

On the average orders of the error term in the circle problem

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Abstract. For a natural number n , let $r(n)$ denote the number of ways of writing n as a sum of two squares, and $P(x)$ the remainder term in the circle problem of Gauss, that is, $P(x) = \sum_{n \leq x} r(n) - \pi x$. The purpose of this paper is to study some properties of the summatory function $\sum_{n \leq x} P(n)^k$ with an arbitrarily fixed natural number k . In particular, we consider the cases $k = 2$ and 3 in detail.

1. Introduction and statement of results

Let $f(n)$ be an arithmetical function, and $E(x)$ the number-theoretic error term defined by

$$E(x) = \sum_{n \leq x} f(n) - g(x), \quad (1.1)$$

where $g(x)$ is a continuously differentiable function. In the previous paper [1], the author studied the differences between the continuous mean values $\int_1^x E(u)^k du$ and the discrete mean values $\sum_{n \leq x} E(n)^k$. Here k denotes a fixed natural number. In particular, he considered the case of the Dirichlet divisor problem in detail. In this paper, we derive similar type of formulas in the case of the circle problem of Gauss. We especially study the differences between the continuous and the discrete mean values of the error term for $k = 2$ and 3 in detail. For a natural number n , let $r(n)$ denote

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the number of ways of writing n as a sum of two squares, and $P(x)$ the error term defined by

$$P(x) = \sum_{n \leq x} r(n) - \pi x. \quad (1.2)$$

The problem called the ‘‘Gaussian circle problem’’ is to determine the best possible estimate of $P(x)$. Many results concerning this problem have been obtained. For instance, we know that this function can be estimated as

$$P(x) = O(x^{131/416}(\log x)^{18627/8320}).$$

This result is due to HUXLEY [7].

As for the mean value theorems of $P(x)$, it is well-known that the formulas

$$\begin{aligned} \int_1^x P(u) du &= -x - \frac{1}{\pi^2} x^{3/4} \sum_{n=1}^{\infty} r(n) n^{-5/4} \sin\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right) \\ &\quad + O(x^{1/4}) \end{aligned} \quad (1.3)$$

and

$$\int_1^x P(u)^2 du = \left(\frac{1}{3\pi^2} \sum_{n=1}^{\infty} r(n)^2 n^{-3/2} \right) x^{3/2} + O(x \log^2 x) \quad (1.4)$$

hold for $x \geq 1$ and $x \geq 2$ respectively. On the discrete mean values of $P(x)$, we can see by applying the general formula of SEGAL [11, Lemma] that

$$\begin{aligned} \sum_{n \leq x} P(n) &= \left(\frac{\pi}{2} - 1\right) x - \frac{1}{\pi^2} x^{3/4} \sum_{n=1}^{\infty} r(n) n^{-5/4} \sin\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right) \\ &\quad + O(x^{1/3}), \end{aligned} \quad (1.5)$$

and thus we see that

$$\sum_{n \leq x} P(n) = \left(\frac{\pi}{2} - 1\right) x + O\left(x^{3/4}\right). \quad (1.6)$$

(Note that the error term in (1.6) can be also represented by $\Omega_{\pm}(x^{3/4})$, namely the error estimate in (1.6) is best possible. (See Section 3 below.)) HARDY [6] investigated the difference between the continuous and

the discrete mean value formulas for the mean square case, and proved that

$$\sum_{n \leq x} P(n)^2 = \int_1^x P(u)^2 du + O(x^{1+\varepsilon}) \tag{1.7}$$

with a sufficiently small $\varepsilon > 0$.

In this paper, we first study the error term in formula (1.7) more closely, especially we improve the estimate of this error term and derive the Ω_{\pm} -estimates of this error term. We obtain the following theorem.

Theorem 1. *Let $P(x)$ be the function defined by (1.2). For $x \geq 2$, we have*

$$\sum_{n \leq x} P(n)^2 = \int_1^x P(u)^2 du + \frac{\pi(\pi - 6)}{6} x + \left\{ \begin{matrix} O \\ \Omega_{\pm} \end{matrix} \right\} (x^{3/4}).$$

Next, we study the difference between the continuous and the discrete mean value formulas in the third power case. We obtain the following theorem.

Theorem 2. *For $x \geq 2$, we have*

$$\sum_{n \leq x} P(n)^3 = \int_1^x P(u)^3 du + \frac{3}{2} A_1 x^{3/2} + O(x \log^2 x),$$

where A_1 denotes the coefficient of the main term in (1.4).

