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On the torsion-free connections on higher order frame bundles

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Abstract. Using the r-jets of flows of vector fields, we show that every torsion-free linear r-th order connection Γ on the tangent bundle of a manifold M determines a reduction of the $(r+1)$-st order frame bundle of M to the general linear group. We deduce that this reduction coincides with the well known reduction determined by the principal connection induced by Γ on the r-th order frame bundle of M.

Introduction

Our starting point is the fact that the principal connections on the r-th order frame bundle P^rM of a manifold M are in bijection with the linear r-th order connections on the tangent bundle TM, i.e. with the linear splittings

$$\Gamma : TM \to J^rTM \tag{1}$$

of the jet projection $J^rTM \to TM$. We shall write

$$\widetilde{\Gamma} : P^rM \to J^1P^rM \tag{2}$$

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for the principal connection corresponding to (1). Its lifting map, denoted by the same symbol,

$$\widetilde{\Gamma} : P^r M \times_M TM \to TP^r M$$

is determined by

$$\widetilde{\Gamma}(u, A) = \mathcal{P}^r X(u), \quad u \in P^r M,\ A \in T_x M,\ x \in M, \tag{3}$$

where $\mathcal{P}^r X$ is the flow prolongation of a vector field $X : M \to TM$ satisfying $j^r_x X = \Gamma(A)$, [3].

The structure group of $P^r M$ is G^r_m, $m = \dim M$. There is a canonical $\mathbb{R}^m \times \mathfrak{g}^{r-1}_m$-valued one-form Θ_r on $P^r M$. P. C. Yuen, [5], introduced the torsion of $\widetilde{\Gamma}$ as the exterior covariant differential

$$D_{\widetilde{\Gamma}}\, \Theta_r, \tag{4}$$

see also [1]. On the other hand, the $(r-1)$-jet at x of the bracket $[X, Y]$ of two vector fields X, Y on M depends on $j^r_x X$ and $j^r_x Y$ only. This defines a map

$$[\ ,\ ]_{r-1} : J^r TM \times_M J^r TM \to J^{r-1} TM. \tag{5}$$

The torsion of Γ can be introduced, [6], as a map

$$\tau_\Gamma : TM \times_M TM \to J^{r-1} TM$$

defined by

$$\tau_\Gamma(A, B) = [\Gamma(A), \Gamma(B)]_{r-1}, \qquad A, B \in T_x M\,. \tag{6}$$

In [3] we deduced that the torsions $D_{\widetilde{\Gamma}}\Theta_r$ and τ_Γ coincide in a natural way.

There is a canonical injection $i_{r+1} : P^{r+1} M \to J^1 P^r M$, see formula (13) below. If $\widetilde{\Gamma}$ is torsion-free, there is a well-known map $\varrho(\widetilde{\Gamma}) : P^1 M \to P^{r+1} M$ defined by the induction

$$i_{r+1} \circ \varrho(\widetilde{\Gamma}) = \widetilde{\Gamma} \circ \varrho(\widetilde{\Gamma}_{r-1}), \tag{7}$$

where $\widetilde{\Gamma}_{r-1}$ is the underlying connection on $P^{r-1} M$, so that $\varrho(\widetilde{\Gamma}_{r-1}) : P^1 M \to P^r M$ by the induction hypothesis. If we consider the canonical injection $GL(m, \mathbb{R}) \hookrightarrow G^{r+1}_m$, [4], p. 130, then $\varrho(\widetilde{\Gamma})(P^1 M)$ is a reduction of

$P^{r+1}M$ to $GL(m,\mathbb{R})$. This establishes the well-known bijection between the torsion-free connections on P^rM and the reductions of $P^{r+1}M$ to $GL(m,\mathbb{R})$, [2], [3].

In Section 1 of the present paper we use the r-jets of flows of vector fields to construct a map $\sigma(\Gamma) : P^1M \to P^{r+1}M$ for every torsion-free linear r-th order connection Γ on TM. Our proof of the fact that $\sigma(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$ is based on two interesting lemmas concerning the r-jets of the bracket of vector fields, the proofs of which we postpone to Section 3. In Section 2 we deduce $\sigma(\Gamma) = \varrho(\widetilde{\Gamma})$, i.e. both constructions of a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$ coincide.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [4].

1. The reduction $\sigma(\Gamma)$

Consider $\Gamma : TM \to J^rTM$. For a linear frame $u \in P^1_xM$, $u = (A_1,\dots,A_m)$, $A_i \in T_xM$, we take vector fields X_i satisfying $j^r_xX_i = \Gamma(A_i)$, $i = 1,\dots,m$. Then

$$\big(Fl^{X_1}_{t^1} \circ \cdots \circ Fl^{X_m}_{t^m}\big)(x)$$

is a local map $\mathbb{R}^m \to M$ and we define

$$\sigma(\Gamma)(u) = j^{r+1}_0\big(Fl^{X_1}_{t^1} \circ \cdots \circ Fl^{X_m}_{t^m}\big)(x) \in P^{r+1}_xM, \tag{8}$$

where $0 \in \mathbb{R}^m$. One verifies easily that $\sigma(\Gamma)(u)$ depends on u and Γ only.

