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Area of reduced polygons

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Abstract. A convex body R of Euclidean space E^d is said to be *reduced* if every convex body $P \subset R$ different from R has thickness smaller than the thickness $\Delta(R)$ of R. We prove that the area of every reduced polygon R is smaller than $\frac{1}{4}\pi \cdot \Delta^2(R)$ and that the factor $\frac{1}{4}\pi$ cannot be lessened. We conjecture that the area of every planar reduced body is at most $\frac{1}{4}\pi \cdot \Delta^2(R)$.

The minimum width of a convex body C of Euclidean d-space E^d is called the *thickness* of C and it is denoted by $\Delta(C)$. We call a convex body $R \subset E^d$ *reduced* if for every convex body $P \subset R$ different from R we have $\Delta(P) < \Delta(R)$ (see [4]). In particular, every convex body of constant width in E^d is a reduced body. We obtain another example of a reduced body dissecting the ball into 2^d congruent subsets by d pairwise perpendicular hyperplanes passing through the center of the ball. If $d = 2$, we call such a set *a quarter of disk*. We know that each reduced body in E^d with smooth boundary is of constant width (see [3]). Let us add that for every $d \geq 3$ there are d-dimensional reduced bodies of a given thickness and with arbitrarily large finite diameter (see [8]).

Every planar strictly convex reduced body is a body of constant width (see [2]). Many other properties of planar reduced bodies are derived in [7]. In particular, there are given upper estimates of the diameter and the perimeter of a reduced body of a given thickness. They are attained only

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for the quarter of disk. The quarter of disk also requires larger disk for covering it than any other reduced body of the same thickness (see [8]). Recall that every reduced polygon has and odd number of vertices (see [7]). The simplest examples of reduced polygons are the regular odd-gons. The only reduced triangles are the regular triangles but for larger number of vertices we have a wide class of reduced odd-gons.

The below presented Theorem gives an estimate of the area of reduced polygons. It is not clear if analogical question about the volume of reduced polytopes makes sense because we do not know if d-dimensional reduced polytopes exist for $d \geq 3$; we only know that simplices are not reduced (see [9]). At the end of this note we formulate a conjecture about the maximum possible area of a planar reduced body. We also present an estimate of the area of reduced bodies which is close to the value presented in this conjecture.

Theorem. *We have*

$$\mathrm{area}(R) < \frac{1}{4}\pi \cdot \Delta^2(R)$$

for every reduced polygon R, and the factor $\frac{1}{4}\pi$ cannot be lessened in general.

This theorem results immediately from the following lemma.

Lemma. *Every reduced non-regular n-gon R has the area smaller than the regular n-gon of width $\Delta(R)$.*

Proof. For convenience of the reader, the below introduced notation is consistent with the notation from [7]. Of course, $n \geq 3$ and n is odd. Consider a reduced n-gon $R = v_1 v_2 \dots v_n$ (Figure 1 gives an illustration for $n = 5$, see also Figure 2). Moreover, if $k \notin \{1, \dots, n\}$, then by v_k we mean the vertex v_m, where $m = k \pmod{n}$ and $m \in \{1, \dots, n\}$. From Theorem 7 of [7] it follows that the projection t_i of v_i on the straight line containing the side $v_{i+(n-1)/2}\, v_{i+(n+1)/2}$ is strictly between the vertices $v_{i+(n-1)/2}$ and $v_{i+(n+1)/2}$. Let s_i be the point of the intersection of segments $v_i t_i$ and $v_{i+(n+1)/2} t_{i+(n+1)/2}$, where $i = 1, \dots, n$. For every $i \in \{1, \dots, n\}$ denote by B_i the "butterfly" being the union of the triangles $v_i s_i t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} s_i t_i$.