Here we shall give some remarks on our present theorems and the differences between $\sum_{n \leq x} P(n)^k$ and $\int_1^x P(u)^k du$ for the case $k \geq 4$.

Remark 1. Concerning the differences between $\sum_{n \leq x} P(n)^k$ and $\int_1^x P(u)^k du$ in higher power cases, applying the same method used in the case of the divisor problem (cf. [1, Section 3]) and the result of the upper bound estimates of $\int_1^x |P(u)|^A du$ (cf. [5], [8] and [9]), we get, for $x \geq 1$,

$$\sum_{n \leq x} P(n)^k = \int_1^x P(u)^k du + \begin{cases} O(x^{(k+3)/4+\varepsilon}) & \text{if } 4 \leq k \leq 10, \\ O(x^{(35k+3)/108+\varepsilon}) & \text{if } k \geq 11. \end{cases} \tag{1.8}$$

To improve these estimates, especially for $k = 4$, we should consider the asymptotic formula for $\int_1^x \psi(u)\psi(u/n_1)\psi(u/n_2)\psi(u/n_3)du$ similar to the case of the divisor problem (cf. Remark in [1, Section 5]),

where n_j are natural numbers with $n_1 \leq n_2 \leq n_3$. Moreover, in the case of the circle problem, we additionally need to use formulas of types $\int_1^x \psi(u)\psi(u/n_1)\psi(u/n_2)\psi(u^{1/2})du$, or $\int_1^x \psi(u)\psi(u/n_1)\psi(u^{1/2})^2du$ by using the method in this paper. It seems to be very complicated, and therefore studies of the error term in (1.8) might be more difficult than those of the case of the divisor problem for $k = 4$, and for higher power cases.

Remark 2. From Theorems 1 and 2, we can deduce the asymptotic formula for $\sum_{n \leq x} P(n)^k$ for $k = 2$ and 3. For example, in the case $k = 2$, from Theorem 1 and (1.4) we have obviously that the asymptotic formula

$$\sum_{n \leq x} P(n)^2 = A_1 x^{3/2} + O(x \log^2 x)$$

holds for $x \geq 2$. Similarly, by applying the asymptotic formulas for the mean value formulas of $P(x)^k$ (cf. [12, Theorem 2]) and the relation in Remark 1, we can obviously derive the asymptotic formulas for $\sum_{n \leq x} P(n)^k$ with $4 \leq k \leq 9$.

Remark 3. Let us put

$$\tilde{P}(x) = \sum'_{n \leq x} r(n) - \pi x + 1,$$

where the symbol $\sum'_{n \leq x}$ indicates that the last term is to be halved if x is an integer. We remark that we often use the function $\tilde{P}(x)$ instead of $P(x)$ as the definition of the error term in the theory of the circle problem. As for the function $\tilde{P}(x)$, we show that the upper bound estimates of $P(x)$ and $\tilde{P}(x)$ are same, since $r(n) \ll n^\varepsilon$ for a sufficiently small $\varepsilon > 0$. Further, in the mean value theorem, we show that the leading orders of $\int_1^x P(u)^k du$ and $\int_1^x \tilde{P}(u)^k du$ are also same for $2 \leq k (\leq 9)$. However, we can show that the behaviour of the function $\sum_{n \leq x} \tilde{P}(n)^k - \int_1^x \tilde{P}(u)^k du$ are different from those of $\sum_{n \leq x} P(n)^k - \int_1^x P(u)^k du$ in the cases $k = 1, 2$ and 3. These results will be discussed in the author's paper [2].

2. Preliminaries

Throughout this paper, k denotes any fixed natural number, and the symbols $O(\)$, \ll and Ω_* have their usual meaning. We denote by χ the

primitive Dirichlet character (mod 4). For a real number x , $\psi(x)$ denotes the periodic Bernoulli function defined by $\psi(x) = x - [x] - 1/2$, where $[x]$ is the greatest integer not exceeding x , and $\psi_1(x)$ is the function defined by $\psi_1(x) = \int_1^x \psi(u)du$ for $x \geq 1$.

As a preparation for proving our theorems, we first consider some identity for the average order of number-theoretic error terms in general setting. In [1, Lemma 1], the author proved an identity for the average order of the error term of general power cases. We present this result as the following lemma.

