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Some groups with n-central normal closures

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Abstract. A group is said to be an n-Kappe group if it satisfies the law $[x^n, y, y] = 1$. We describe the structural similarities between n-central groups and n-Kappe groups. In particular, we characterize 2-Kappe, 3-Kappe and metabelian p-Kappe groups. We show that in each of these three cases, these groups are closely related to groups with n-central normal closures.

1. Introduction

Given an integer n, a group G is said to be *n-central* if the factor group $G/Z(G)$ is a group of exponent n. The study of n-central groups was introduced in [3] and it is also the subject of [9] and [13]. Note that the variety of n-central groups is determined by the semigroup law $x^n y = yx^n$, which is equivalent to another semigroup law $(xy)^n = (yx)^n$. The consideration of semigroups satisfying such conditions is the topic of [14].

For a group G define *the set of right* 2*-Engel elements* by $R_2(G) = \{a \in G : [a, x, x] = 1 \text{ for every } x \in G\}$. A well-known result of W. Kappe [10] says that $R_2(G)$ is always a characteristic subgroup of G. Thus we define a group G to be an *n-Kappe group* if $G/R_2(G)$ is a group of exponent n. These groups arise naturally in connection with n-Bell groups (a group is said to be *n-Bell* if it satisfies the identity $[x^n, y] = [x, y^n]$). For instance,

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it is shown in [1] that every n-Bell group is $n(n-1)$-Kappe and that every finite n-Bell group is isomorphic to $A \times B \times C$, where A is an n-Kappe group, B is an $(n-1)$-Kappe group and C is a 2-Engel group.

The main purpose of this paper is the investigation of structural similarities between n-central and n-Kappe groups. We use the results of [3], [9] and [13] as guidelines for dealing with the question of n-Kappe groups for special values of n and for special classes of groups. In particular, we are mostly concerned with soluble n-Kappe groups. It is proved in [13] that for every integer n there is an integer $m > 1$ depending only on n such that every locally soluble n-central group is m-abelian (a group is said to be *m-abelian* when it satisfies the law $(xy)^m = x^m y^m$). Since m-abelian groups are closely related to m-Bell groups [1], the following result is not unexpected.

Theorem 1. *Let G be a locally soluble n-Kappe group. Then there exists an integer $m = f(n) > 1$ such that G is m-Bell and m-Levi group.*

Recall that a group is said to be *m-Levi* [7], if it satisfies the law $[x^m, y] = [x, y]^m$. Note also that the proof of Theorem 1 depends on the solution of the restricted Burnside problem [21], [22]. In the case of metabelian groups, we prove that every metabelian n-Kappe group is $2n^2$-Bell and $2n^2$-Levi; moreover, when n is odd, we can replace $2n^2$ by n^2. It is also shown by an example that this result is best possible at least in the case when n is an odd prime.

In [9], there is a characterization of metabelian p-central p-groups, which is extended to a characterization of metabelian p-central groups in [13]. The corresponding result for metabelian p-Kappe groups is the following.

Theorem 2. *Let p be an odd prime and let G be a metabelian group. The following conditions are equivalent.*

(a) *G is a p-Kappe group.*

(b) $[x, {}_{p+1}y] = [x, {}_p y, x] = [x, y, y]^p = 1$ *for any* $x, y \in G$.

(c) *G is nilpotent of class $\leq p + 1$ and $E_2(G)^p = 1$.*

Here we use the notation $E_n(G) = \langle [x, {}_n y] : x, y \in G \rangle$, where the commutator $[x, {}_n y]$ is defined inductively by $[x, {}_0 y] = x$ and $[x, {}_{n+1} y] =$

$\big[[x, {}_n y], y\big]$ for $n \geq 0$. The crucial step in proving this is a classification of polycyclic n-Kappe groups as those groups which can be embedded in a direct product of a finite soluble n-Kappe group and a finitely generated torsion-free group of class ≤ 2. This enables the reduction of the problem to the consideration of p-groups. Consequently, we prove that every metabelian p-Kappe group G has p-central normal closures. By this we mean that the normal closure a^G of element a in G is p-central for every $a \in G$. The converse is not true in general; the group W constructed in Example 1 of [9] is a metabelian group of exponent p^2 with p-central normal closures, yet it is not nilpotent.

Surprisingly, the case $p = 2$ is essentialy different from the case of p-central groups. We prove that every 2-Kappe group is metabelian, yet there are 2-Kappe groups which are not nilpotent. Nevertheless, we obtain the following characterization of 2-Kappe groups.

Theorem 3. *Let G be a group. The following statements are equivalent.*

(a) *G is a 2-Kappe group.*

(b) $[x, y, y, y] = [x, y, y, x] = [x, y, y]^2 = 1$ *for any $x, y \in G$.*

(c) *Every 2-generator subgroup of G is nilpotent of class ≤ 3 and $E_2(G)^2 = 1$.*

As a direct consequence we show that every 2-Kappe group has 2-central normal closures. On the other hand, there are two-generator groups of class 4 with 2-central normal closures, hence the converse does not hold in general. Beside that, we compute the nilpotency classes of free r-generator 2-Kappe groups. The result is as follows.

Theorem 4. *Let $r > 1$ and let G_r be the free r-generator 2-Kappe group. Then G_r is nilpotent of class $r + 1$.*

In [9] it is proved that every 3-central group is nilpotent of class ≤ 4; in fact, every 3-Bell group is also nilpotent of class ≤ 4 by [8]. Turning our attention to 3-Kappe groups, we obtain the following result.

