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Composition operators between weighted inductive limits of spaces of holomorphic functions

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Abstract. Composition operators between weighted inductive limits of Banach spaces of holomorphic functions defined on open subsets of the complex plane are studied. The continuity and compactness of composition operators between weighted Banach spaces of type H^∞ on arbitrary open subsets of $\mathbb{C}$ is treated first.

1. Introduction: Notation and preliminaries

Weighted Banach spaces of holomorphic functions and their countable inductive limits arise in several areas of analysis. The references [4]–[7], [14], [15] are examples of recent literature on this subject. Composition operators between weighted Banach spaces of holomorphic functions have been studied by Bonet, Domański, Lindström, Taskinen, Contreras and Hernández-Díaz in [10]–[12], [16]. Bonet and Friz have studied composition operators between weighted Fréchet spaces of holomorphic functions in [13]. There is a vast literature about composition operators on Banach spaces of holomorphic functions. We refer the reader to [17], [22].

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Our aim in this paper is to study composition operators between countable inductive limits of weighted Banach spaces of holomorphic functions. To do this, we study some properties of weighted Banach spaces of holomorphic functions on arbitrary open subsets of $\mathbb{C}$ and composition operators between them. The results obtained in this way might be of independent interest. The techniques of our proofs are related to methods developed in [5], [12], [14]. Weighted inductive limits of spaces of entire functions appear as Fourier Laplace transforms of spaces of ultradistributions. To our knowledge, this is the first attempt to study composition operators on inductive limits of spaces of holomorphic functions. Our main results are Theorems 8 and 14.

Our notation for locally convex spaces and functional analysis is standard. We refer the reader to [19]–[21]. If A is a subset of a locally convex space E, we denote by $\Gamma(A)$ its absolutely convex hull. Given a sequence $(E_n)_n$ of Banach spaces such that $E_n \hookrightarrow E_{n+1}$ continuously for each n, we denote by $E := \operatorname{ind}_n E_n$ its inductive limit, i.e. its union endowed with the strongest locally convex topology for which the injections $E_n \hookrightarrow E$ are continuous. These spaces are called (LB)-spaces [2]. E is called *regular* if every bounded subset B of E is contained and bounded in some E_n, and E is called *boundedly retractive* if for for each bounded subset B of E there is n such that E_n contains B and E and E_n endow the same topology on B. A linear mapping $T : E \to F$ between two locally convex spaces is said to be *compact* if there exists a 0-neighbourhood U in E such that $T(U)$ is relatively compact in F, *Montel* if it maps bounded sets into relatively compact sets and *bounded* if there is a 0-neighbourhood U in E such that $T(U)$ is bounded. If E is a Banach space T is bounded (Montel) if and only if it is continuous (compact).

If G is an open subset of $\mathbb{C}$, we denote by $H(G)$ the space of all holomorphic functions on G endowed with the topology τ_0 of uniform convergence on the compact subsets of G. We denote by $\mathbb{D}$ the open unit disc centered at zero. Let G_1 and G_2 be two open subsets of $\mathbb{C}$. If E and F are two spaces of holomorphic functions defined on G_1 and G_2 respectively and $\varphi : G_2 \to G_1$ is a holomorphic function such that $f \circ \varphi \in F$ for each $f \in E$, then the composition operator with symbol φ is $C_\varphi : E \to F$, $f \mapsto f \circ \varphi$.

For our notation on *weighted spaces* and *weighted inductive limits*, see [8] and [3]. A *weight* on G is a function $v : G \to \mathbb{R}$ which is strictly positive and continuous. The weighted spaces of holomorphic functions with O and o-growth conditions are defined as

$$Hv(G) := \big\{ f \in H(G) : \|f\|_v := \sup_{z \in G} v(z)|f(z)| < \infty \big\}$$

and

$$Hv_0(G) := \{ f \in H(G) : vf \text{ vanishes at infinity on } G \}.$$

Both of them are Banach spaces endowed with the norm $\|\cdot\|_v$. This norm topology is stronger than the one induced by τ_0 in these spaces. We recall that $g : G \to \mathbb{C}$ *vanishes at infinity on* G if for each $\varepsilon > 0$ there exists a compact set $K \subset G$ such that $|g(z)| < \varepsilon$ for each $z \in G \setminus K$. We denote by Bv and Bv_0 the closed unit balls of these spaces. We remark that Bv is compact for the compact open topology τ_0. Given a weight v on G, the associated weight is defined by

$$\widetilde{v}(z) := \frac{1}{\|\delta_z\|_{Hv(G)'}},$$

where δ_z is the evaluation at z and

$$\|\delta_z\|_{Hv(G)'} = \max\{ |f(z)| : f \in H(G), |f| \le 1/v \}.$$

Observation 1. If $Hv(G) \neq \{0\}$, then $0 < v(z) \le \widetilde{v}(z) < \infty$ for each $z \in G$, $Hv(G) = H\widetilde{v}(G)$ and the norms $\|\cdot\|_v$ and $\|\cdot\|_{\widetilde{v}}$ coincide.

