A_2-A_1)}{q}}\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)y^{\frac{(1-\lambda)+q(A_2-A_1)}{q}}g(y)\,dy$$

$$> \left(\int_0^\infty y^{-\frac{p((1-\lambda)+q(A_2-A_1))}{q}}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)}g^q(y)\,dy\right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p\right)^{\frac{1}{p}}$$

$$\cdot\Big(\int_0^\infty y^{1-\lambda+q(A_2-A_1)}g^q(y)\,dy\Big)^{\frac{1}{q}}$$

$$> L\Big(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty y^{1-\lambda+q(A_2-A_1)}g^q(y)\,dy\Big)^{\frac{1}{q}},$$

and so we the have reverse in (2). While the reverse in (2) is valid, the reverse inequality in (3) holds, too. □

Another way of generalizing Theorem C and Theorem 1 arises from the substitution $x_i = C_i u_i^{\alpha_i}$, $i = 1, 2, \dots, n$. More precisely, we have the following

**Theorem 3.** *If $n \geq 2$ is an integer, $\lambda > n-2$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i, C_i > 0$, $i = 1, 2, \dots, n$, then*

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(C_1{x_1}^{\alpha_1} + C_2{x_2}^{\alpha_2} + \cdots + C_n{x_n}^{\alpha_n})^\lambda} dx_1 dx_2 \dots dx_n$$

$$< K_1\Big(\int_0^\infty x^{\alpha_i(n-\lambda+p_i(A_i-A_{i+1})-p_i)+p_i-1}{f_i}^{p_i}(x)dx\Big)^{\frac{1}{p_i}},$$

*where $K_1 = \prod_{i=1}^n {\alpha_i}^{\frac{1}{p_i}-1} \prod_{i=1}^n C_i^{\frac{n-\lambda}{p_i}+A_i-A_{i+1}-1} K$ and $A_i \in \big(\frac{n-\lambda-1}{p_{i-1}}, \frac{1}{p_{i-1}},\big)$, $i = 1, 2, \dots n$. If $0 < p_1 < 1$ and $p_i < 0$, $i = 2, 3, \dots, n$, then we obtain the reverse inequality for any $A_2 \in \big(\frac{n-\lambda-1}{p_1}, \frac{1}{p_1}\big)$ and $A_i \in (\frac{1}{p_{i-1}}, \frac{n-\lambda-1}{p_{i-1}})$, $i \neq 2$.*

Putting $C_1 = C_2 = \cdots = C_n = 1$, as a special case we obtain the result of Brnetić and Pečarić ([1]). It is obvious that the numbers $A_i = \frac{n-\lambda}{2p_{i-1}}$, $i = 1, 2, \dots, n$ satisfy the conditions of Theorem 3, so we obtain

**Corollary 1.** *If $n \geq 2$ is an integer, $\lambda > n-2$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$ with $p_i > 0$, $i = 1, 2, \dots, n$, then*

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(C_1{x_1}^{\alpha_1} + C_2{x_2}^{\alpha_2} + \cdots + C_n{x_n}^{\alpha_n})^\lambda} dx_1 dx_2 \dots dx_n$$

$$< K_2 \prod_{i=1}^n \Big(\int_0^\infty x^{\alpha_i(n-\lambda)(\frac{1}{2}+\frac{p_i}{2p_{i-1}})+p_i(1-\alpha_i)-1}{f_i}^{p_i}(x)dx\Big)^{\frac{1}{p_i}}$$

*where $K_2 = \prod_{i=1}^n {\alpha_i}^{\frac{1}{p_i}-1} \prod_{i=1}^n C_i^{\frac{n-\lambda}{p_i}+A_i-A_{i+1}-1} \frac{\Gamma^2(\frac{2-n+\lambda}{2})}{\Gamma(\lambda)}$. If $0 < p_1 < 1$ and $p_i < 0$, $i = 2, 3, \dots, n$ then we obtain the reverse inequality.*

Gavrea considered the special case ($C_i = \alpha_i = 1,\ i = 1, 2, \dots, n$) of Corollary 1 in [3], but without the reverse inequality.