Theorem 5. *Let G be a 3-Kappe group. Then we have:*

(a) *G is nilpotent of class ≤ 6.*

(b) *Every two-generator subgroup of G is nilpotent of class ≤ 4.*

The bounds for the nilpotency classes in Theorem 5 are best possible, as calculations using the Nilpotent Quotient Algorithm [17] show. With the help of this result, we are able to obtain a characterization of 3-Kappe groups which yields that every 3-Kappe group has 3-central normal closures. This raises a question whether every n-Kappe group has n-central normal closures. We show that this is not true in general, since there exists a metabelian 4-Kappe group $G = \langle a, b \rangle$, where a^G is not 4-central. Moreover, there is a 4-Kappe group with derived length 3 which contains a non-nilpotent normal closure.

In [7], the following sets were the objects of investigation:

$$\mathcal{E}(G) = \{n \in \mathbb{Z} : (xy)^n = x^n y^n \text{ for all } x, y \in G\},$$

$$\mathcal{B}(G) = \{n \in \mathbb{Z} : [x^n, y] = [x, y^n] \text{ for all } x, y \in G\},$$

$$\mathcal{L}(G) = \{n \in \mathbb{Z} : [x^n, y] = [x, y]^n \text{ for all } x, y \in G\}.$$

These sets are semigroups under multiplication and they always contain zero. The main result of [7] is an arithmetic characterization of the sets $\mathcal{E}(G)$, $\mathcal{B}(G)$ and $\mathcal{L}(G)$. It is shown there that each of these sets always forms what is called a Levi system, which is, roughly speaking, a union of idempotent residue classes modulo a certain integer m, which depends on G. Using this information, we determine $\mathcal{B}(G)$ and $\mathcal{L}(G)$ where G is the free 2-Kappe, free 3-Kappe, free 4-Kappe or free metabelian p-Kappe group, respectively. It is interesting to note that these sets coincide with $\mathcal{E}(G)$, where G is the free 2-central, free 3-central, free 4-central or free metabelian p-central group, respectively; see [3] and [13].

The notation is mainly taken from [19]. The standard commutator identities [18, Part 1, Section 2.1] will be used without further reference.

2. Proofs of results

At the beginning we state some well-known results which we use throughout the paper. The first lemma is about 2-Engel groups; it was proved by F. W. Levi [11]. Recall that a group is said to be *n-Engel* if it satisfies the law $[x, {}_n y] = 1$.

Lemma 1 ([11]).

(a) *If G is a 2-Engel group, then* $\gamma_3(G)^3 = \gamma_4(G) = 1$.

(b) *Every 2-generator 2-Engel group is nilpotent of class* ≤ 2.

(c) *Every group of exponent three is 2-Engel.*

The next result collects some facts about right 2-Engel elements of a given group.

Lemma 2. *Let G be any group, $x, y, z \in G$ and $a \in R_2(G)$.*

(a) *The group a^G is abelian.*

(b) $[a, [x, y]] = [a, x, y]^2$.

(c) $[a, x]^{rs} = [a^r, x^s]$ *for all integers r and s.*

(d) $[a, x, y, z]^2 = 1$, *hence* $a^2 \in Z_3(G)$.

Proof. The assertions (a) and (b) are proved in [10] and (c) follows directly from (a). The identity $[a, x, y, z]^2 = 1$ is proved in [18, Part 2, p. 43]. □

The following lemma facilitates computations in metabelian groups. We will use it without any further reference.

Lemma 3 ([9]). *Let G be a metabelian group, $x, y, z \in G$ and $c, d \in G'$. Then we have:*

(a) $[c, x, y] = [c, y, x]$.

(b) $[x, y, z] = [y, x, z]^{-1}$.

(c) $[cd, x] = [c, x][d, x]$.

(d) $[x, y^n] = \prod_{1 \leq i \leq n} [x, {}_i y]^{\binom{n}{i}}$.

(e) $(xy^{-1})^n = x^n \cdot \prod_{0 < i+j < n} [x, {}_i y, {}_j x]^{\binom{n}{i+j+1}} \cdot y^{-n}$.

A group G is said to be *an Engel group* when for every $x, y \in G$ there exists a nonnegative integer $n = n(x, y)$ such that $[x, {}_n y] = 1$. The next result is elementary:

Lemma 4. *Let G be a group. If the factor group $G/R_2(G)$ is locally nilpotent, then G is locally nilpotent.*

Proof. Since $G/R_2(G)$ is locally nilpotent, it is also an Engel group, hence G is an Engel group. Beside that, the group G is (2-Engel)-by-(locally nilpotent), hence it is locally soluble and therefore locally nilpotent by a result of Gruenberg; see [18, Part 2, p. 60]. □

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Clearly we may assume that G is a two-generator soluble n-Kappe group. For any $x \in G$ we have $x^n \in R_2(G)$, hence $x^{2n} \in Z_3(G)$ by Lemma 2. This implies that $G/Z_3(G)$ is a finitely generated soluble group of exponent $2n$, therefore $|G : Z_3(G)| < \infty$. By a theorem of Baer [19, 14.5.1], $\gamma_4(G)$ is a finite group of exponent bounded by a function of n. Since $G/E_2(G)$ is a 2-Engel group, we conclude that $\gamma_4(G) \leq E_2(G)$ by Lemma 1. For any $x, y \in G$ and for any integer l we have $[x^l, y, y] \equiv [x, y, y]^l \mod \gamma_4(G)$, hence $[x, y, y]^n \in \gamma_4(G)$. We conclude that the abelian factor group $E_2(G)/\gamma_4(G)$ is of exponent n, hence $E_2(G)$ is a group of finite exponent k. By the solution of the restricted Burnside problem [21], [22], k depends on n only.