Proof. It is clear that $0 < v \le \widetilde{v} \le \infty$ on G (see [5, 1.12]). If $f \in Hv(G)$ is a nonzero function, then for each $z_0 \in G$ there exists $k \in \mathbb{N} \cup \{0\}$ such that the function $g(z) := f(z)/(z - z_0)^k$ is holomorphic and $g(z_0) \neq 0$. The continuity of v implies that $g \in Hv(G)$. Thus $\delta_{z_0}(g) = g(z_0) \neq 0$ and $\widetilde{v}(z_0) < \infty$. The equality between the Banach spaces and their norms is shown in [5, 1.12]. □

$H\widetilde{v}_0(G)$ is always a closed subspace of $Hv_0(G)$, but these two spaces do not coincide in general. If $G = \mathbb{C}$ and $v(z) = 1/\max(1, |z|^{n+1/2})$ for $z \in \mathbb{C}$ with $n \in \mathbb{N}$, then $\widetilde{v}(z) = 1/\max(1, |z|^n)$ (cf. [5, 1.3]). Therefore $g(z) = z^n \in Hv_0(G) \setminus H\widetilde{v}_0(G)$.

A weight v defined on a balanced open set G (i.e $G = \mathbb{C}$ or G is an open disc centered at zero) is called radial if $v(z) = v(\lambda z)$ for each $z \in G$ and each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Bierstedt and Summers showed in [9] that $Hv(G)$ is canonically isomorphic to the bidual $Hv_0(G)''$ if and only if the unit ball Bv of $Hv(G)$ coincides with the closure $\overline{Bv_0}^{co}$ of the unit ball Bv_0 of Hv_0 in the compact open topology τ_0. We consider this biduality condition in many cases throughout this paper. If G is balanced and v radial, Bierstedt, Bonet and Galbis showed in [4, 1.5 (c)] that, if $Hv_0(G)$ contains the polynomials, then the polynomials are dense in $Hv_0(G)$ and $Bv = \overline{Bv_0}^{co}$.

It follows from results in [4] and [12] that if v is radial, G is balanced and $Hv_0(G)$ contains the polynomials, then $Hv_0(G) = H\widetilde{v}_0(G)$. This result leads naturally to the question whether there exists a relation between the biduality condition $Bv = \overline{Bv_0}^{co}$ and the equality $Hv_0(G) = H\widetilde{v}_0(G)$. We see below that the two conditions are not related in general.

Example 2. Let $G := \{z \in \mathbb{C} : 0 < |z| < 2\}$ and let $v_n : G \to (0,\infty)$ be defined by $v_n(z) = 1$ if $0 < |z| \leq 1$ and $v_n(z) = (2 - |z|)^n$ if $1 < |z| < 2$, $n \in \mathbb{N}$. It is easy to see that $f_1(z) := 1 \in Bv_n$, and, for $1 \leq |z_0| < 2$, $f_2(z) := 1/(2 - (\overline{z}_0/|z_0|)z)^n \in Bv_n$. Therefore $v_n = \widetilde{v_n}$ and $H(v_n)_0(G) = H(\widetilde{v_n})_0(G)$.

Each $g \in H(v_n)_0(G)$ can be holomorphically extended to 0 by defining $g(0) = 0$. If we assume $g \in B(v_n)_0$, then $|g(z)| \leq 1$ for each $z \in \mathbb{D}$. Thus, we can apply Schwarz's Lemma [1, 2.1.29] to obtain $|g(z)| \leq |z|$ for each $z \in D$. This yields, for $0 < |z| < 1$,

$$1 = \widetilde{v_n}(z) = \max\{|g(z)| : g \in Bv_n\} > \sup\{|g(z)| : g \in B(v_n)_0\}$$

and $\overline{B(v_n)_0}^{co} \subsetneq Bv_n$.

Remark 3. Example 2 was already given in [14, p. 95]. It is clear that, contrary to the final assertion in this article, $H(v_n)_0(G) = H(\widetilde{v_n})_0(G)$ for each $n \in \mathbb{N}$. Thus, the example already given in the paper [14] shows that [14, Theorem 3 (a)] as stated is false. However, [14, Theorem 3 (b)] is correct as a careful inspection of the given proof shows. This is precisely the argument which inspires our proof of (iii) → (iv) in Theorem 8 and Theorem 10 below.