Lemma 1 ([1, Lemma 1]). *Let $E(x)$ and $g(x)$ be the functions defined in (1.1), and assume that $g(x)$ is continuously differentiable. For a fixed natural number k , we have*

$$\begin{aligned} \sum_{n \leq x} E(n)^k &= \left(\frac{1}{2} - \psi(x)\right) E(x)^k + \int_1^x E(u)^k du \\ &\quad + k \int_1^x \left(\frac{1}{2} - \psi(u)\right) g'(u) E(u)^{k-1} du. \end{aligned}$$

Next, we prepare some integral formulas involving the ψ -function:

Lemma 2 ([1, Lemma 3]). *For a natural number n and a real number $y \geq 1$, we have*

$$\begin{aligned} \int_1^y \psi(u)\psi\left(\frac{u}{n}\right) du &= \frac{1}{12n} y - \frac{1}{6n} \psi(y)^3 + \frac{1}{2} \psi(y)^2 \psi\left(\frac{y}{n}\right) - \frac{1}{8} \psi\left(\frac{y}{n}\right) \\ &\quad + \frac{1}{24n} \psi(y) - \frac{1}{12n}. \end{aligned}$$

Lemma 3 ([1, Lemma 5]). *Let n_1 and n_2 be natural numbers with $n_1 \leq n_2$. We have, for $y \geq 1$,*

$$\int_1^y \psi(u)\psi\left(\frac{u}{n_1}\right)\psi\left(\frac{u}{n_2}\right) du = \frac{n_2}{24n_1} \psi\left(\frac{y}{n_2}\right)^2 - \frac{n_2}{96n_1} + O(1)$$

uniformly in y , n_1 and n_2 .

Lemma 4. *For a real number $y \geq 1$, we have*

$$\int_1^y \psi(u)\psi(u^{1/2})du = \frac{1}{12} y^{1/2} + \psi(y^{1/2})\psi_1(y) + C - \frac{1}{12} y^{-1/2}\psi(y)^3$$

$$+ \frac{1}{48} y^{-1/2} \psi(y) + O(y^{-3/2})$$

uniformly in y , where C is a certain absolute constant.

PROOF. We can easily see that

$$\int_1^y \psi(u)[u^{1/2}]du = [y^{1/2}]\psi_1(y).$$

Then from the definition of the ψ -function, we have

$$\begin{aligned} \int_1^y \psi(u)\psi(u^{1/2})du &= \psi(y^{1/2})\psi_1(y) - \frac{1}{2} \int_1^y u^{-1/2}\psi_1(u)du \\ &= \psi(y^{1/2})\psi_1(y) + \frac{1}{12} y^{1/2} - \frac{1}{12} y^{-1/2}\psi(y)^3 + \frac{1}{48} y^{-1/2}\psi(y) \\ &\quad - \frac{1}{24} \int_1^y u^{-3/2}\psi(u)^3du + \frac{1}{96} \int_1^y u^{-3/2}\psi(u)du + C' \end{aligned}$$

with a certain absolute constant C' , since

$$\int_1^y \psi_1(u)du = -\frac{1}{12} y + \frac{1}{6} \psi(y)^3 - \frac{1}{24} \psi(y) + \frac{1}{12}. \tag{2.1}$$

The assertion of the lemma can be proved by this formula, since the integrals $\int_1^\infty u^{-3/2}\psi(u)^3du$ and $\int_1^\infty u^{-3/2}\psi(u)du$ are convergent absolutely, and the partial integrals $\int_y^\infty u^{-3/2}\psi(u)^3du$ and $\int_y^\infty u^{-3/2}\psi(u)du$ are estimated by $O(y^{-3/2})$ by applying the estimates $\int_1^y \psi(u)^3du \ll 1$ and $\psi_1(y) \ll 1$ respectively. \square

Lemma 5. For a real number $y \geq 1$ and a natural number n with $n \leq y^{1/2}$, we have

$$\int_1^y \psi(u)\psi(u^{1/2})\psi\left(\frac{u}{n}\right)du = \frac{1}{12n} y^{1/2}\psi(y^{1/2})^2 - \frac{1}{144n} y^{1/2} + C(n) + O(1),$$

where $C(n)$ is a function depending only on n , whose explicit definition is given by equation (2.2) below.