Now let $x, y \in G$. Expansion of $[x^n, xy, xy] = 1$ implies $[x^n, y, x] = 1$, hence $[x^n, y] \in Z(\langle x, y\rangle)$. By Lemma 2 (c) we obtain $[x^n, y] \equiv [x, y]^n \mod E_2(G)$, which yields $[x, y]^n = [x^n, y]e$ for some $e \in E_2(G)$. This gives $[x, y]^{nk} = [x^n, y]^k e^k = [x^n, y]^k = [x^{nk}, y]$. By symmetry we have $[x, y]^{nk} = [x, y^{nk}]$, hence G is an (nk)-Bell group and also an (nk)-Levi group. □

Note that the proof of Theorem 1 gives a very crude explicit bound for m such that every soluble n-Kappe group is m-Bell and m-Levi. We can substantially improve this bound at least in the case of metabelian groups.

Lemma 5. *Every metabelian n-Kappe group G is also a $2n^2$-Levi group and a $2n^2$-Bell group. Furthermore, if n is odd, then G is also n^2-Levi and n^2-Bell.*

Proof. Let $x, y \in G$. As G is metabelian, we get $1 = [[x, y]^n, y, y] = [x, y, y, y]^n$, hence $\exp E_3(G)$ divides n. This yields

$$1 = [x, y^n, y]^n = \prod_{i=1}^{n} [x,{}_{i+1}y]^{n\binom{n}{i}} = [x, y, y]^{n^2},$$

hence $\exp E_2(G) \mid n^2$. As $[x, y^n] \in Z(\langle x, y\rangle)$, we obtain

$$[x, y^{2n^2}] = [x, y^n]^{2n} = \prod_{i=1}^{n} [x, {}_i y]^{2n\binom{n}{i}} = [x, y]^{2n^2} = [x^{2n^2}, y],$$

therefore G is $2n^2$-Bell and $2n^2$-Levi. When n is odd, it divides $\binom{n}{2}$, hence a similar manipulation as above gives $[x, y^{n^2}] = [x, y]^{n^2} = [x^{n^2}, y]$, which proves the second part. □

The following observation is of significant importance for our next results:

Proposition 1. *Let G be a finitely generated soluble n-Kappe group. Then G is an extension of a periodic soluble n-Kappe group by a finitely generated torsion-free group of class ≤ 2.*

Proof. As in the proof of Theorem 1, we conclude that $|\gamma_4(G)| < \infty$, hence the elements of finite order form a characteristic subgroup T of the group G. Since $\exp E_2(G) < \infty$, the factor group G/T is a torsion-free 2-Engel group. But we also have $\gamma_3(G/T)^3 = 1$ by Lemma 1, hence G/T is of class ≤ 2. This proves the result. □

It is easily seen that every torsion-free locally soluble n-central group is abelian [3]. The situation is similar for n-Kappe groups. More precisely, we have:

Corollary 1. *Every locally soluble torsion-free n-Kappe group is nilpotent of class ≤ 2.*

Corollary 2. *Let G be a polycyclic group. Then G is an n-Kappe group if and only if it is isomorphic to a subgroup of a direct product of a finite soluble n-Kappe group and a finitely generated torsion-free group of class ≤ 2.*

Proof. Let G be a polycyclic n-Kappe group and let T be its torsion subgroup. By Corollary 1, we may assume that $T \neq 1$. Since T is finitely generated, it is finite. The well-known result of Hirsch [19, 5.4.17] says that G is residually finite, so for every non-trivial element a of T there exists a normal subgroup $N_a \lhd G$ of finite index such that $a \notin N_a$. Let $N = \bigcap_{a \in T\setminus\{1\}} N_a$. Clearly, $|G : N| < \infty$ and $N \cap T = 1$. Hence G can be naturally embedded into $(G/N) \times (G/T)$; here G/N is a finite soluble n-Kappe group, whereas G/T is a finitely generated torsion-free group of class ≤ 2 by Proposition 1. The converse statement is obvious. □

This result is particularly useful in the situation when G is a finitely generated nilpotent p-Kappe group. In this case, G can be naturally embedded into a direct product of a finite p-Kappe p-group and a finitely generated 2-Engel group. As a consequence, we are able to obtain a characterization of metabelian p-Kappe groups given by Theorem 2.

Proof of Theorem 2. Assume G is a metabelian p-Kappe group, let $x, y \in G$ and put $H = \langle x, y\rangle$. The factor group $H/R_2(H)$ is a metabelian two-generator group of exponent p. By a result of Meier–Wunderli [12], $H/R_2(H)$ is nilpotent of class $\leq p-1$, hence the group H satisfies the identity of the form $[x_1, \dots, x_p, x_{p+1}, x_{p+1}] = 1$, where $x_i \in H$. In particular, G is $(p+1)$-Engel and also satisfies the identity $[x, {}_p y, x] = 1$. Now we have $1 = [y, x^p, x] = \prod_{i=1}^{p} [y, {}_{i+1} x]^{\binom{p}{i}} = [y, x, x]^p$, hence (a) implies (b).