Example 4. Let $G := \{z \in \mathbb{C} : 0 < |z| < 2\}$, $v_n(z) = |z|^{3/2}$ if $0 < |z| < 1$ and $v_n(z) = (2 - |z|)^n$ if $1 \leq |z| < 2$. A similar argument to the one used in the previous example shows that $\widetilde{v_n}(z) = (2 - |z|)^n$ for $1 \leq |z| < 2$. The function $f_0(z) := 1/z$ is in Bv_n and consequently $\widetilde{v_n}(z) \leq |z|$ for $0 < |z| < 1$. If $g \in Bv$, then $h(z) := z^2 g(z)$ satisfies $h(z) \leq |z|^{1/2} \leq 1$ for each $0 < |z| \leq 1$. Then we can extend h holomorphically as $h(0) = 0$, and the Schwarz's Lemma yields $|h(z)| \leq |z|$, or equivalently $|g(z)| \leq 1/|z|$. Hence $\widetilde{v_n}(z) = |z|$ for $0 < |z| < 1$. Thus, $f_0(z) = 1/z \in H(v_n)_0(G) \setminus H(\widetilde{v_n})_0(G)$.

Every function $f \in Hv_n(G)$ admits a Laurent development around zero of the form $f(z) = \sum_{n=-1}^{\infty} a_n z^n$. We fix $f \in Bv_n$ and $z_0 \in G$. Let $g(z) := zf(z)$. We proceed as in the proof of [4, 1.5]. We denote by $p_k(z)$ $(k = 0, 1, \dots)$ the Taylor Polynomial of g centered at zero of degree k and by $[C_m(g)](z)$ $(m = 0, 1, \dots)$ the Cesàro means of the Taylor polynomials of g about zero; that is

$$[C_m(g)](z) = \frac{1}{m+1} \sum_{i=0}^{m} \left(\sum_{k=0}^{i} p_k(z) \right), \quad z \in G.$$

We apply [4, 1.1] to obtain

$$|[C_m(g)](z_0)| \leq \max_{|\lambda|=1} |g(\lambda z_0)| = \max_{|\lambda|=1} |z_0| \, |f(\lambda z_0)|.$$

Since v_n is radial, it follows

$$v_n(z_0) \frac{|[C_m(g)](z_0)|}{|z_0|} \leq \max_{|\lambda|=1} v_n(\lambda z_0) |f(\lambda z_0)| \leq 1.$$

Hence $h_m(z) := [C_m(g)](z)/z \in Bv$. Moreover $h_m(z) = \sum_{k=-1}^{m} b_k z^k$ for some $b_k \in \mathbb{C}$, $k = -1, 0, \dots, m$. This yields $h_m(z) \in H(v_n)_0(G)$. Thus, $(h_m)_m$ is a sequence in $B(v_n)_0$ which is pointwise (or τ_0) convergent to f. Therefore $\overline{B(v_n)_0}^{co} = Bv_n$.

2. Composition operators between weighted Banach spaces of holomorphic functions on arbitrary open subsets of $\mathbb{C}$.

Let G_1 and G_2 open and connected subsets of $\mathbb{C}$, let v and w (continuous) weights on G_1 and G_2 respectively and let $\varphi : G_2 \to G_1$ a holomorphic function. We consider the composition operator $C_\varphi : H(G_1) \to H(G_2)$, $C_\varphi(f) = f \circ \varphi$. The following result is an extension of [12, 2.1].

Proposition 5. *The following conditions are equivalent for the composition operator C_φ:*

(a) $C_\varphi : Hv(G_1) \to Hw(G_2)$ *is continuous,*

(b) $C_\varphi(Hv(G_1)) \subset Hw(G_2)$,

(c) $\sup_{z\in G_2} \widetilde{w}(z)/\widetilde{v}(\varphi(z)) < \infty$,

(d) $\sup_{z\in G_2} w(z)/\widetilde{v}(\varphi(z)) < \infty$.

Proof. (a) $\iff$ (b) follows from the Closed Graph Theorem since $C_\varphi : (H(G_1), \tau_0) \to (H(G_2), \tau_0)$ is continuous. The inequality $w \le \widetilde{w}$ on G_2 yields (c) $\implies$ (d).

To see (d) $\implies$ (a), since $\|f\|_v = \|f\|_{\widetilde{v}}$, we have

$$\|C_\varphi(f)\|_w = \sup_{z\in G_2} w(z)|C_\varphi f(z)| \le \sup_{z\in G_2} \frac{w(z)}{\widetilde{v}(\varphi(z))}\|f\|_v.$$

To show (a) $\implies$ (c), assume that (a) holds, suppose that (c) fails and choose a sequence $(z_n)_n \subset G_2$ such that $\widetilde{w}(z_n) > n\widetilde{v}(\varphi(z_n))$ and a sequence $(f_n)_n \subset Hv(G_1)$ such that $1 = \|f_n\|_v = \|f_n\|_{\widetilde{v}} = \widetilde{v}(\varphi(z_n))f_n(\varphi(z_n))$ for each $n \in \mathbb{N}$. By (a), there exists $C > 0$ such that, for each $n \in \mathbb{N}$, $\sup_{z\in G_2} \widetilde{w}(z)|f_n(\varphi(z))| = \sup_{z\in G_2} w(z)|f_n \circ \varphi(z)| < C$. For each $n \in \mathbb{N}$

$$\widetilde{w}(z_n)|f_n(\varphi(z_n))| = \frac{\widetilde{w}(z_n)}{\widetilde{v}(\varphi(z_n))}\widetilde{v}(\varphi(z_n))|f_n(\varphi(z_n))| > n,$$

a contradiction. □

Observation 6. (1) Condition (d) is optimal in the above proposition in the sense that we cannot replace $\widetilde{v}$ by v. For instance, for G and v as in Example 4, take $G_1 = G_2 = G$, $w = \widetilde{v}$ and $\varphi(z) = z$.