As before, it is interesting to describe the special case $n = 2$ in equivalent forms. So we have

**Theorem 4.** *If $\lambda > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, $A, B, \alpha, \beta > 0$, then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\lambda}\,dx\,dy$$
$$< L_1 \left( \int_0^\infty x^{\alpha(1-\lambda+p(A_1-A_2)+1-p)+p-1} f^p(x)\,dx \right)^{\frac{1}{p}} \tag{7}$$
$$\cdot \left( \int_0^\infty x^{\beta(1-\lambda+q(A_2-A_1)+1-q)+q-1} g^q(x)\,dx \right)^{\frac{1}{q}},$$

*and*

$$\int_0^\infty y^{\beta((\lambda-1)(p-1)+p(A_1-A_2)+1)-1} \left( \int_0^\infty \frac{f(x)}{(Ax^\alpha + By^\beta)^\lambda}\,dx \right)^p dy$$
$$< {L_1}^p \int_0^\infty x^{\alpha(1-\lambda)+p\alpha(A_1-A_2)-(p-1)(\alpha-1)} f^p(x)\,dx, \tag{8}$$

*where* $L_1 = \alpha^{-\frac{1}{q}} \beta^{-\frac{1}{p}} A^{\frac{1}{p}-\frac{1}{q}-\frac{\lambda}{q}+A_1-A_2} B^{\frac{1}{q}-\frac{1}{p}-\frac{\lambda}{q}+A_2-A_1} L$, *while* $A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$ *and* $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$. *If $0 < p < 1$, $q < 0$ and $A_1 \in (\frac{1}{q}, \frac{1-\lambda}{q})$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$, then the reverse inequalities in* (7) *and* (8) *are valid, as well as the following inequality:*

$$\int_0^\infty x^{\alpha((\lambda-1)(q-1)+q(A_2-A_1)+1)-1} \left( \int_0^\infty \frac{g(y)}{(Ax^\alpha + By^\beta)^\lambda}\,dy \right)^q dx$$
$$< {L_1}^q \int_0^\infty y^{\beta(1-\lambda)+q\beta(A_2-A_1)-(q-1)(\beta-1)} q^q(y)\,dy. \tag{9}$$

*These inequalities are also equivalent.*

Remark that Theorem 4 is a generalization of Theorem D from the Introduction.

Now we shall consider some special cases. Namely, if we put $A_1 = A_2 = \frac{2-\lambda}{pq}$ in Theorem 4 (they satisfy the conditions), we obtain

**Corollary 2.** *If $\lambda > 2 - \min\{p,q\}$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, $A, B, \alpha, \beta > 0$, then the following inequalities hold and are equivalent:*

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\lambda}\, dx\, dy \\ & < L_2 \left( \int_0^\infty x^{\alpha(1-\lambda)+(p-1)(1-\alpha)} f^p(x)\, dx \right)^{\frac{1}{p}} \\ & \quad \cdot \left( \int_0^\infty x^{\beta(1-\lambda)+(q-1)(1-\beta)} g^q(x)\, dx \right)^{\frac{1}{q}},\end{aligned} \tag{10}$$

*and*

$$\begin{aligned}\int_0^\infty & y^{\beta(\lambda-1)(p-1)+\beta-1} \left( \int_0^\infty \frac{f(x)}{(Ax^\alpha + By^\beta)^\lambda}\, dx \right)^p dy \\ & < L_2{}^p \int_0^\infty x^{\alpha(1-\lambda)-(p-1)(\alpha-1)} f^p(x)\, dx,\end{aligned} \tag{11}$$

*where* $L_2 = \alpha^{-\frac{1}{q}} \beta^{-\frac{1}{p}} A^{\frac{1}{p}-\frac{1}{q}-\frac{\lambda}{q}} B^{\frac{1}{q}-\frac{1}{p}-\frac{\lambda}{q}} B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$. *If* $0 < p < 1$, $q < 0$ *and* $2 - p < \lambda < 2 - q$, *then the reverse inequalities in* (10) *and* (11) *are valid, as well as the following inequality*

$$\begin{aligned}\int_0^\infty & x^{\alpha(\lambda-1)(q-1)+\alpha-1} \left( \int_0^\infty \frac{g(y)}{(Ax^\alpha + By^\beta)^\lambda}\, dy \right)^q dx \\ & < L_2{}^q \int_0^\infty y^{\beta(1-\lambda)-(q-1)(\beta-1)} g^q(y)\, dy.\end{aligned} \tag{12}$$

*These inequalities are also equivalent.*

If we put $A = B = \alpha = \beta = 1$ in Corollary 2, then we obtain the result of Yang ([7]), but without reverse inequalities. Further, $\alpha = \beta = 1$ was considered by the same author in [8].