Assume (b). Then we have $[y, x^p, y] = \prod_{i=1}^{p} [y, {}_i x, y]^{\binom{p}{i}} = 1$, hence (a) and (b) are equivalent.

Next we prove that (a) and (b) imply (c). Since G is metabelian, $[x, y, y]^p = 1$ implies $E_2(G)^p = 1$. To prove that G is nilpotent of class $\leq p+1$, we may obviously assume that G is finitely generated. By Lemma 4 and a result of Meier–Wunderli [12], G is nilpotent, hence it is also polycyclic. By Corollary 2, G is isomorphic to a subgroup of a direct product of a finite p-Kappe p-group and a finitely generated 2-Engel group. Therefore we may restrict ourselves without loss of generality to the case when G is a finite p-Kappe p-group. As G satisfies any two-variable identity of the form $[y_1, \dots, y_p, y, y] = 1$, where $y_i \in \{x, y\}$, it follows that every two-generator subgroup of the group G is of class $\leq p+1$. In particular, G satisfies the identity $[x, y, y, y, {}_{p-2} x] = 1$. By the result of Gupta and Newman [2], the factor group $\gamma_{p+2}(G)/\gamma_{p+3}(G)$ has exponent e dividing $2(p+2)(p-2)!$. Since e is prime to p, we have $\gamma_{p+2}(G) = \gamma_{p+3}(G)$, hence G is nilpotent of class $\leq p+1$.

As (c) clearly implies (b), the theorem is proved. □

We have proved that for an odd prime p every metabelian p-Kappe group is nilpotent of class $\leq p+1$. We will show that this bound for the nilpotency class is best possible. Before embarking on an appropriate example, we briefly recall the notion of *a polycyclic presentation* of a finitely generated nilpotent group. This presentation is given by a finite number of generators $g_1, \ldots, g_r$ and relations of the form

$$g_i^{m_i} = w_{ii}(g_{i+1}, \ldots, g_r) \quad \text{for } i \in I,$$

$$[g_j, g_i] = w_{ij}(g_{j+1}, \ldots, g_r) \quad \text{for } 1 \leq i < j \leq r.$$

Here m_i are positive integers, $w_{ij}(g_k, \ldots, g_r)$ are group words in the generators $g_k, \ldots, g_r$ and I is a (possibly empty) set of indices. It is straightforward to see that every group with this kind of presentation is nilpotent. Conversely, let G be a finitely generated nilpotent group. By refining the lower central series of G one can obtain a normal series $G = G_1 > G_2 > \cdots > G_{r+1} = 1$ with cyclic factors. Such a polycyclic series gives a rise to a sequence of generators of G by choosing a generator g_i for each cyclic factor G_i/G_{i+1}. Let I be the set of all indices i such that G_i/G_{i+1} is finite. Then G has a presentation of the above form. In a group given by a polycyclic presentation each element in the group can be written as *a normal word* $g_1^{e_1} \cdots g_r^{e_r}$ with $e_i \in \mathbb{Z}$ and $0 \leq e_i < m_i$ for $i \in I$. In general, this presentation is not unique. A polycyclic presentation with the property that the normal form of each element is uniquely determined is called *consistent*.

Example 1. Let p be any odd prime, let F be the free group of rank two and consider the group $G = F/F''\gamma_3(F)^p\gamma_{p+2}(F)$. The group G is metabelian of class $p+1$ and $\gamma_3(G)^p = 1$. By Theorem 2, G is a two-generator p-Kappe group. It is not difficult to see that the group G has a consistent polycyclic presentation with generators a, b, x and x_{ij}, where $i, j \geq 0$, $i + j \in \{1 \ldots, p-1\}$, and the relations are $[a, b] = x$, $[x, a] = x_{10}$, $[x, b] = x_{01}$, $[x_{ij}, a] = x_{i+1,j}$ and $[x_{ij}, b] = x_{i,j+1}$ for $i + j < p - 1$, $[x_{ij}, a] = [x_{ij}, b] = 1$ for $i + j = p - 1$, $[x_{ij}, x_{kl}] = [x_{ij}, x] = 1$ and $x_{ij}^p = 1$; here $x_{ij} = [a, b, {}_ia, {}_jb]$. By Lemma 5, G is p^2-Bell and p^2-Levi.

Let $k > 1$ be the smallest integer such that G is a k-Levi group. Since G is metabelian, we have

$$[a, b]^k = [a^k, b] = \prod_{i=1}^{k} [a, b, {}_{i-1}a]^{\binom{k}{i}} = [a, b]^k \prod_{i=1}^{k-1} [a, b, {}_ia]^{\binom{k}{i+1}}.$$

Suppose $k \leq p$ and let e_i be an integer between 0 and $p-1$ such that $e_i = -\binom{k}{i+1} \mod p$. Then the above equation yields

$$x_{k-1,0} = \prod_{i=1}^{k-2} x_{i0}^{e_i}.$$

Since the left and the right side of this equation are written in the normal form, this is clearly impossible because of the consistency of the presentation. Hence $k > p$, which together with the class restriction yields

$$\prod_{i=1}^{p-1} x_{i0}^{\binom{k}{i+1}} = 1.$$

Now the consistency of the presentation implies that p divides $\binom{k}{i+1}$ for every $i = 1, \ldots, p-1$. The smallest possible value for k is p^2, hence G is not n-Levi for any $1 < n < p^2$.