$C_\varphi : Hv(G_1) \to Hw(G_2)$ is an isometric isomorphism which satisfies $\sup_{z\in G_2} w(z)/v(\varphi(z)) = \infty$.

(2) The same example shows that there exist continuous composition operators $C_\varphi : Hv(G_1) \to Hw(G_2)$ such that the restriction $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ is not continuous because $C_\varphi(Hv_0(G_1)) \nsubseteq Hw_0(G_2)$.

(3) In general the continuity of $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ does not imply that of $C_\varphi : Hv(G_1) \to Hw(G_2)$. For $G_1 = G_2 = \mathbb{D}$ (the unit disc in $\mathbb{C}$), $v(z) := 1$, $w(z) := 1/|1-z|$ and $\varphi(z) := z$ we have $Hv_0(G_1) = \{0\}$ and then $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ is continuous. In this case $Hv(G_1)$ is the Hardy space $H^\infty(\mathbb{D})$ that is not contained in $Hw(G_2)$.

Proposition 7. (1) *If $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ is continuous and $\overline{Bv_0}^{co} = Bv$, then $C_\varphi : Hv(G_1) \to Hw(G_2)$ is continuous.*

(2) *Suppose that $C_\varphi : Hv(G_1) \to Hw(G_2)$ is continuous and $Hv_0(G_1) = H\widetilde{v}_0(G_1)$. If either*

(i) *$\varphi^{-1}(K)$ is relatively compact in G_2 for each compact subset K of G_1 or*

(ii) *w vanishes at ∞ on G_2,*

then $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ is continuous.

Proof. (1) Let $M > 0$ such that $C_\varphi(Bv_0) \subset MBw_0$. We fix $f \in Bv$. By hypothesis, there exists a sequence $(f_n)_n \subset Bv_0$ which converges to f for the compact open topology. Then $\|(C_\varphi(f_n))\|_w \le M$ for each n. Passing to the limit, we obtain $w(z)|f \circ \varphi(z)| \le M$ for each $z \in G_2$. This yields $C_\varphi(Bv) \subset MBw(G_2)$.

(2) If (i) holds, we apply the equivalence between (a) and (c) in Proposition 5 to obtain M such that $\widetilde{w} \le M(\widetilde{v} \circ \varphi)$ on G_2. We fix $\varepsilon > 0$. If $f \in H\widetilde{v}_0(G_1)$ there exists $K \subset G_1$ such that $\widetilde{v}(\varphi(z))|f(\varphi(z))| < \varepsilon/M$ for every $z \in G_2$ such that $\varphi(z) \in G_1 \setminus K$. This implies $\widetilde{w}(z)|f \circ \varphi(z)| < \varepsilon$ for each $z \in G_2 \setminus \varphi^{-1}(K)$. Hence $C_\varphi(Hv_0(G_1)) = C_\varphi(H\widetilde{v}_0(G_1)) \subset H\widetilde{w}_0(G_2) \subset Hw_0(G_2)$.

Now we assume (ii) and we apply the equivalence between (a) and (d) in Proposition 5 to choose $M \ge 1$ such that $w \le M(\widetilde{v} \circ \varphi)$ on G_2. We fix $f \in Hv_0(G_1) = H\widetilde{v}_0(G_1)$. For each $\varepsilon > 0$ there exists a compact subset $K_1 \subset G_1$ such that $\widetilde{v}(z)|f(z)| < \varepsilon/M$ for each $z \in G_1 \setminus K_1$. By our assumption on w, we can choose a compact subset $K_2 \subset G_2$ such that $w(z) < \varepsilon / \max_{\lambda \in K_1}(1+|f(\lambda)|)$ for each $z \in G_2 \setminus K_2$. An easy computation shows that $w(z)|f \circ \varphi(z)| < \varepsilon$ for every $z \in G_2 \setminus K_2$. □

Theorem 8. *Consider the following assertions:*

(i) $C_\varphi : Hv(G_1) \to Hw_0(G_2)$ *is compact.*

(ii) $C_\varphi : Hv(G_1) \to Hw(G_2)$ *is compact and* $C_\varphi(Hv_0(G_1)) \subset Hw_0(G_2)$.

(iii) $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ *is compact.*

(iv) *For each* $\varepsilon > 0$ *there exists a compact subset* $K_2 \subset G_2$ *such that* $\frac{w(z)}{\widetilde{v}(\varphi(z))} < \varepsilon$ *for every* $z \in G_2 \setminus K_2$.