Another interesting case that we shall consider is $A_1 = \frac{2-\lambda}{2q}$ and $A_2 = \frac{2-\lambda}{2p}$. We obtain

**Corollary 3.** *If $\lambda > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, $A, B, \alpha, \beta > 0$ then* *the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\lambda}\,dx\,dy$$
$$< L_3\Big(\int_0^\infty x^{-\frac{\alpha\lambda p}{2}+p-1} f^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_0^\infty x^{-\frac{\beta\lambda q}{2}+q-1} g^q(x)\,dx\Big)^{\frac{1}{q}}, \tag{13}$$

*and*

$$\int_0^\infty y^{\frac{\beta\lambda p}{2}-1}\Big(\int_0^\infty \frac{f(x)}{(Ax^\alpha + By^\beta)^\lambda}\,dx\Big)^p dy < L_3{}^p \int_0^\infty x^{-\frac{\alpha\lambda p}{2}+p-1} f^p(x)\,dx,$$

*where* $L_3 = \alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{\lambda}{2}-\frac{2\lambda}{q}}B^{-\frac{\lambda}{2}}B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)$. *If* $0 < p < 1$ *and* $q < 0$, *then the reverse inequalities in* (13) *and* (14) *are valid, as well as the following inequality:*

$$\int_0^\infty x^{\frac{\alpha\lambda q}{2}-1}\Big(\int_0^\infty \frac{g(y)}{(Ax^\alpha + By^\beta)^\lambda}\,dy\Big)^q dx < L_3{}^q \int_0^\infty y^{-\frac{\beta\lambda q}{2}+q-1} g^q(y)\,dy.$$

*These inequalities are also equivalent.*

If we put $A = B = \alpha = \beta = 1$ in Corollary 3, we obtain the result of Yang ([6]).

On the other hand, if we put $A_1 = \frac{1-b}{q} - \frac{1}{pq}$ and $A_2 = \frac{1-c}{p} - \frac{1}{pq}$ in Theorem 4, we obtain

**Corollary 4.** *If* $\lambda > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$, $A, B, \alpha, \beta > 0$, $0 < b + \frac{1}{p} < \lambda$ *and* $0 < c + \frac{1}{q} < \lambda$, *then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax^\alpha + By^\beta)^\lambda}\,dx\,dy$$
$$< L_4\Big(\int_0^\infty x^{\alpha(p-1)(1-b)+\alpha(c-\lambda)+(p-1)(1-\alpha)} f^p(x)\,dx\Big)^{\frac{1}{p}} \tag{14}$$
$$\cdot \Big(\int_0^\infty x^{\beta(q-1)(1-c)+\beta(b-\lambda)+(q-1)(1-\beta)} g^q(x)\,dx\Big)^{\frac{1}{q}},$$

*and*

$$\int_0^\infty y^{\beta(\lambda-b)(p-1)+\beta c-1}\left(\int_0^\infty \frac{f(x)}{(Ax^\alpha+By^\beta)^\lambda}\,dx\right)^p dy$$
$$< L_4{}^p \int_0^\infty x^{\alpha(p-1)(1-b)+\alpha(c-\lambda)+(p-1)(1-\alpha)} f^p(x)\,dx \quad (15)$$

*where*

$$L_4 = \alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{c}{p}-\frac{\lambda+b}{q}}B^{\frac{b}{q}-\frac{\lambda+c}{p}}\frac{1}{\Gamma(\lambda)}$$
$$\cdot\left(\Gamma\left(c+\frac{1}{q}\right)\Gamma\left(\lambda-c-\frac{1}{q}\right)\right)^{\frac{1}{p}}\left(\Gamma\left(b+\frac{1}{p}\right)\Gamma\left(\lambda-b-\frac{1}{p}\right)\right)^{\frac{1}{q}}.$$