PROOF. First, we consider the integral $\int_1^y \psi(u)\psi(u/n)[u^{1/2}]du$. We have by using Lemma 2 that

$$\int_1^y \psi(u)\psi\left(\frac{u}{n}\right)[u^{1/2}]du$$

$$\begin{aligned} &= \sum_{j=1}^{[y^{1/2}]-1} j \int_{j^2}^{(j+1)^2} \psi(u)\psi\left(\frac{u}{n}\right) du + [y^{1/2}] \int_{[y^{1/2}]^2}^y \psi(u)\psi\left(\frac{u}{n}\right) du \\ &= [y^{1/2}] \int_1^y \psi(u)\psi\left(\frac{u}{n}\right) du - \frac{1}{72n} [y^{1/2}] ([y^{1/2}] - 1) (2[y^{1/2}] + 5). \end{aligned}$$

From the definition of the ψ -function and the above expression, we have

$$\begin{aligned} \int_1^y \psi(u)\psi(u^{1/2})\psi\left(\frac{u}{n}\right) du &= \int_1^y u^{1/2}\psi(u)\psi\left(\frac{u}{n}\right) du - \frac{1}{2} \int_1^y \psi(u)\psi\left(\frac{u}{n}\right) du \\ &\quad - \int_1^y \psi(u)\psi\left(\frac{u}{n}\right) [u^{1/2}] du \\ &= \psi(y^{1/2}) \int_1^y \psi(u)\psi\left(\frac{u}{n}\right) du - \frac{1}{2} \int_1^y u^{-1/2} \int_1^u \psi(t)\psi\left(\frac{t}{n}\right) dt du \\ &\quad + \frac{1}{72n} [y^{1/2}] ([y^{1/2}] - 1) (2[y^{1/2}] + 5). \end{aligned}$$

Consider the double integral in the above formula. By using Lemma 2, we have

$$\begin{aligned} &\int_1^y u^{-1/2} \int_1^u \psi(t)\psi\left(\frac{t}{n}\right) dt du \\ &= \frac{1}{18n} y^{3/2} - \frac{1}{6n} y^{1/2} + \frac{1}{2} \int_1^y u^{-1/2} \psi(u)^2 \psi\left(\frac{u}{n}\right) du \\ &\quad - \frac{1}{6n} \int_1^y u^{-1/2} \psi(u)^3 du - \frac{1}{8} \int_1^y u^{-1/2} \psi\left(\frac{u}{n}\right) du + O(1). \end{aligned}$$

Since $\int_1^y \psi(u)^3 du = O(1)$ uniformly in y , we have $\int_1^y u^{-1/2} \psi(u)^3 du = O(1)$. By using integration by parts and Lemma 3, we have

$$\int_1^y u^{-1/2} \psi(u)^2 \psi\left(\frac{u}{n}\right) du = \frac{1}{24} n^{1/2} \int_{1/n}^{y/n} u^{-3/2} \psi_1(u) du + O(1),$$

since $\psi(u)^2 = 2\psi_1(u) + 1/4$ and $n \leq y^{1/2}$ *. Also we have

$$\int_1^y u^{-1/2} \psi\left(\frac{u}{n}\right) du = \frac{1}{2} n^{1/2} \int_{1/n}^{y/n} u^{-3/2} \psi_1(u) du - n\psi_1\left(\frac{1}{n}\right) + O(1)$$

*We note that in the first paragraph in this section we have defined the function $\psi_1(x)$ for $x \geq 1$. But from the periodicity of this function, we can remove this condition from the definition of this function. In fact, we see that $\psi_1(1/n) = \psi(1 + 1/n)$ for every natural number n .

for $n \leq y^{1/2}$. Hence by substituting these formulas and

$$\begin{aligned} \frac{1}{72n}[y^{1/2}]([y^{1/2}] - 1)(2[y^{1/2}] + 5) &= \frac{1}{36n}y^{3/2} - \frac{1}{12n}y\psi(y^{1/2}) \\ &+ \frac{1}{12n}y^{1/2}\psi(y^{1/2})^2 - \frac{13}{144n}y^{1/2} + O(1), \end{aligned}$$

and by using $\psi(u)^2 = 2\psi_1(u) + 1/4$, we have

$$\begin{aligned} \int_1^y \psi(u)\psi(u^{1/2})\psi\left(\frac{u}{n}\right) du &= \frac{1}{12n}y^{1/2}\psi(y^{1/2})^2 - \frac{1}{144n}y^{1/2} - \frac{1}{16}n\psi_1\left(\frac{1}{n}\right) \\ &+ \frac{1}{48}n^{1/2} \int_{1/n}^{y/n} u^{-3/2}\psi_1(u)du + O(1) \end{aligned}$$

for $n \leq y^{1/2}$. Furthermore, since $\int_{y/n}^\infty u^{-3/2}\psi_1(u)du = O(y^{-1/4})$ for $n \leq y^{1/2}$, we obtain

$$\int_1^y \psi(u)\psi(u^{1/2})\psi\left(\frac{u}{n}\right) du = \frac{1}{12n}y^{1/2}\psi(y^{1/2})^2 - \frac{1}{144n}y^{1/2} + C(n) + O(1)$$

with $n \leq y^{1/2}$ and

$$C(n) = -\frac{1}{16}n\psi_1\left(\frac{1}{n}\right) + \frac{1}{48}n^{1/2} \int_{1/n}^\infty u^{-3/2}\psi_1(u)du. \tag{2.2}$$