Then (i) $\implies$ (ii), (ii) $\implies$ (iii) *and* (iv) $\implies$ (i). *If we assume* $\overline{Bv_0}^{co} = Bv$, *then* (iii) $\implies$ (iv) *and all the conditions are equivalent.*

Proof. (i) $\implies$ (ii) and (ii) $\implies$ (iii) are trivial.

(iv) $\implies$ (i): Let $f \in Hv(G_1) = H\widetilde{v}(G_1)$ satisfy $\|f\|_v = \|f\|_{\widetilde{v}} \leq 1$ and let $\varepsilon > 0$. Select $K_2 \subset G_2$ as in (iv). For each $z \in G_2 \setminus K_2$

$$\begin{aligned}|w(z)C_\varphi f(z)| &= w(z)|f(\varphi(z))| \\ &\leq \sup_{z \in G_2 \setminus K_2} \frac{w(z)}{\widetilde{v}(\varphi(z))} \widetilde{v}(\varphi(z))|f(\varphi(z))| < \varepsilon.\end{aligned}$$

This implies $C_\varphi(Hv(G_1)) \hookrightarrow Hw_0(G_2)$. To see that C_φ is compact we use the following claim whose proof we omit because it is analogous to the one of [17, 3.11].

Claim. *C_φ is compact if and only if for each sequence $(f_n)_n$ which is bounded in $Hv(G_1)$ and convergent to 0 in $(H(G_1), \tau_0)$ the sequence $C_\varphi(f_n)$ converges to 0 in $Hw_0(G_2)$.*

Let $\varepsilon > 0$ and let $(f_n)_n$ be a sequence in Bv which tends to 0 in $H(G_1)$ endowed with the compact open topology. We apply (iv) to get a compact subset $K_2 \subset G_2$ such that $w(z) \leq \frac{\varepsilon}{2}\widetilde{v}(\varphi(z))$ whenever $z \in G_2 \setminus K_2$. Hence

$$w(z)|f_n(\varphi(z))| \leq \frac{w(z)}{\widetilde{v}(\varphi(z))}\widetilde{v}(\varphi(z))|f_n(\varphi(z))| < \frac{\varepsilon}{2} \tag{1}$$

for each $z \in G_2 \setminus K_2$ and for each $n \in \mathbb{N}$. The compactness of $\varphi(K_2)$ permits us to choose $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$|f_n(\rho)| \leq \frac{\varepsilon}{2 \max_{z \in K_2} w(z)} \tag{2}$$

for each $\rho \in \varphi(K_2)$. From (1) and (2) it follows that, for each $n \geq n_0$,

$$\begin{aligned} \|C_\varphi f_n\|_w &= \sup_{z \in G_2} w(z)|f_n(\varphi(z))| \\ &\leq \sup_{z \in G_2 \setminus K_2} w(z)|f_n(\varphi(z))| + \sup_{z \in K_2} w(z)|f_n(\varphi(z))| < \varepsilon. \end{aligned}$$

We assume $\overline{Bv_0}^{co} = Bv$ to show (iii) $\Longrightarrow$ (iv). We suppose that $C_\varphi : Hv_0(G_1) \to Hw_0(G_2)$ is compact. For each $g \in Hw_0(G_2)$ and each $z \in G_2$ we have $|\langle w(z)\delta_z, g\rangle| = w(z)|g(z)| \leq \|g\|_w$ (we denote by δ_z the evaluation at z). Hence, by the Banach–Steinhaus theorem, the set $A := \{w(z)\delta_z : z \in G_2\}$ is bounded in $Hw_0(G_1)'$. We apply that the transpose map $C_\varphi^t : Hw_0(G_2)'_b \to Hv_0(G_1)'_b$ is compact to obtain that $C_\varphi^t(A)$ is relatively compact. This set coincides with $\{w(z)\delta_{\varphi(z)} : z \in G_2\}$. We denote by B its absolutely convex closed hull. Since B is compact in $Hv_0(G_1)$, the norm topology on it coincides with the one induced by $\sigma(Hv_0(G_1)', Hv_0(G_1))$. We fix $\varepsilon > 0$. There exist $f_1, ..., f_s \in Hv_0(G_1)$ such that

$$B \cap \{f_1, \ldots, f_s\}^\circ \subset \{\psi \in B : \|\psi\|_{Hv_0(G_1)'} < \varepsilon\}.$$

By hypothesis, $f_j \circ \varphi \in Hw_0(G_2)$, $j = 1, \ldots, s$. Let K_2 be a compact subset of G_2 such that

$$w(z)|f_j \circ \varphi(z)| < 1, \quad j = 1, \ldots, s, \; z \in G_2 \setminus K_2.$$

Thus, for each $z \in G_2 \setminus K_2$ we have $w(z)\delta_{\varphi(z)} \in B \cap \{f_1, \ldots, f_s\}^\circ$. This yields

$$\|w(z)\delta_{\varphi(z)}\|_{Hv_0(G_1)'} < \varepsilon, \quad z \in G_2 \setminus K_2. \tag{3}$$