*If* $0<p<1$ *and* $q<0$ *then the reverse inequalities in* (16) *and* (17) *are valid as well as the inequality*

$$\int_0^\infty x^{\alpha(\lambda-c)(q-1)+\alpha b-1}\left(\int_0^\infty \frac{g(y)}{(Ax^\alpha+By^\beta)^\lambda}\,dy\right)^q dx$$
$$< L_4{}^q \int_0^\infty y^{\beta(q-1)(1-c)+\beta(b-\lambda)+(q-1)(1-\beta)} g^q(y)\,dy. \quad (16)$$

## 3. Non-homogeneous discrete case

Now we shall consider the discrete case. Our main result that generalizes all results of Brnetić and Pečarić ([2]), is the following

**Theorem 5.** *If* $\{a_n\}$ *and* $\{b_n\}$ *are nonnegative real sequences,* $\lambda>0$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $A,B,\alpha,\beta>0$, *then the following inequalities hold:*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{(Am^\alpha+Bn^\beta)^\lambda} < L_1\left(\sum_{m=1}^\infty m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}a_m{}^p\right)^{\frac{1}{p}}$$
$$\cdot\left(\sum_{n=1}^\infty n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)}b_n{}^q\right)^{\frac{1}{q}}, \quad (17)$$

*and*

$$\sum_{n=1}^{\infty} n^{\beta(\lambda-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\right)^p < L_1{}^p\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}a_m{}^p, \tag{18}$$

*where* $L_1$ *is defined in Theorem 4 and* $A_1 \in (\max\{\frac{1-\lambda}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q})$, $A_2 \in (\max\{\frac{1-\lambda}{p}, \frac{\beta-1}{\beta p}\}, \frac{1}{p})$. *In particular, the inequalities* (19) *and* (20) *are equivalent.*

Proof. First we will prove inequality (19). If we apply Hölder's inequality we obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\\
&\quad=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\cdot\frac{a_m(Am^{\alpha})^{A_1+\frac{1}{q\alpha}-\frac{1}{q}}}{(Bn^{\beta})^{A_2+\frac{1}{p\beta}-\frac{1}{p}}}\cdot\frac{b_n(Bn^{\beta})^{A_2+\frac{1}{p\beta}-\frac{1}{p}}}{(Am^{\alpha})^{A_1+\frac{1}{q\alpha}-\frac{1}{q}}}\\
&\quad<\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m{}^p}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\cdot\frac{(Am^{\alpha})^{pA_1+\frac{p}{q\alpha}-\frac{p}{q}}}{(Bn^{\beta})^{pA_2+\frac{1}{\beta}-1}}\right)^{\frac{1}{p}}\\
&\qquad\cdot\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n{}^q}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\cdot\frac{(Bn^{\beta})^{qA_2+\frac{q}{p\beta}-\frac{q}{p}}}{(Am^{\alpha})^{qA_1+\frac{1}{\alpha}-1}}\right)^{\frac{1}{q}}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m{}^p}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\cdot\frac{(Am^{\alpha})^{pA_1+\frac{p}{q\alpha}-\frac{p}{q}}}{(Bn^{\beta})^{pA_2+\frac{1}{\beta}-1}}\\
&\quad=\sum_{m=1}^{\infty}a_m{}^p(Am^{\alpha})^{pA_1+\frac{p}{q\alpha}-\frac{p}{q}}\sum_{n=1}^{\infty}\frac{1}{(Am^{\alpha}+Bn^{\beta})^{\lambda}\cdot(Bn^{\beta})^{pA_2+\frac{1}{\beta}-1}}\\
&\quad\le\sum_{m=1}^{\infty}a_m{}^p(Am^{\alpha})^{pA_1+\frac{p}{q\alpha}-\frac{p}{q}}\int_0^{\infty}\frac{dy}{(Am^{\alpha}+By^{\beta})^{\lambda}\cdot(By^{\beta})^{pA_2+\frac{1}{\beta}-1}}\\
&\quad=\beta^{-1}B^{-\frac{1}{\beta}}A^{1-\lambda+p(A_1-A_2)+(p-1)(\frac{1}{\alpha}-1)}\\
&\qquad\cdot\sum_{m=1}^{\infty}a_m{}^p m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}\int_0^{\infty}\frac{t^{-A_2p}}{(1+t)^{\lambda}}\,dt
\end{aligned}$$

$$= \beta^{-1} B^{-\frac{1}{\beta}} A^{1-\lambda+p(A_1-A_2)+(p-1)(\frac{1}{\alpha}-1)} \frac{\Gamma(1-A_2p)\Gamma(\lambda-1+A_2p)}{\Gamma(\lambda)}$$

$$\cdot \sum_{m=1}^{\infty} {a_m}^p m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}.$$