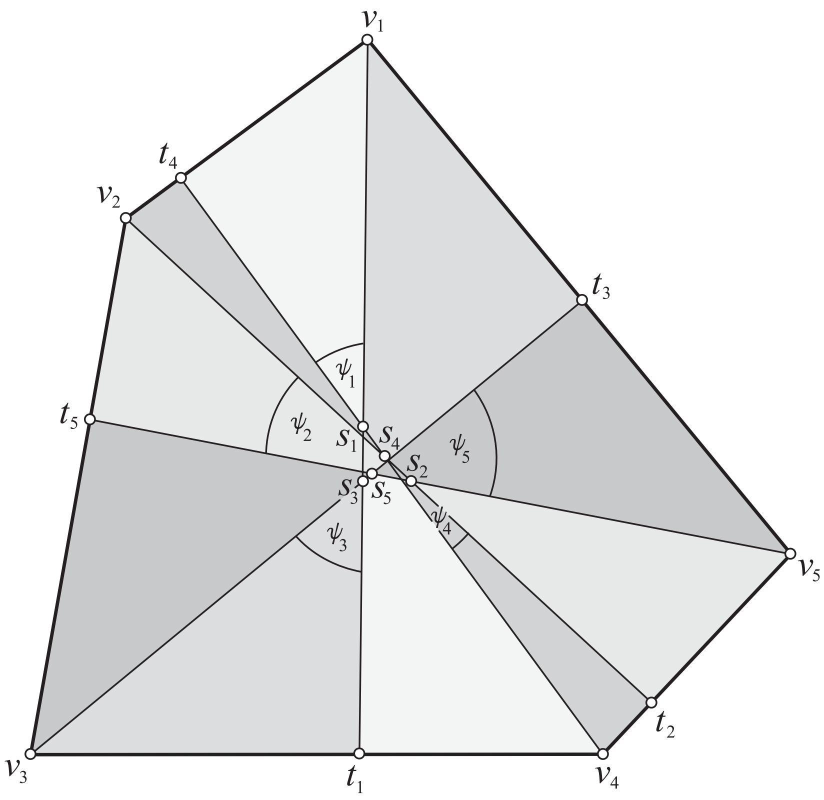

Figure 1

Of course, $B_1 \cup \cdots \cup B_n \subset R$. In order to show the opposite inclusion, we will show that every point $p \in R$ belongs to $B_1 \cup \cdots \cup B_n$. We present every B_i as the union of chords which pass through s_i. All the chords of successively $B_1, B_{1+(n+1)/2}, B_2, B_{2+(n+1)/2}, \ldots, B_{(n+1)/2}$ are in straight lines which step by step rotate changing the centers of rotation; those centers successively are $s_1, s_{1+(n+1)/2}, s_2, s_{2+(n+1)/2}, \ldots, s_{(n+1)/2}$. We assume that all the chords are oriented with origins in segments $v_1v_2, v_2v_3, \ldots,$ $v_{(n-1)/2}v_{(n+1)/2}, v_{(n+1)/2}t_1$. When we start from the chord v_1t_1, after total rotation by π we arrive to the chord t_1v_1 which has opposite direction. After the rotation by π, point p is in the opposite closed side of the oriented chord (i.e. it changes the position between left and right sides). Since the described change of the chord is continuous, there is a position of the chord which contains p. Hence $p \in B_1 \cup \dots \cup B_n$. Consequently, $R = B_1 \cup \dots \cup B_n$.

Denote by ψ_i the angle $\angle v_i s_i t_{i+(n+1)/2}$. From Theorem 3 of [7] we conclude that $|v_i t_{i+(n+1)/2}| = \Delta(R) \cdot \tan \frac{\psi_i}{2}$. Since the triangle $v_i s_i t_{i+(n+1)/2}$ has right angle at $t_{i+(n+1)/2}$, we obtain $|s_i t_{i+(n+1)/2}| = \Delta(R) \cdot \tan \frac{\psi_i}{2} \cdot \cot \psi_i = \Delta(R) \cdot \tan \frac{\psi_i}{2} \cdot \frac{1-\tan^2(\psi_i/2)}{2\tan(\psi_i/2)} = \frac{1}{2}\Delta(R) \cdot (1 - \tan^2 \frac{\psi_i}{2})$. Hence the area of each of the rectangular triangles $v_i s_i t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} s_i t_i$ is $\frac{1}{4}\Delta^2(R) \cdot (1 - \tan^2 \frac{\psi_i}{2}) \tan \frac{\psi_i}{2}$. So the area of B_i equals to $\frac{1}{2}\Delta^2(R) \cdot (1 - \tan^2 \frac{\psi_i}{2}) \tan \frac{\psi_i}{2}$. In other words, the area is $\frac{1}{2}\Delta^2(R) \cdot f(\psi_i)$, where $f(\psi) = (1 - \tan^2 \frac{\psi}{2}) \tan \frac{\psi}{2}$.