Proposition 1. *If Γ is torsion-free, then $\sigma(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$.*

The proof will be based on the following two lemmas, the proofs of which we postpone to the last section.

Consider two vector fields X and Y on M. Then $\big(Fl^X_t \circ Fl^Y_\tau\big)(x)$ is a local map $\mathbb{R}^2 \to M$, so that $j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) \in (T^{r+1}_2M)_x$ is a $(2, r+1)$-velocity on M.

Lemma 1. *If $j^{r-1}_x[X,Y] = 0$, then*

$$j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) = j^{r+1}_{0,0}\big(Fl^Y_\tau \circ Fl^X_t\big)(x). \tag{9}$$

Further, $\big(Fl_t^X \circ Fl_t^Y\big)(x)$ is a local map $\mathbb{R} \to M$, so that $j_0^{r+1}\big(Fl_t^X \circ Fl_t^Y\big)(x) \in (T_1^{r+1}M)_x$ is a $(1, r+1)$-velocity on M.

Lemma 2. *If* $j_x^{r-1}[X,Y] = 0$, *then*

$$j_0^{r+1}\big(Fl_t^X \circ Fl_t^Y\big)(x) = j_0^{r+1}\big(Fl_t^{X+Y}\big)(x). \tag{10}$$

Now we prove Proposition 1 by using Lemmas 1 and 2. We shall also use the well known formula

$$Fl_{at}^X = Fl_t^{aX}, \qquad a \in \mathbb{R}. \tag{11}$$

Take $g = (a_j^i) \in GL(m,\mathbb{R})$, $u = (A_i) \in P_x^1 M$ and consider $ug = (a_i^j A_j)$. Write $\Gamma(A_i) = j_x^r X_i$. Since Γ is torsion-free, by (10), (11) and (9) we obtain gradually

$$\begin{aligned}
\sigma(\Gamma)(ug) &= j_0^{r+1}\big(Fl_{t^1}^{a_1^1 X_1 + \dots + a_1^m X_m} \circ \dots \circ Fl_{t^m}^{a_m^1 X_1 + \dots + a_m^m X_m}\big) \\
&= j_0^{r+1}\big(Fl_{t^1}^{a_1^1 X_1} \circ \dots \circ Fl_{t^1}^{a_1^m X_m} \circ \dots \circ Fl_{t^m}^{a_m^1 X_1} \circ \dots \circ Fl_{t^m}^{a_m^m X_m}\big) \\
&= j_0^{r+1}\big(Fl_{a_1^1 t^1}^{X_1} \circ \dots \circ Fl_{a_1^m t^1}^{X_m} \circ \dots \circ Fl_{a_m^1 t^m}^{X_1} \circ \dots \circ Fl_{a_m^m t^m}^{X_m}\big) \\
&= j_0^{r+1}\big(Fl_{a_1^1 t^1 + \dots + a_m^1 t^m}^{X_1} \circ \dots \circ Fl_{a_1^m t^1 + \dots + a_m^m t^m}^{X_m}\big).
\end{aligned}$$

This proves Proposition 1.

2. The main result

We shall use the following form of the canonical injection $i_{r+1} : P^{r+1}M \to J^1 P^r M$. We have $P^r M \subset T_m^r M$, where T_m^r is the functor of (m,r)-velocities. Clearly, $j_0^r f \in T_m^r M$, $f : \mathbb{R}^m \to M$, can be expressed in the form

$$j_0^r f = \big(T_m^r f\big)(e), \qquad e = j_0^r \,\mathrm{id}_{\mathbb{R}^m}. \tag{12}$$

Write $E_i = \frac{\partial}{\partial t}\big|_0 j_0^r \tau_t^i \in T_e T_m^r \mathbb{R}^m$, where $\tau_t^i : \mathbb{R}^m \to \mathbb{R}^m$ is the translation $\bar{t}^1 = t^1, \dots, \bar{t}^i = t^i + t, \dots, \bar{t}^m = t^m$. If we consider $j_0^{r+1}\psi \in P^{r+1}M$, then

$$\big(TT_m^r \psi\big)(E_i) \tag{13}$$

is an m-tuple of tangent vectors at $j_0^r \psi \in P^r M$. The linear span of these vectors defines $i_{r+1}(j_0^{r+1}\psi) \in J^1 P^r M$.

Proposition 2. *If Γ is a torsion-free linear r-th order connection on TM and $\widetilde{\Gamma}$ is the corresponding principal connection on P^rM, then* $\sigma(\Gamma) = \varrho(\widetilde{\Gamma})$.