Analogusly, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{{b_n}^q}{(Am^\alpha+Bn^\beta)^\lambda} \cdot \frac{(Bn^\beta)^{qA_2+\frac{q}{p\beta}-\frac{q}{p}}}{(Am^\alpha)^{qA_1+\frac{1}{\alpha}-1}}$$

$$< \alpha^{-1} A^{-\frac{1}{\alpha}} B^{1-\lambda+q(A_2-A_1)+(q-1)(\frac{1}{\beta}-1)} \frac{\Gamma(1-A_1q)\Gamma(\lambda-1+A_1q)}{\Gamma(\lambda)}$$

$$\cdot \sum_{n=1}^{\infty} {b_n}^q n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)},$$

so the inequality (19) holds.

Further, it is obvious that the functions

$$f(x) = \frac{1}{(Ax^\alpha+Bn^\beta)^\lambda \cdot (x^\alpha)^{qA_1+\frac{1}{\alpha}-1}} \qquad \text{and}$$

$$g(y) = \frac{1}{(Am^\alpha+By^\beta)^\lambda \cdot (y^\beta)^{pA_2+\frac{1}{\beta}-1}}$$

are decreasing if the conditions $qA_1+\frac{1}{\alpha}-1 \geq 0$ and $pA_2+\frac{1}{\beta}-1 \geq 0$ are satisfied. Also, taking into account that the gamma function is defined for positive reals, we obtain $A_1 \in \big(\max\{\frac{1-\lambda}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q}\big)$ and $A_2 \in \big(\max\{\frac{1-\lambda}{p}, \frac{\beta-1}{\beta p}\}, \frac{1}{p}\big)$.

Let us show that the inequalities (19) and (20) are equivalent. Suppose that the inequality (19) is valid. By putting

$$b_n = n^{\beta(\lambda-1)(p-1)+p\beta(A_1-A_2)+\beta-1} \left( \sum_{m=1}^{\infty} \frac{a_m}{(Am^\alpha+Bn^\beta)^\lambda} \right)^{p-1},$$

taking into account that $\frac{1}{p}+\frac{1}{q}=1$ and using (19), we have

$$\sum_{n=1}^{\infty} n^{\beta(\lambda-1)(p-1)+p\beta(A_1-A_2)+\beta-1} \left( \sum_{m=1}^{\infty} \frac{a_m}{(Am^\alpha+Bn^\beta)^\lambda} \right)^p$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} \\
&< L_1\left(\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)} {a_m}^p\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\sum_{n=1}^{\infty} n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)} {b_n}^q\right)^{\frac{1}{q}}, \\
&= L_1\left(\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)} {a_m}^p\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\sum_{n=1}^{\infty} n^{\beta(\lambda-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\right)^p\right)^{\frac{1}{q}},
\end{aligned}
$$

so we obtain the inequality (20).

Now, suppose that inequality (20) is valid. By applying Hölder's inequality and (20) we have

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} \\
&= \sum_{n=1}^{\infty}\left(n^{-\frac{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)}{q}}\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\right) \\
&\quad \cdot n^{\frac{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)}{q}} b_n \\
&< \sum_{n=1}^{\infty}\left(n^{-\frac{p}{q}(\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta))}\left(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\right)^p\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\sum_{n=1}^{\infty} n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)} {b_n}^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{n=1}^{\infty} n^{\beta(\lambda-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\right)^p\right) \\
&\quad \cdot \left(\sum_{n=1}^{\infty} n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)} {b_n}^q\right)^{\frac{1}{q}}
\end{aligned}
$$

$$< L_1\left(\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)} {a_m}^p\right)^{\frac{1}{p}}$$
$$\cdot\left(\sum_{n=1}^{\infty} n^{\beta(1-\lambda)+\beta q(A_2-A_1)+(q-1)(1-\beta)} {b_n}^q\right)^{\frac{1}{q}},$$

so we obtain (19). This completes the proof. □

Remark that Theorem 5 is a generalization of Theorem E from Introduction.