The proof of the lemma is complete. □

In order to prove our theorems, we make use of the representation of the error term $P(x)$ as a sum of the ψ -function. We prove the following lemma.

Lemma 6. *Let χ be the primitive Dirichlet character (mod 4). We have*

$$P(x) = -4 \sum_{n \leq x} \chi(n)\psi\left(\frac{x}{n}\right) + 2 \sum_{n \leq x} \chi(n) - 2 + O(x^{-1}).$$

We also have that the function $P(x)$ can be written down alternatively as

$$\begin{aligned} P(x) &= -4 \sum_{n \leq x^{1/2}} \chi(n)\psi\left(\frac{x}{n}\right) + 4\psi(x^{1/2}) \sum_{n \leq x^{1/2}} \chi(n) \\ &+ 4 \sum_{n \leq x^{1/2}} \sum_{m \leq x/n} \chi(m) - 4x \int_{x^{1/2}}^\infty u^{-2} \sum_{n \leq u} \chi(n)du + O(x^{-1}). \end{aligned}$$

PROOF. It is easy to see that

$$\begin{aligned}
 P(x) &= -4x \sum_{n>x} \chi(n)n^{-1} - 4 \sum_{n\leq x} \chi(n)\psi\left(\frac{x}{n}\right) - 2 \sum_{n\leq x} \chi(n) \\
 &= -4 \sum_{n\leq x} \chi(n)\psi\left(\frac{x}{n}\right) + 2 \sum_{n\leq x} \chi(n) - 4x \int_x^\infty u^{-2} \sum_{n\leq u} \chi(n)du.
 \end{aligned}$$

Consider the integral in the above formula. We apply the fact

$$\sum_{n\leq y} \chi(n) = \frac{1}{2} - \psi\left(\frac{y-1}{4}\right) + \psi\left(\frac{y-3}{4}\right) \tag{2.3}$$

(see [3, Lemma 4.7]) to this formula to obtain

$$\int_x^\infty u^{-2} \sum_{n\leq u} \chi(n)du = \frac{1}{2}x^{-1} + O(x^{-2}).$$

This completes the proof of the first assertion of the lemma.

Next we prove the second assertion of this lemma. We see that

$$\begin{aligned}
 \sum_{n\leq x} \chi(n)\psi\left(\frac{x}{n}\right) &= \sum_{n\leq x^{1/2}} \chi(n)\psi\left(\frac{x}{n}\right) + \sum_{x^{1/2}<n\leq x} \chi(n)\psi\left(\frac{x}{n}\right) \\
 &= \sum_{n\leq x^{1/2}} \chi(n)\psi\left(\frac{x}{n}\right) + \frac{1}{2} \sum_{n\leq x} \chi(n) - \psi(x^{1/2}) \sum_{n\leq x^{1/2}} \chi(n) \\
 &\quad - \sum_{n\leq x^{1/2}} \sum_{m\leq x/n} \chi(m) + \frac{1}{2} + x \int_{x^{1/2}}^\infty u^{-2} \sum_{n\leq u} \chi(n)du + O(x^{-1})
 \end{aligned}$$

by (2.3). By substituting this formula into the first formula of this lemma, the second formula can be proved, and the proof of this lemma is complete. □

We will use the first formula in Lemma 6 to prove Theorem 1, and use the second one to prove Theorem 2.