Using a similar argument, we can prove that $[a^n, b] \neq [a, b^n]$ for any $1 < n < p^2$, hence G is not n-Bell for any $1 < n < p^2$.

The following definition is taken from [7]: Let $q_1, q_2, \ldots, q_t$ be integers, $q_i > 1$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$. Let $B(q_1, q_2, \ldots, q_t)$ be the set of integers which is the union of 2^t residue classes modulo q_i satisfying each a system of congruences $m \equiv \delta_i \mod q_i$, where $i = 1, \ldots, t$ and $\delta_i \in \{0, 1\}$. It is proved in [7] that each of the sets $\mathcal{E}(G)$, $\mathcal{B}(G)$ and $\mathcal{L}(G)$ is equal either to $\mathbb{Z}$, $\{0, 1\}$ or to some $B(q_1, q_2, \ldots, q_t)$ with $q_i > 2$. This enables us to formulate the following result:

Corollary 3. *Let p be an odd prime. Then we have:*

(a) *If G is a metabelian p-Kappe group, then G has p-central normal closures.*

(b) *If G is a metabelian p-Kappe group, then $\gamma_3(G)^{3p} = \gamma_4(G)^p = 1$.*

(c) *Let G be a free metabelian p-Kappe group with two or more generators. Then $\mathcal{B}(G) = \mathcal{L}(G) = B(p^2)$.*

Proof. (a) Since the class of metabelian groups with p-central normal closures forms a finitely based variety of groups (see [15] and [16, Theorem 36.11]), we may assume that G is finitely generated, hence it is polycyclic. By Corollary 2, G is isomorphic to a subgroup of $P \times N$, where P is a finite p-Kappe p-group and N is a finitely generated 2-Engel group. Therefore it suffices to show that both P and N have p-central normal closures. As for the group P, this follows directly from [9, Theorem 14] and from Theorem 2. Since N is 2-Engel, it has abelian normal closures by Lemma 2, thus we have the result.

To prove (b), observe that $G/E_2(G)$ is a 2-Engel group, which yields $\gamma_3(G/E_2(G))^3 = \gamma_4(G/E_2(G)) = 1$ by Lemma 1. Therefore we have $\gamma_3(G)^3 \leq E_2(G)$ and $\gamma_4(G) \leq E_2(G)$, hence $\gamma_3(G)^{3p} = \gamma_4(G)^p = 1$ by Theorem 2.

(c) By Lemma 5, G is p^2-Bell and p^2-Levi. Now Example 1 shows that $n = p^2$ is the smallest positive integer such that $G^n \leq R_2(G)$ and $n \in \mathcal{B}(G)$ ($n \in \mathcal{L}(G)$). By Corollary 1 of [7], we have $k^2 \equiv k \mod p^2$ for every $k \in \mathcal{B}(G)$ ($k \in \mathcal{L}(G)$). This congruence has two solutions, namely $k \equiv 0 \mod p^2$ and $k \equiv 1 \mod p^2$, which proves that $\mathcal{B}(G) \subseteq B(p^2)$ and $\mathcal{L}(G) \subseteq B(p^2)$.

Let t be an arbitrary integer. For $x, y \in G$ we have $[x^p, y] \in Z(\langle x, y\rangle)$, hence $[x^{pt}, y] = [x^p, y]^t$. Replacing x by x^p, we get $[x^{p^2t}, y] = [x^{p^2}, y]^t = [x, y]^{p^2t} = [x, y^{p^2t}]$, as $p^2 \in \mathcal{B}(G) \cap \mathcal{L}(G)$. Since $[x^{p^2t}, y] \in Z(\langle x, y\rangle)$, we get

$$[x^{p^2t+1}, y] = [x^{p^2t}, y][x, y] = [x, y]^{p^2t+1} = [x, y^{p^2t+1}].$$

Thus we have proved that $B(p^2) \subseteq \mathcal{B}(G)$ and $B(p^2) \subseteq \mathcal{L}(G)$, as required. □

Turning our attention to 2-Kappe groups, we first prove Theorem 3 which characterizes 2-Kappe groups in terms of certain Engel words:

Proof of Theorem 3. Suppose that G is a 2-Kappe group. Then $G/R_2(G)$ has exponent two and hence it is abelian. Thus G satisfies the law $[x, y, z, z] = 1$. In particular, G is 3-Engel, hence every 2-generator subgroup is metabelian. Now we have $1 = [y, x^2, x] = [y, x, x]^2$ and $1 = [x^2, y, y] = [x, y, y, x]$, hence (a) implies (b).

Assume G satisfies the laws $[x, y, y, y] = [x, y, y, x] = [x, y, y]^2 = 1$. Then every 2-generator subgroup of G is nilpotent of class ≤ 3. We claim that $E_2(G) \leq R_2(G)$. For this, let $x, y, z \in G$. Since G is 3-Engel, the subgroup $H = \langle x, y, z\rangle$ is nilpotent of class ≤ 5. Expanding the identity

$[z, xy, xy, xy] = 1$ modulo $\gamma_5(H)$, we get

$$[z, x, x, y][z, x, y, x][z, y, x, x][z, x, y, y][z, y, x, y][z, y, y, x] \equiv 1 \mod \gamma_5(H).$$

Replacing x by x^{-1} in this equtation, we obtain

$$[z, x, x, y]^2[z, x, y, x]^2[z, y, x, x]^2 \equiv 1 \mod \gamma_5(H),$$

hence $[z, x, y, x]^2 \equiv 1 \mod \gamma_5(H)$.