Since $Bv = \overline{Bv_0}^{co}$, we conclude $\|\delta_{\varphi(z)}\|_{Hv_0(G_1)'} = \|\delta_{\varphi(z)}\|_{Hv(G_1)'} = \frac{1}{\tilde{v}(\varphi(z))}$, and therefore (3) is equivalent to condition (iv). □

Remark 9. For each $n \in \mathbb{N}$, let G and v_n be as in the Example 2, and let w_n be defined on $\mathbb{D}$ as an extension of v_n such that $w_n(0) = 1$. It is immediate that $Hv_n(G) \simeq Hw_n(\mathbb{D})$ isometrically as Banach spaces. Since

$H(w_n)_0(\mathbb{D})$ contains the polynomials, $\overline{B(w_n)_0}^{co} = Bw_n$ by [4, 1.5 (c)]. Therefore Theorem 8 implies that the injections $Hw_n(\mathbb{D}) \hookrightarrow Hw_m(\mathbb{D})$ are compact whenever $n < m$, and then the injections $Hv_n(G) \hookrightarrow Hv_m(G)$ are also compact. Since $v_n \geq v_m$ we also have $H(v_n)_0(G) \hookrightarrow H(v_m)_0(G)$. However $v_m/\widetilde{v_n}$ is identically 1 on a punctured neighbourhood of 0. This shows that, in general, (ii) does not imply (iv) in Theorem 8.

3. Composition operators on weighted inductive limits

Here $\mathcal{V} = (v_n)_n$ will always denote a decreasing sequence of weights on an open subset G of $\mathbb{C}$. The *weighted (LB)-space of holomorphic functions with O-growth conditions* associated with $\mathcal{V}$ is the locally convex inductive limit

$$\mathcal{V}H(G) := \mathrm{ind}_n \, Hv_n(G);$$

An alternative description of this space can be given by considering the sequence $\widetilde{\mathcal{V}} = (\widetilde{v_n})_n$, i.e. $\mathcal{V}H(G) \simeq \widetilde{\mathcal{V}}H(G)$ topologically.

The weighted (LB)-space of holomorphic functions with o-growth conditions is defined analogously by

$$\mathcal{V}_0H(G) := \mathrm{ind}_n \, H(v_n)_0(G).$$

We remark that in general $\mathcal{V}_0H(G) \supsetneq \widetilde{\mathcal{V}}_0H(G)$ algebraically.

Our first result is connected with the problem of the *projective descriptions for this space* (cf. [3]). Given a decreasing sequence $\mathcal{V} = (v_n)_n$ of weights, Bierstedt, Meise and Summers introduced the system of weights $\overline{V}$ associated with V by

$$\overline{V} := \{\overline{v} : G \to [0,\infty) : \overline{v} \text{ upper semicontinuous and } \overline{v}/v_n \text{ bounded in } G \text{ for each } n \in \mathbb{N}\}.$$

The projective hull $H\overline{V}(G)$ of the inductive limit is defined by

$$H\overline{V}(G) := \left\{ f \in H(G) : \|f\|_{\overline{v}} := \sup_{z \in G} \overline{v}|f(z)| < \infty \, \forall \overline{v} \in \overline{V} \right\}$$

endowed with the locally convex topology defined by the system of seminorms $\{\|\cdot\|_{\overline{v}}, \overline{v} \in \overline{V}\}$. In [8] it is proved that $\mathcal{V}H(G) = H\overline{V}(G)$ holds algebraically, the injection $\mathcal{V}H(G) \hookrightarrow H\overline{V}(G)$ is continuous and both spaces have the same bounded sets. A decreasing sequence $\mathcal{V} = (v_n)_n$ of weights on G is said to satisfy property (S) if for each $n \in \mathbb{N}$ there exists $m > n$ such that v_n/v_m vanishes at infinity on G. If $\mathcal{V}$ satisfies (S), then $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ is a (DFS) space. $\mathcal{V}H(G)$ being a (DFS) space, even Montel, is a sufficient condition to have the topological equality $\mathcal{V}H(G) = H\overline{V}(G)$, and this happens whenever $\mathcal{V} = (v_n)_n$ or $\widetilde{V} = (\widetilde{v_n})_n$ satisfies (S) (cf. [8, 5]). The next result constitutes an extension of [5, 3.5], and it is a partial answer to Problem 1 in [3]. It should be compared with [14, Theorem 3 (b)].