3. Proof of Theorem 1

Here we prove Theorem 1. In Lemma 1, we put $k = 2$, $f(n) = r(n)$, namely $g(x) = \pi x$ and $E(x) = P(x)$. Then we have

$$\begin{aligned} \sum_{n \leq x} P(n)^2 &= \int_1^x P(u)^2 du - \pi x - \frac{1}{\pi} x^{3/4} \sum_{n=1}^{\infty} r(n) n^{-5/4} \sin \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right) \\ &\quad - 2\pi \int_1^x \psi(u) P(u) du + O(x^{2/3}) \end{aligned} \quad (3.1)$$

by formula (1.3). Now we consider the integral $\int_1^x \psi(u) P(u) du$ in (3.1). By substituting the first formula in Lemma 6, and then by using Lemma 2, the Pólya–Vinogradov inequality, the estimate $P(x) = O(x^{1/3})$ and Lemma 6 once again, we have

$$\begin{aligned} \int_1^x \psi(u) P(u) du &= -4 \int_1^x \psi(u) \sum_{n \leq u} \chi(n) \psi \left(\frac{u}{n} \right) du + O(\log x) \\ &= -\frac{1}{3} x \sum_{n \leq x} \chi(n) n^{-1} - 2 \left(\psi(x)^2 - \frac{1}{4} \right) \sum_{n \leq x} \chi(n) \psi \left(\frac{x}{n} \right) \\ &\quad + O(\log x) = -\frac{1}{12} \pi x + O(x^{1/3}). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \sum_{n \leq x} P(n)^2 &= \int_1^x P(u)^2 du + \frac{\pi(\pi - 6)}{6} x \\ &\quad - \frac{1}{\pi} x^{3/4} \sum_{n=1}^{\infty} r(n) n^{-5/4} \sin \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right) + O(x^{2/3}). \end{aligned}$$

The O -estimate of the error term in Theorem 1 can be followed easily from this formula. Thus, to complete the proof of Theorem 1, it suffices to show that

$$-\sum_{n=1}^{\infty} r(n) n^{-5/4} \sin \left(2\pi \sqrt{nx} + \frac{\pi}{4} \right) = \Omega_{\pm}(1). \quad (3.2)$$

To prove these estimates, we apply the methods used in [4, Section 6] and

[10, Section 4]. Let $J(u)$ be the function defined by

$$J(u) = - \sum_{n=1}^{\infty} r(n)n^{-5/4} \sin \left(u\sqrt{n} + \frac{\pi}{4} \right).$$

We first prove the Ω_+ -estimate in (3.2). For this purpose, it suffices to show that

$$\limsup_{u \rightarrow \infty} J(u) > 0. \tag{3.3}$$

Let M be a sufficiently large positive integer, and $\delta > 0$ a given number satisfying $\delta < M^{-1/2}$. We write the integer n with $n \leq M$ as $n = \nu^2 q$ with a square-free integer q . Then by using Kronecker's approximation theorem, there exist arbitrarily large numbers u satisfying

$$\left| \frac{1}{2\pi} u\sqrt{q} - \frac{1}{2} - m_q \right| < \delta_q$$

for certain integers m_q and a positive real number δ_q with $\delta_q < \delta$ (cf. [10, p. 408]). Hence we deduce that, for $n \leq M$,

$$\sin \left(u\sqrt{n} + \frac{\pi}{4} \right) = \lambda_n \sin \frac{\pi}{4} + O(\delta\sqrt{n})$$

with $\lambda_n = 1$ if $n \equiv 0 \pmod{4}$ and $\lambda_n = -1$ otherwise. Therefore we obtain

$$\limsup_{u \rightarrow \infty} J(u) \geq -\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \lambda_n r(n)n^{-5/4} + O(M^{-1/4}). \tag{3.4}$$

Now we consider the value of the infinite series $\sum_{n=1}^{\infty} \lambda_n r(n)n^{-5/4}$. Since $r(4n) = r(n)$ for any natural number n , we have

$$\sum_{n=1}^{\infty} \lambda_n r(n)n^{-s} = (2^{1-2s} - 1) \sum_{n=1}^{\infty} r(n)n^{-s} \tag{3.5}$$

for $\Re s > 1$. Hence by substituting $s = 5/4$ into (3.5), we see that the value of the infinite series on the right-hand side of (3.4) is negative, and therefore we obtain (3.3). The proof of the Ω_+ -estimate is complete.

Next we prove the Ω_- -estimate of $J(u)$, namely we show that

$$\liminf_{u \rightarrow \infty} J(u) < 0. \tag{3.6}$$

Similarly to the above, we apply Kronecker’s theorem to find u such that

$$\left| \frac{1}{2\pi}u\sqrt{q} - m_q \right| < \delta_q$$

with the same notation as above, so that

$$\sin \left(u\sqrt{n} + \frac{\pi}{4} \right) = \sin \frac{\pi}{4} + O(\delta\sqrt{n}).$$

By using this inequality, we obtain (3.6). The proof of the Ω_- -estimate is complete, and therefore (3.2) is established. This completes the proof of Theorem 1.