The Hall–Witt identity [18, Part 1, Section 2.1] gives $[z, [y, x, x]] \equiv [z, y, x, x][z, x, y, x]^{-2}[z, x, x, y] \mod \gamma_5(H)$, thus $[z, [y, x, x]] \equiv [z, y, x, x][z, x, x, y] \mod \gamma_5(H)$. From this we conclude that $[y, x, x, z] \equiv [z, y, x, x][z, x, x, y] \mod \gamma_5(H)$. Expansion of $[yz, x, x, yz] = 1$ gives $[z, x, x, y][y, x, x, z] \equiv 1 \mod \gamma_5(H)$, which further implies $[z, y, x, x] \equiv 1 \mod \gamma_5(H)$. Replacing z by $[z, y]$, using the class restriction and relabeling the variables, we get $[x, y, y, z, z] = 1$, hence $[x, y, y] \in R_2(G)$.

For $x, y, z, w \in G$ we now have $[[x, y, y], [z, w, w]] = [x, y, y, [z, w], w]^2 = [[x, y, y]^2, [z, w], w] = 1$ by Lemma 2, thus $E_2(G)$ is abelian. This implies $E_2(G)^2 = 1$, hence we have (c).

Now assume (c). Then we have $[x^2, y, y] = [x, y, y]^2[x, y, x, y] = 1$, hence (c) implies (a). □

As a consequence, the following properties of 2-Kappe groups are derived:

Corollary 4. *Let G be a 2-Kappe group.*

(a) *G has 2-central normal closures.*

(b) *G is metabelian and we have $\gamma_3(G)^6 = \gamma_4(G)^2 = 1$.*

(c) *Suppose that G is a free 2-Kappe group with two or more generators. Then $\mathcal{B}(G) = \mathcal{L}(G) = B(4)$.*

Proof. (a) This follows from [9, Theorem 8] and from Theorem 3.

(b) Let $x, y, z, w \in G$. Since $[x, y] \in R_2(G)$, we get $[[x, y], [z, w]] = [x, y, z, w]^2$ by Lemma 2. Besides that, $\gamma_4(G) \leq E_2(G)$ by Lemma 1. Since $E_2(G)$ is abelian, we deduce that $\gamma_4(G)^2 = 1$, hence G is metabelian. The relation $\gamma_3(G)^6 = 1$ follows now similarly as in the proof of Corollary 3.

(c) By (a) and [3], G has 4-abelian normal closures, hence we get

$$[x^4, y] = x^{-4}(x^y)^4 = (x^{-1}x^y)^4 = [x, y]^4 = [x, y^4],$$

therefore G is 4-Bell and 4-Levi. Now, the construction in Example 1 also works for $p = 2$, hence $n = 4$ is the smallest positive integer such that $G^n \leq R_2(G)$ and $n \in \mathcal{B}(G)$ ($n \in \mathcal{L}(G)$). The rest of the proof now follows along the lines of the proof of (c) in Corollary 3. □

Theorem 3 also facilitates the computation of the nilpotency class of the free 2-Kappe group G_r of rank r:

Proof of Theorem 4. From Theorem 3 it follows that G_2 is of class ≤ 3. Now let $r > 2$ and suppose that the class of G_k is $\leq k+1$ for $k < r$. Let X be a generating set of the group G_r and consider the commutator of the form $c = [x_1, x_2, \ldots, x_{r+2}]$, where $x_i \in X$. Using the induction hypothesis and the identity $[x, y, z, z] = 1$, it suffices to consider the commutator of the form $c = [x_1, x_2, x_1, x_2, x_3, \ldots, x_{r+2}]$, where $x_3, \ldots, x_{r+2}$ are pairwise distinct and are not equal to x_1 or x_2. Since G_2 is of class ≤ 3, we have $c = 1$, hence G_r is of class $\leq r+1$.

Suppose there exists an $r > 1$ such that G_r is nilpotent of class $\leq r$. By a result of Heineken [4], every 2-Kappe group would be nilpotent. On the other hand, consider the group $G = C_2 \wr A$, the restricted wreath product of a cyclic group of order two and an infinite elementary abelian 2-group A. Clearly G is a metabelian group. Let B be the base group of G, let $b \in B$ and $x \in G$. Since $x^2 \in B$, we have $1 = [b, x^2] = b^{x^2}b$, hence $[b, x, x] = b^{x^2-2x+1} = b^{x^2}b = 1$. This proves that $B \leq R_2(G)$. As G/B is a group of exponent 2, we deduce that G is a 2-Kappe group. But G is not nilpotent, since the fact that A is infinite implies $Z(G) = 1$. This contradiction shows that the class of G_r is $r+1$ precisely. □

Note that Theorem 4 yields the existence of non-nilpotent 2-Kappe groups, which is not the case for 2-central groups [9, Theorem 7]. The situation is quite different for 3-Kappe groups.