Theorem 10. *If $V = (v_n)_n$ is a decreasing sequence of weights on G such that $Bv_n = \overline{B(\widetilde{v_n})_0}^{co}$ for each $n \in \mathbb{N}$, then $\mathcal{V}H(G)$ is a (DFS) space if and only if the sequence $\widetilde{V} = (\widetilde{v_n})_n$ satisfies (S).*

Proof. $\mathcal{V}H(G) = \widetilde{\mathcal{V}}H(G)$ is a (DFS) space if and only if for each $n \in \mathbb{N}$ there exists $m > n$ such that $i : H\widetilde{v_n} \to H\widetilde{v_m}$ is compact. Since $H(\widetilde{v_n})_0 \subseteq H(\widetilde{v_m})_0$, the result is an immediate consequence of the equivalence between the conditions (ii) and (iv) in Theorem 8 and [5, 1.2 (v)]. $\square$

Remark 11. If we take $\mathcal{V} = (v_n)_n$ and G as in Example 4, a similar argument to the one used in Remark 9 (a) shows that the injections $i : Hv_n(G) \to Hv_m(G)$ are compact whenever $n < m$. Thus, the space $\mathcal{V}H(G)$ is a (DFS) space such that $Bv_n = \overline{B(v_n)_0}^{co}$ for each $n \in \mathbb{N}$ and $\widetilde{\mathcal{V}}$ does not satisfy (S) because $\widetilde{v_m}(z)/\widetilde{v_n}(z) = 1$ for each $z \in \mathbb{D} \setminus \{0\}$ and for each $m, n \in \mathbb{N}$.

Let G_1 and G_2 be two complex domains, let $\mathcal{V} = (v_n)_n$ and $\mathcal{W} = (w_n)_n$ two sequences of weights on G_1 and G_2 respectively and let $\varphi : G_2 \to G_1$ a holomorphic mapping.

Proposition 12. (1) *The following conditions are equivalent:*

(a) $C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}H(G_2)$ *is continuous,*

(b) $C_\varphi(\mathcal{V}H(G_1)) \subset \mathcal{W}H(G_2)$*,*

(c) *for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $C_\varphi : Hv_n(G_1) \to Hw_m(G_2)$ is continuous,*

(d) *for each* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that* $\sup_{z \in G_2} \frac{w_m(z)}{\tilde{v_n}(z)} < \infty$.

(2) *The following conditions are equivalent:*

(a) $C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}H(G_2)$ *is bounded,*

(b) *there exists* $m \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ $C_\varphi : Hv_n(G_1) \to Hw_m(G_2)$ *is continuous,*

(c) *there exists* $m \in \mathbb{N}$ *such that* $\sup_{z \in G_2} \frac{w_m(z)}{\tilde{v_n}(z)} < \infty$ *for each* $n \in \mathbb{N}$.

Proof. In (1) the equivalence between the conditions (a) and (b) is a consequence of the Closed Graph Theorem. Given a mapping T between two (LB) spaces $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$, a straightforward application of the Grothendieck factorization theorem [20, Theorem 24.33] shows that T is continuous if and only if for each n there exists m such that $T : E_n \to F_m$ is continuous. The conclusion follows from Proposition 5.

To prove (2) we observe that, in the general case, a mapping between (LB)-spaces $T : E := \operatorname{ind}_n E_n \to F := \operatorname{ind}_n F_n$, F being regular, is bounded if and only if there exists m such that $T : E_n \to F_m$ is continuous for each $n \in \mathbb{N}$. Indeed, if $T : E \to F$ is bounded then the regularity of F implies that there exists m such that $T : E \to F_m$ is bounded, and then the conclusion follows from the continuity of the inclusion $E_n \hookrightarrow E$ for each n. Conversely, if such m can be found, for each $n \in \mathbb{N}$ there exists α_n such that $\alpha_n T(B_n) \subset B$, B_n being the unit ball of E_n and B being the unit ball of F_m. Therefore, if we denote by U the absolutely convex hull of $\cup_n \alpha_n B_n$, we have that U is a 0-neighbourhood in E such that $T(U) \subset B$, and T is bounded. Now, the equivalences are immediately obtained from Proposition 5. □

A sequence $\mathcal{V} = (v_n)_n$ of weights on G is said to be *regularly decreasing* if for each n there exists $m > n$ such that for each subset Y of G

$$\inf_Y \frac{v_m}{v_n} > 0 \Longrightarrow \inf_Y \frac{v_k}{v_n} > 0 \quad \text{for all } k \geq m.$$

To obtain results about composition operators which are compact or Montel we require the range space $\mathcal{W}H(G_2)$ to be boundedly retractive. It is well known that if a sequence $\mathcal{V}$ of weights on G is regularly decreasing, then both $\mathcal{V}H(G)$ and $\mathcal{V}_0H(G)$ are boundedly retractive and therefore $\mathcal{V}_0H(G)$ is complete [8].