Let us show that the above function $f(\psi)$ is concave down in our domain $(0, \frac{\pi}{2})$. We find the first derivative $f'(\psi) = \frac{1}{2}(1 - 3\tan^2 \frac{\psi}{2}) \cos^{-2} \frac{\psi}{2}$ and the second derivative $f''(\psi) = \frac{1}{4}[-6\tan \frac{\psi}{2} + 2(1 - 3\tan^2 \frac{\psi}{2}) \cos \frac{\psi}{2} \sin \frac{\psi}{2}] \cos^{-4} \frac{\psi}{2} = -\frac{1}{2} \sin \frac{\psi}{2}(3 - \cos^2 \frac{\psi}{2} + 3\sin^2 \frac{\psi}{2}) \cos^{-5} \frac{\psi}{2} = -\sin \frac{\psi}{2}(1 + 2\sin^2 \frac{\psi}{2}) \cos^{-5} \frac{\psi}{2}$. Clearly, the last expression is negative for $0 < \psi < \frac{\pi}{2}$, which implies that the function $f(\psi)$ is concave down in $(0, \frac{\pi}{2})$.

Of course, $\psi_1 + \dots + \psi_n = \pi$. Since $R = B_1 \cup \dots \cup B_n$, we see that the area of R is at most $\frac{1}{2}\Delta^2(R) f(\psi_1) + \dots + \frac{1}{2}\Delta^2(R) f(\psi_n) = \frac{n}{2} \cdot \Delta^2(R) \cdot [\frac{1}{n} f(\psi_1) + \dots + \frac{1}{n} f(\psi_n)]$. From the earlier established concavity down of the function $f(\psi)$ and from the classic Jensen's inequality [5] we conclude that the area of R is at most $\frac{n}{2} \cdot \Delta^2(R) \cdot f(\frac{1}{n}\psi_1 + \dots + \frac{1}{n}\psi_n) = \frac{n}{2} \cdot \Delta^2(R) \cdot f(\frac{\pi}{n})$, which is equal to the area of the regular n-gon of width $\Delta(R)$.

Since $f(\psi)$ is not linear, the equality in the Jensen's inequality is possible only for $\psi_1 = \dots = \psi_n$. This is why every non-regular reduced n-gon has the area smaller than the regular n-gon. □

It is obvious that as n tends to infinity, the area of the regular n-gon of thickness 1 (which, by Lemma, is the reduced n-gon of thickness 1 and the maximum possible area) tends to $\frac{1}{4}\pi$.

On the other hand, for every odd integer $n \geq 3$ and every $\varepsilon > 0$ there exists a non-degenerated reduced n-gon of thickness 1 whose area is less than $\frac{1}{3}\sqrt{3} + \varepsilon$ (the number $\frac{1}{3}\sqrt{3}$ here equals to the area of the regular triangle of thickness 1). In such an n-gon for $n \in \{5, 7, \dots\}$, the angles ψ_1, ψ_2, $\psi_{(n+3)/2}$ (as defined in the proof of Lemma) are almost 60° and the rest from the angles $\psi_1, \dots, \psi_n$ are close to 0° (the case of a reduced heptagon is illustrated in Figure 2). So the shape of such a reduced n-gon is "close" to the shape of the regular triangle and the limit case is the n-gon degenerated to the regular triangle of thickness 1. Let us add that the area of every planar reduced body R is at least $\frac{1}{3}\sqrt{3} \cdot \Delta^2(R)$. This follows by the well known PÁL's theorem ([10]) that every body of minimum width w has area at least $\frac{1}{3}\sqrt{3} \cdot w^2$.