Proof. We proceed by induction. If Γ_{r-1} and $\widetilde{\Gamma}_{r-1}$ are the underlying connections in the order $r-1$, then

$$\sigma(\Gamma_{r-1}) = \varrho(\widetilde{\Gamma}_{r-1}) \tag{14}$$

by the induction hypothesis. Consider $u = (A_1, \dots, A_m) \in P^1_x M$ and write

$$v = \sigma\big(\Gamma_{r-1}\big)(u) = \varrho\big(\widetilde{\Gamma}_{r-1}\big)(u).$$

By (13), $i_{r+1}\big(j_0^{r+1}\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(x)\big)$ is the linear span of the vectors

$$TT^r_m\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(E_i). \tag{15}$$

Using the basic properties of flows, Lemma 1 and (12), we deduce that (15) is equal to

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_0 T^r_m\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_i}_{t+t^i} \circ \dots \circ Fl^{X_m}_{t^m}\big)(e)\\
&\quad= \frac{\partial}{\partial t}\Big|_0 \big(Fl^{\mathcal{T}^r_m X_i}_{t} \circ Fl^{\mathcal{T}^r_m X_1}_{t^1} \circ \dots \circ Fl^{\mathcal{T}^r_m X_m}_{t^m}\big)(e)\\
&\quad= \mathcal{T}^r_m X_i\big(T^r_m(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t_m})(e)\big) = \mathcal{T}^r_m X_i(v),
\end{aligned}$$

where $\mathcal{T}^r_m X_i$ denotes the flow prolongation of X_i. By (3) and by the induction hypothesis, this m-tuple spans $\varrho(\widetilde{\Gamma})(v)$. □

3. The proofs of Lemmas 1 and 2

In general, if we have two maps $f, g : \mathbb{R}^m \to \mathbb{R}^m$, it suffices to verify the condition $j^r_0 f = j^r_0 g$ on all curves of the form $x^i = a^i t$, $i = 1, \dots, m$, [4]. By the flow property (11), Lemma 1 follows from the fact that $j^{r-1}_x[X, Y] = 0$ implies

$$j_0^{r+1}\big(Fl^X_t \circ Fl^Y_t\big)(x) = j_0^{r+1}\big(Fl^Y_t \circ Fl^X_t\big)(x) \in T^{r+1}_1 M. \tag{16}$$

But this is a direct consequence of Lemma 2. So it suffices to prove Lemma 2. We have the following 3 cases.

I. If $X(x)=Y(x)=0$, then the $(r+1)$-jets of the flows of X and Y are in the group of all invertible $(r+1)$-jets of M into M with source x and target x and we have a well known result concerning Lie groups.

II. If $X(x)\neq 0$, we can consider such local coordinates on M that $X=\frac{\partial}{\partial x^1}$. Then $j^{r-1}_x[\frac{\partial}{\partial x^1},Y]=0$ means

$$D_\alpha\frac{\partial Y^i(x)}{\partial x^1}=0\,,\qquad 0\leq\|\alpha\|\leq r-1, \tag{17}$$

where Y^i are the coordinate components of Y and D_α denotes the partial derivative with respect to a multiindex α of the range m.

The flow $\psi^i(t,x)$ of the vector field $\frac{\partial}{\partial x^1}+Y$ satisfies

$$\frac{\partial\psi^i(t,x)}{\partial t}=\delta^i_1+Y^i\big(\psi(t,x)\big). \tag{18}$$

If $\eta^i(t,x)$ denotes the flow of Y, then the coordinate expression of $Fl^X_t\circ Fl^Y_t$ is

$$\mu^i(t,x)=\delta^i_1 t+\eta^i(t,x). \tag{19}$$

Hence

$$\frac{\partial\mu^i(t,x)}{\partial t}=\delta^i_1+\frac{\partial\eta^i(t,x)}{\partial t}=\delta^i_1+Y^i\big(\eta(t,x)\big). \tag{20}$$

From (19) we obtain

$$\frac{\partial^k\mu^i(t,x)}{\partial t^k}=\frac{\partial^k\eta^i(t,x)}{\partial t^k},\qquad k\geq 2. \tag{21}$$

For $t=0$, (18) and (20) yield directly $\frac{\partial\psi^i(0,x)}{\partial t}=\frac{\partial\mu^i(0,x)}{\partial t}$. Then we find by direct evaluation

$$\frac{\partial^2\psi^i(t,x)}{\partial t^2}=\frac{\partial Y^i\big(\psi(t,x)\big)}{\partial x^j}\,\frac{\partial\psi^j(t,x)}{\partial t}, \tag{22}$$

$$\frac{\partial^2\mu^i(t,x)}{\partial t^2}=\frac{\partial Y^i\big(\eta(t,x)\big)}{\partial x^j}\,\frac{\partial\eta^j(t,x)}{\partial t}. \tag{23}$$

Hence (17) implies

$$\frac{\partial^2\psi^i(0,x)}{\partial t^2}=\frac{\partial^2\mu^i(0,x)}{\partial t^2}. \tag{24}$$

By iteration we deduce (10) for every r.

III. The case $Y(x) \neq 0$ can be reduced to II by using $Fl_t^X \circ Fl_t^Y = \big(Fl_{-t}^Y \circ Fl_{-t}^X\big)^{-1}$.

This proves Lemmas 1 and 2.

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