Lemma 13. *Let $T : E := \mathrm{ind}_n E_n \to F := \mathrm{ind}_n F_n$ be a linear mapping between two Hausdorff (LB) spaces such that F is boundedly retractive. Then T is compact if and only if there exists $m \in \mathbb{N}$ such that $T : E_n \to F_m$ is compact for each $n \in \mathbb{N}$, and T is Montel if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T : E_n \to F_m$ is compact.*

Proof. We show the characterization of the compact operators. First suppose that m as in the statement exists. Let B_n and B the unit balls of E_n and F_m respectively. Even without the assumption that F is boundedly retractive, we have that $(T(B_n))_n$ is a sequence of relatively compact subsets of F_m and then we can get a sequence $(\alpha_n)_n$ of positive numbers such that $\alpha_n T(B_n) \subset (1/n)B$. The set $K := \bigcup_n \alpha_n T(B_n)$ is easily seen to be relatively (sequentially) compact in F_m. Hence $U := \Gamma(\bigcup_n \alpha_n B_n)$ is a 0-neighbourhood in E such that $T(U)$ is relatively compact in F. Conversely, if $T : E \to F$ is compact and F is boundedly retractive, then there is $m \in \mathbb{N}$ and a 0-neighbourhood U in E such that $T(U)$ is a compact subset of F_m. Hence $T : E \to F_m$ is compact. This implies that $T : E_n \to F_m$ is compact for each $n \in \mathbb{N}$. A similar argument works for Montel operators. □

Theorem 14. *Assume that $\mathcal{W}H(G_2)$ is boundedly retractive.*

(1) *Consider the following conditions:*

(i) *$C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}_0H(G_2)$ is compact.*

(ii) *$C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}H(G_2)$ is compact and $C_\varphi : \mathcal{V}_0H(G_1)) \to \mathcal{W}_0H(G_2)$ is bounded.*

(iii) *$C_\varphi : \mathcal{V}_0H(G_1) \to \mathcal{W}_0H(G_2)$ is compact.*

(iv) *There exists $m \in \mathbb{N}$ such that $C_\varphi : H(v_n)_0(G_1) \to H(w_m)_0(G_2)$ is compact for each $n \in \mathbb{N}$.*

(v) *There exists $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ and for each $\varepsilon > 0$ there exists a compact subset K_2 of Ω_2 such that*

$$\sup_{z \in G_2 \setminus K_2} \frac{w_m(z)}{\widetilde{v_n} \circ \varphi(z)} < \varepsilon.$$

Then (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), (iii) $\Longrightarrow$ (iv) *and* (v) $\Longrightarrow$ (i). *If in addition we assume that $\overline{B(v_n)_0}^{co} = Bv_n$ for each $n \in \mathbb{N}$, then* (iv) $\Longrightarrow$ (v) *and all the conditions are equivalent.*

(2) *Consider the following conditions:*

(i) $C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}_0H(G_2)$ *is Montel.*

(ii) $C_\varphi : \mathcal{V}H(G_1) \to \mathcal{W}H(G_2)$ *is Montel and* $C_\varphi(\mathcal{V}_0H(G_1)) \hookrightarrow \mathcal{W}_0H(G_2)$*.*

(iii) $C_\varphi : \mathcal{V}_0H(G_1) \to \mathcal{W}_0H(G_2)$ *is Montel.*

(iv) *For each* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that* $C_\varphi : H(v_n)_0(G_1) \to H(w_m)_0(G_2)$ *is compact.*

(v) *For each* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that for each* $\varepsilon > 0$ *there exists a compact subset* K_2 *of* Ω_2 *such that*

$$\sup_{z \in G_2 \setminus K_2} \frac{w_m(z)}{\widetilde{v_n} \circ \varphi(z)} < \varepsilon.$$

Then (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), (iii) $\Longrightarrow$ (iv) *and* (v) $\Longrightarrow$ (i)*. If in addition we assume that* $\overline{B(v_n)_0}^{co} = Bv_n$ *for each* $n \in \mathbb{N}$*, then (iv)* $\Longrightarrow$ *(v) and all the conditions are equivalent.*

Proof. We only prove (1), since (2) can be obtained similarly.

(i) $\Longrightarrow$ (ii) is a consequence of the continuous injections $\mathcal{V}_0H(G_2) \hookrightarrow \mathcal{V}H(G_2)$ and $\mathcal{W}_0H(G_2) \hookrightarrow \mathcal{W}H(G_2)$, and (iii) $\Longrightarrow$ (iv) follows from Proposition 12 since $\mathcal{W}_0H(G)$ is boundedly retractive. Now, (v) $\Longrightarrow$ (i) follows from Theorem 8 and Proposition 12.

(ii) $\Longrightarrow$ (iii): By Lemma 13, there exists $m \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $C_\varphi : Hv_n(G_1) \to Hw_m(G_2)$ is compact. Since $C_\varphi : \mathcal{V}_0H(G_1)) \to \mathcal{W}_0H(G_2)$ is bounded and $\mathcal{W}_0H(G)$ is boundedly retractive, we can get $p > m$ such that $C_\varphi : H(v_n)_0(G_1) \to H(w_p)_0(G_2)$ is continuous for each $n \in \mathbb{N}$. Therefore $C_\varphi : H(v_n)_0(G_1) \to H(w_p)_0(G_2)$ is compact for each $n \in \mathbb{N}$, and the result follows from Lemma 13.

If we assume $\overline{B(v_n)_0}^{co} = Bv_n$ for each $n \in \mathbb{N}$, then (iv) $\Longrightarrow$ (v) follows from Theorem 8. □

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