We note that the error term in (1.6) can be replaced by $\Omega_{\pm}(x^{3/4})$, that is, the error estimate in (1.6) is best possible. This result is a direct consequence of formulas (1.5) and (3.2).

4. Proof of Theorem 2

By Lemma 1 and (1.4), we have

$$\sum_{n \leq x} P(n)^3 = \int_1^x P(u)^3 du + \frac{3}{2} \pi A_1 x^{3/2} - 3\pi \int_1^x \psi(u) P(u)^2 du + O(x \log^2 x),$$

where A_1 is the coefficient of the main term in formula (1.4). Now we treat the second integral on the right-hand side of the above formula. We see from the second formula in Lemma 6 that

$$\begin{aligned} P(u)^2 &= 16 \sum_{m_1, m_2 \leq u^{1/2}} \chi(m_1 m_2) \psi \left(\frac{u}{m_1} \right) \psi \left(\frac{u}{m_2} \right) \\ &\quad - 32\psi(u^{1/2}) \sum_{m, n \leq u^{1/2}} \chi(mn) \psi \left(\frac{u}{m} \right) \\ &\quad + 16 \left(\sum_{n \leq u^{1/2}} \sum_{m \leq u/n} \chi(m) \right)^2 - 32 \sum_{m, n_1 \leq u^{1/2}} \chi(m) \psi \left(\frac{u}{m} \right) \sum_{n_2 \leq u/n_1} \chi(n_2) \\ &\quad + 32\psi(u^{1/2}) \sum_{m, n_1 \leq u^{1/2}} \chi(m) \sum_{n_2 \leq u/n_1} \chi(n_2) - 32uI(u)\psi(u^{1/2}) \sum_{n \leq u^{1/2}} \chi(n) \end{aligned}$$

$$\begin{aligned}
 & - 32uI(u) \sum_{n \leq u^{1/2}} \sum_{m \leq u/n} \chi(n) + 32uI(u) \sum_{n \leq u^{1/2}} \chi(n) \psi\left(\frac{u}{n}\right) \\
 & + 16u^2I(u)^2 + O(1) = \sum_{j=1}^9 Q_j(u) + O(1),
 \end{aligned}$$

say, where

$$I(u) = \int_{u^{1/2}}^{\infty} t^{-2} \sum_{a \leq t} \chi(a) dt.$$

And we see that

$$\int_1^x \psi(u)P(u)^2 du = \sum_{j=1}^9 \int_1^x \psi(u)Q_j(u) du + O(x).$$

It is easy to see that $\int_1^x \psi(u)Q_1(u) du = O(x)$ by Lemma 3, and $\int_1^x \psi(u)Q_6(u) du = O(x)$ by Lemmas 4 and 5, integration by parts, the Pólya–Vinogradov inequality and the formula $I(u) = u^{-1/2}/2 + O(u^{-1})$.

Consider the case $j = 2$. We have

$$\begin{aligned}
 & \int_1^x \psi(u)Q_2(u) du \\
 & = -32 \int_1^x \psi(u)\psi(u^{1/2}) \left\{ \sum_{m \leq n \leq u^{1/2}} \sum_{n \leq m \leq u^{1/2}} \right\} \chi(mn) \psi\left(\frac{u}{m}\right) du \\
 & \quad + 32 \int_1^x \psi(u)\psi(u^{1/2}) \sum_{m \leq u^{1/2}} \chi(m)^2 \psi\left(\frac{u}{m}\right) du \\
 & = -32\{Q_{21} + Q_{22} - Q_{23}\},
 \end{aligned}$$

say. We have by Lemma 5 and the Pólya–Vinogradov inequality that

$$\begin{aligned}
 Q_{21} & = \sum_{n \leq x^{1/2}} \chi(n) \sum_{m \leq n} \chi(m) \int_{n^2}^x \psi(u)\psi(u^{1/2})\psi\left(\frac{u}{m}\right) du \\
 & = \sum_{n \leq x^{1/2}} \chi(n) \sum_{m \leq n} \chi(m) \left(\frac{1}{12m} x^{1/2} \psi(x^{1/2})^2 - \frac{1}{144m} x^{1/2} \right) + O(x) \\
 & = O(x),
 \end{aligned}$$

since $\sum_{m \leq y} \chi(m)m^{-1} = O(1)$. Similarly we have $Q_{22} = O(x)$ and $Q_{23} = O(x^{1/2} \log x)$. Therefore we obtain $\int_1^x \psi(u)Q_2(u)du = O(x)$.