Proof of Theorem 5. Since every group of exponent three is nilpotent by Lemma 1, it follows from Lemma 4 that every 3-Kappe group is locally nilpotent. Let G be any finitely generated 3-Kappe group. Then G is nilpotent, hence it is polycyclic. By Corollary 2, G can be embedded into a direct product of a finite 3-Kappe 3-group and a finitely generated 2-Engel group. Therefore we may assume that G is a finite 3-Kappe 3-group. Now, since $G/R_2(G)$ is a group of exponent 3, it follows from Lemma 1 that $G/R_2(G)$ is 2-Engel, hence G satisfies the law $[x, y, y, z, z] = 1$. As G does not contain involutions, G is nilpotent of class ≤ 6 by [5]. Using [4], we conclude that every 3-Kappe group is nilpotent of class ≤ 6, hence (a) is proved.

Let $G = \langle a, b\rangle$ be a 2-generator 3-Kappe group. We may assume that $\gamma_6(G) = 1$. Since every group of exponent three is 2-Engel by Lemma 1 (c), the factor group $G/R_2(G)$ is nilpotent of class ≤ 2 by Lemma 1 (b), hence G satisfies the law $[x, y, z, w, w] = 1$. Let $x_1, \ldots, x_5 \in \{a, b\}$. Expanding the identity $[x_1, x_2, x_3, x_4x_5, x_4x_5] = 1$ and using the class restriction, we obtain

$$[x_1, x_2, x_3, x_4, x_5] = [x_1, x_2, x_3, x_5, x_4]^{-1}. \tag{1}$$

Beside that, we also have

$$[x_1, x_2, x_3, x_4, x_5] = [x_2, x_1, x_3, x_4, x_5]^{-1}. \tag{2}$$

Now the equations (1) and (2) imply that in order to establish that the class of G is ≤ 4, we only have to show that the commutator $[a, b, a, b, a]$ is trivial. To see this, we expand the commutator $[ab, ba]$ in two ways to obtain

$$[ab, ba] = [ab, a][ab, b][ab, b, a] = [a, b, b][a, b, a][a, b, b, a]$$

and

$$[ab, ba] = [a, ba][a, ba, b][b, ba] = [a, b][a, b, a][a, b, b][a, b, a, b][b, a].$$

As $[a, b, a]^{-1} = [b, a, a]^{[a,b]}$ and $[a, b, b]^{-1} = [b, a, b]^{[a,b]}$, this further gives

$$[a, b, b, a] = [b, a, a]^{[a,b]}[b, a, b]^{[a,b]}[b, a, a]^{-1}[b, a, b]^{-1}[a, b, a, b].$$

Because of the class restriction $[b, a, a]$ commutes with $[b, a, b]$. Since we have $[b, a, a]^{[a,b]} = [b, a, a][b, a, a, [a, b]]$ and $[b, a, b]^{[a,b]} = [b, a, b][b, a, b, [a, b]]$, we get $[a, b, b, a] = [a, b, a, b][b, a, a, [a, b]][b, a, b, [a, b]]$, hence $[a, b, a, b] \equiv [a, b, b, a] \mod \gamma_5(G)$. This means that $[a, b, a, b, a] = 1$, which concludes the proof. □

Example 2. The bounds for the nilpotency class in Theorem 5 are best possible. The group constructed in Example 1 for $p = 3$ is a two-generator metabelian 3-Kappe group of class 4 precisely. To obtain a 3-Kappe group of class 6, one can use the Nilpotent Quotient Algorithm [17] implemented in GAP [20]. Starting with the free group F of rank three, we use the Nilpotent Quotient Algorithm to compute a consistent polycyclic presentation of the factor group $H = F/\gamma_7(F)$. In order to obtain a quotient group of H which satisfies the identical relation $[x^3, y, y] = 1$, we use a result of Higman [6] which says that a finitely generated nilpotent group of class $\leq c$ given by a polycyclic presentation with generating sequence $g_1, \dots, g_r$ satisfies an identity $w(x_1, \dots, x_k) = 1$ if $w(h_1, \dots, h_k) = 1$ for all normal words $h_1, \dots, h_k$ for which the sum of weights in given generators is at most c. This enables us to enforce the identity $[x^3, y, y] = 1$ on the group H by simply adding a certain finite set of instances of this identity to the presentation of H (this procedure is also a part of the Nilpotent Quotient Algorithm). The resulting quotient group is a 3-Kappe group of class 6 with derived length 3.

Corollary 5.

(a) *A group G is 3-Kappe if and only if $\langle x, y\rangle$ is nilpotent of class ≤ 4 and $[x, y, y]^3 = 1$ for any $x, y \in G$.*

(b) *Let G be a free 3-Kappe group with two or more generators. Then $\mathcal{B}(G) = \mathcal{L}(G) = B(9)$.*

(c) *Every 3-Kappe group has 3-central normal closures.*

Proof. As every 2-generator 3-Kappe group is metabelian and nilpotent of class ≤ 4, (a) and (b) follow directly from Theorem 2 and Corollary 3.