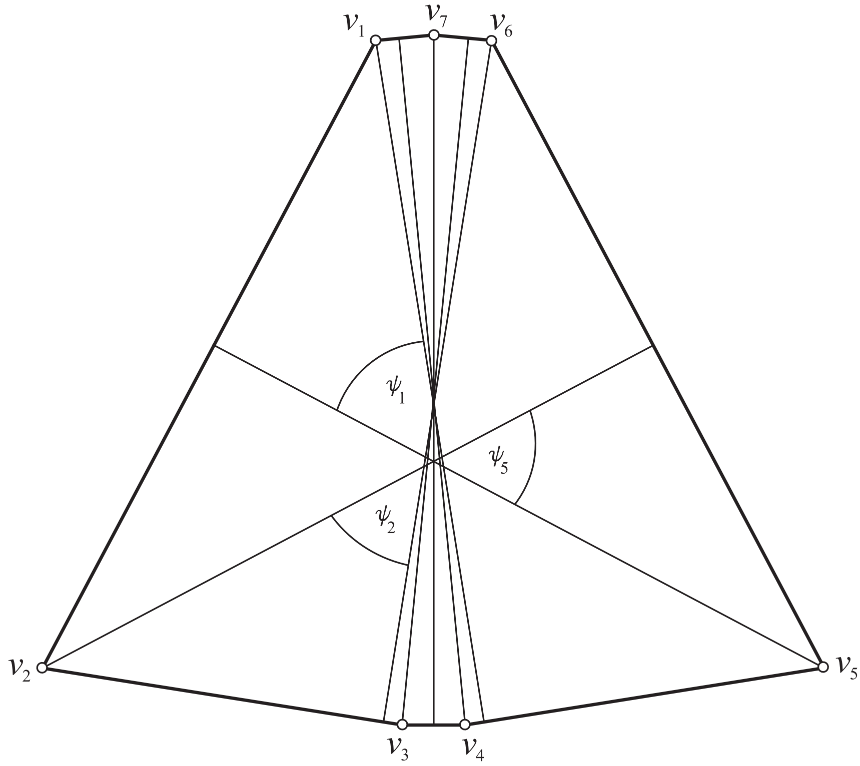

Figure 2

Proposition. *Every planar reduced body R has area at most $\Delta^2(R)$.*

PROOF. Recall the isoperimetric inequality

$$\text{area}\,(C) \leq \frac{1}{2} \cdot \text{perim}\,(C)\Delta(C) - \frac{1}{4}\pi \cdot \Delta^2(C)$$

for every convex body C with $\text{area}\,(C) \geq \frac{1}{4}\pi \cdot \Delta^2(C)$ given in [6] and [1].

Take also into account the inequality

$$\text{perim}\,(R) \leq (2 + \frac{1}{2}\pi) \cdot \Delta(R)$$

proved in [7]. If $\text{area}\,(R) \geq \frac{1}{4}\pi \cdot \Delta^2(R)$, by those two inequalities we obtain that $\text{area}\,(R) \leq \frac{1}{2} \cdot \text{perim}\,(R) \cdot \Delta(R) - \frac{1}{4}\pi \cdot \Delta^2(R) \leq \frac{1}{2}(2 + \frac{1}{2}\pi) \cdot \Delta(R) \cdot \Delta(R) - \frac{1}{4}\pi \cdot \Delta^2(R) = \Delta^2(R)$. In the opposite case we have $\text{area}(R) < \frac{1}{4}\pi \cdot \Delta^2(R) \leq \Delta^2(R)$. □

Conjecture. *Every planar reduced body R has area at most $\frac{1}{4}\pi \cdot \Delta^2(R)$ with equality only for each disk and each quarter of disk.*

References

[1] D. C. Benson, Sharpened forms of the plane isoperimetric inequality, *Amer. Math. Monthly* **77** (1970), 29–34.

[2] B. V. Dekster, Reduced strictly convex plane figure is of constant width, *J. Geom.* **26** (1986), 77–81.

[3] H. Groemer, Extremal convex sets, *Monatsh. Math.* **96** (1986), 29–39.

[4] E. Heil, Kleinste konvexe Körper gegebener dicke, *Fachbereich Mathematik, Technische Hochschule Darmstadt*, 1978, *Preprint # 453.*

[5] J. L. W. V. Jensen, Sur les fonctions convexes et les inégalites entre les valeurs moyennes, *Acta Math.* **30** (1906), 175–193.

[6] T. Kubota, Einige Unglaichheitsbeziehungen über Eilinien und Eiflächen, *The Science Reports of the Tohoku Imperial University, Ser.* **1**, 12 (1923), 45–65.

[7] M. Lassak, Reduced convex bodies in the plane, *Israel J. Math.* **70** (1990), 247–256.

[8] M. Lassak, On the smallest disk containing a planar reduced body, *Arch. Math.* **80** (2003), 553–560.

[9] H. Martini and K. J. Swanepoel, Non-planar simplices are not reduced, *Publ. Math. Debrecen* **64** (2004), 101–106.

[10] J. Pál, Ein Minimalumproblem für Ovale, *Math. Ann.* **83** (1921), 311–319.

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