We shall consider the same special cases for Theorem 5. First, if we put $A_1 = A_2 = \frac{2-\lambda}{pq}$ in Theorem 5, then we obtain

**Corollary 5.** *If $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, $A,B>0$, $0<\alpha,\beta\le 1$ and $2-\min\{p,q\}<\lambda\le 2+\min\{\frac{p}{\alpha}-p,\frac{q}{\beta}-q\}$, then the following inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^\alpha+Bn^\beta)^\lambda}$$
$$< L_2\left(\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+(p-1)(1-\alpha)} {a_m}^p\right)^{\frac{1}{p}}\cdot\left(\sum_{n=1}^{\infty} n^{\beta(1-\lambda)+(q-1)(1-\beta)} {b_n}^q\right)^{\frac{1}{q}},$$

*and*

$$\sum_{n=1}^{\infty} n^{\beta(\lambda-1)(p-1)+\beta-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(Am^\alpha+Bn^\beta)^\lambda}\right)^p$$
$$< {L_2}^p\sum_{m=1}^{\infty} m^{\alpha(1-\lambda)+(p-1)(1-\alpha)} {a_m}^p$$

*where $L_2$ is defined as in Corollary 2.*

If we put $\alpha=\beta=1$, we obtain, as in the integral case, the result of Yang ([8]).

Now we shall consider the case $A_1=\frac{2-\lambda}{2q}$ and $A_2=\frac{2-\lambda}{2p}$. We obtain

**Corollary 6.** *If $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, $0<\lambda\le\min\{\frac{2}{\alpha},\frac{2}{\beta}\}$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $A,B,\alpha,\beta>0$, then the following* *inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}$$
$$< L_3\Big(\sum_{m=1}^{\infty} m^{-\frac{\alpha\lambda p}{2}+p-1}{a_m}^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} n^{-\frac{\beta\lambda q}{2}+q-1}{b_n}^q\Big)^{\frac{1}{q}},$$

*and*

$$\sum_{n=1}^{\infty} n^{\frac{\beta\lambda p}{2}-1}\Big(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\Big)^p < {L_3}^p\sum_{m=1}^{\infty} m^{-\frac{\alpha\lambda p}{2}+p-1}{a_m}^p$$

*where $L_3$ is defined as in Corollary 3.*

Note that Corollary 6 is a generalization of YANG's result in [7].

Finally, if we put $A_1=\frac{1-b}{q}-\frac{1}{pq}$ and $A_2=\frac{1-c}{p}-\frac{1}{pq}$ in Theorem 5, we obtain the following

**Corollary 7.** *If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, $A,B,\alpha,\beta>0$, $0<b+\frac{1}{p}\le \min\{\frac{1}{\alpha},\lambda\}$ and $0<c+\frac{1}{q}\le\min\{\frac{1}{\beta},\lambda\}$, then the following inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_4\Big(\sum_{m=1}^{\infty} m^{\alpha(p-1)(1-b)+\alpha(c-\lambda)+(p-1)(1-\alpha)}{a_m}^p\Big)^{\frac{1}{p}}$$
$$\cdot\Big(\sum_{n=1}^{\infty} n^{\beta(q-1)(1-c)+\beta(b-\lambda)+(q-1)(1-\beta)}{b_n}^q\Big)^{\frac{1}{q}},$$

*and*

$$\sum_{n=1}^{\infty} n^{\beta(\lambda-b)(p-1)+\beta c-1}\Big(\sum_{m=1}^{\infty}\frac{a_m}{(Am^{\alpha}+Bn^{\beta})^{\lambda}}\Big)^p$$
$$< {L_4}^p\sum_{m=1}^{\infty} m^{\alpha(p-1)(1-b)+\alpha(c-\lambda)+(p-1)(1-\alpha)}{a_m}^p,$$

*where $L_4$ is defined as in Corollary 4.*

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*(Received November 5, 2003; revised July 16, 2004)*