Let G be a 3-Kappe group and let $x, y, z, w \in G$. By means of expansion in the free nilpotent group of class 6 and rank 4, we obtain

$$[x^y, x^z, x^w, x] = c_1^{-1} c_2 c_3 c_4^{-1} c_5^{-1} c_6^{-1} c_7 c_8,$$

where $c_1 = [x, z, {}_3x]$, $c_2 = [x, y, {}_3x]$, $c_3 = [x, z, x, w, {}_2x]$, $c_4 = [x, z, {}_2x, w, x]$, $c_5 = [x, y, z, {}_3x]$, $c_6 = [x, y, x, w, {}_2x]$, $c_7 = [x, y, x, z, {}_2x]$, $c_8 = [x, y, {}_2x, w, x]$. As $\langle x, y\rangle$ and $\langle x, z\rangle$ are of class ≤ 4, we have $c_1 = c_2 = 1$. By Lemma 1, G satisfies the law $[x_1, x_2, x_3, x_4, x_5, x_5] = 1$, hence $c_3 = c_5 = c_6 = c_7 = 1$. Consider now the identity $[x, y, x, x, xw, xw] = 1$. Expanding and using the class restriction, we conclude that $c_8 = 1$, and consequently $c_4 = 1$. This yields that G satisfies the law $[x^y, x^z, x^w, x] = 1$. Using Theorem 4.3

of [15], we see that a^G is nilpotent of class ≤ 3 for any $a \in G$. Observing (a), we get $[a^x, a]^3 = [a, x, a]^3 = 1$ for any $x, a \in G$, hence $\gamma_2(a^G)^3 = 1$. It follows from [9, Theorem 9] that a^G is 3-central for any $a \in G$, hence the result follows. □

In view of Corollaries 3, 4 and 5, it seems appropriate to ask whether every (metabelian) n-Kappe group has n-central normal closures. The next example shows that this is not true for metabelian 4-Kappe groups.

Example 3. Let D be a group with commuting generators x, y_1, y_2, z_1, z_2, z_3, $w_1, \ldots w_4$, $v_1, \ldots, v_5$ which satisfy the following additional relations: $y_i^4 = v_{i+1}v_{i+2}$ for $i = 1, 2$, $z_i^2 = v_i v_{i+1} v_{i+2}$ for $i = 1, 2, 3$ and $w_i^2 = v_j^2 = 1$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 5$. Let $A = [D]\langle a\rangle$ be the semidirect product of D with the infinite cyclic group $\langle a\rangle$ where a induces the following automorphism on D: $[x, a] = y_2$, $[y_i, a] = z_{i+1}$, $[z_i, a] = w_{i+1}$, $[w_i, a] = v_{i+1}$ and $[v_i, a] = 1$. Let $G = [A]\langle b\rangle$, where b is an element of infinite order acting on A in the following way: $[a, b] = x$, $[x, b] = y_1$, $[y_i, b] = z_i$, $[z_i, b] = w_i$, $[w_i, b] = v_i$ and $[v_i, b] = 1$. Clearly, $G = \langle a, b\rangle$ is metabelian and nilpotent of class 6. It is lengthy to prove that G is a 4-Kappe group; we only give an outline of this verification. First of all, note that $v_i \in Z(G)$ for $i = 1, \ldots, 5$ and $w_i \in \gamma_5(G) \leq Z_2(G)$ for $i = 1, \ldots, 4$. This also implies $z_i^2 \in Z(G)$ for $i = 1, 2, 3$. For $g, h \in G$ we obtain $[y_i^2, g, h] = [y_i, g, h]^2 \in \gamma_5(G)^2 = 1$, hence $y_1^2, y_2^2 \in Z_2(G)$. In particular, we have $(G')^2 \leq R_2(G)$. Now, if g is an arbitrary element of G, then a repeated use of Lemma 3 (e) gives $g^4 = a^{4m} b^{4n} c_1^2 c_2$, where m and n are integers, $c_1 \in G'$ and $c_2 \in \gamma_5(G)$. Hence it suffices to show that a^4 and b^4 are 2-Engel elements. Using Lemma 3, we conclude that this reduces to proving that the commutators $[a^4, b, b]$, $[a^4, b, a]$, $[b^4, a, a]$ and $[b^4, a, b]$ are trivial. This follows readily from the presentation of G, hence G is a 4-Kappe group. On the other hand, we have $[[a, b]^4, a] = [a, b, a]^4 = y_2^4 \neq 1$, hence a^G is not 4-central.

We conclude this paper by some remarks on 4-Kappe groups:

Remark. Since every 2-generator metabelian group of exponent four is nilpotent of class ≤ 4, it follows in particular that every metabelian 4-Kappe group is 6-Engel. Observing Corollary 1 of [9], we conclude that every normal closure of an element of a metabelian 4-Kappe group is nilpotent of class ≤ 5. In fact, this bound is sharp as Example 3 shows; namely, we have $[a^b, {}_4a] \neq 1$, hence the class of a^G is 5 precisely. On the other hand, there exists a 4-Kappe group with derived length 3 which is not a Fitting group. The appropriate example can already be found in [18, Part 2, p. 4] and will be briefly restated here. Consider the group $G = (C_2 \wr A) \wr C_2$, where C_2 is the cyclic group of order two and A is an infinite elementary abelian 2-group. Following the lines of the second part of the proof of Theorem 4, we conclude that G is a 4-Kappe group. On the other hand, there is an element $x \in G$ such that x^G is not nilpotent of any class [18].

Remark. By Lemma 4, every 4-Kappe group is locally nilpotent. Thus it is possible to obtain the polycyclic presentation of the free 2-generator 4-Kappe group G with the help of the Nilpotent Quotient Algorithm [17]. The group G is of class 8 with derived length 3; the construction is similar to that from Example 2. It can be seen from the presentation of G that $n = 16$ is the smallest positive integer greater than 1 such that G is n-Bell and n-Levi (note that the group G constructed in Example 3 is also 16-Bell and 16-Levi and it is not n-Bell (n-Levi) for any $1 < n < 16$). As above, we conclude that $\mathcal{B}(G) = \mathcal{L}(G) = B(16)$.

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