Consider the case $j = 9$. Since (2.1) can be rewritten as

$$\int_1^y \psi_1(u)du = -\frac{1}{12}y + O(1)$$

uniformly in $y \geq 1$, we have

$$I(u) = \frac{1}{2}u^{-1/2} + 4u^{-1}\psi_1\left(\frac{u^{1/2}-1}{4}\right) - 4\psi_1\left(\frac{u^{1/2}-3}{4}\right) + O(u^{-3/2}).$$

Hence by applying this formula and integration by parts, we have

$$\begin{aligned} \int_1^x \psi(u)Q_9(u)du &= 64 \int_1^x u^{1/2}\psi(u)\psi_1\left(\frac{u^{1/2}-1}{4}\right) du \\ &\quad - 64 \int_1^x u^{1/2}\psi(u)\psi_1\left(\frac{u^{1/2}-3}{4}\right) du + O(x). \end{aligned}$$

As for the first integral on the right-hand side in the above, by using the definition of $\psi_1(y)$ and by interchanging the integrations we have

$$\begin{aligned} \int_1^x u^{1/2}\psi(u)\psi_1\left(\frac{u^{1/2}-1}{4}\right) du &= \int_1^x u^{1/2}\psi(u) \int_1^{(u^{1/2}-1)/4} \psi(t)dt du \\ &= - \int_1^{(x^{1/2}-1)/4} \psi(t) \int_1^{(4t+1)^2} u^{1/2}\psi(u)du dt + O(x^{1/2}) = O(x). \end{aligned}$$

Similarly, we can see that second one can be also estimated by $O(x)$, and we have $\int_1^x \psi(u)Q_9(u)du = O(x)$.

For $j = 7$, we have

$$\begin{aligned} \int_1^x \psi(u)Q_7(u)du &= -32 \left\{ \int_1^x uI(u)\psi(u) \sum_{m,n \leq u^{1/2}} \chi(m)du \right. \\ &\quad \left. + \int_1^x uI(u)\psi(u) \sum_{n \leq u^{1/2}} \sum_{u^{1/2} < m \leq u/n} \chi(m)du \right\}. \quad (4.1) \end{aligned}$$

Since $\psi_1(m) = 0$ for any natural number m , we have by using the Pólya-Vinogradov inequality and the similar method used in the estimate of the

case $j = 9$ that the first part on the right-hand side in (4.1) is estimated by $O(x)$. In the second one, we see that this part is equal to

$$\begin{aligned}
 &= 4 \sum_{m \leq x^{1/2}} \chi(m) \sum_{n \leq m} \int_{mn}^{m^2} \psi(u) \left\{ \psi_1 \left(\frac{u^{1/2} - 1}{4} \right) - \psi_1 \left(\frac{u^{1/3} - 3}{4} \right) \right\} du \\
 &+ 4 \sum_{x^{1/2} < m \leq x} \chi(m) \sum_{n \leq m} \int_{mn}^x \psi(u) \left\{ \psi_1 \left(\frac{u^{1/2} - 1}{4} \right) - \psi_1 \left(\frac{u^{1/3} - 3}{4} \right) \right\} du + O(x)
 \end{aligned}$$

by partial summation, (2.1) and the Pólya–Vinogradov inequality. Now we treat the part $\sum_{m \leq x^{1/2}} \chi(m) \sum_{n \leq m} \int_{mn}^{m^2} \psi(u) \psi_1((u^{1/2} - 1)/4) du$. By substituting the definition of $\psi_1(u)$ and by interchanging the summations and integrations, we see that this part equal to

$$\begin{aligned}
 &= \sum_{n \leq x^{1/2}} \sum_{n \leq m \leq x^{1/2}} \chi(m) \int_{(m^{1/2}n^{1/2}-1)/4}^{(m-1)/4} \psi(t) \psi_1((4t + 1)^2) dt + O(x) \\
 &= \sum_{n \leq x^{1/2}} \int_{(n^{1/2}x^{1/4}-1)/4}^{(x^{1/2}-1)/4} \psi(t) \psi_1((4t + 1)^2) \sum_{4t+1 \leq m \leq x^{1/2}} \chi(m) dt + O(x) \\
 &= O(x).
 \end{aligned}$$

We also see that the other three part can be estimated by $O(x)$ by applying the same argument. Hence we have $\int_1^x \psi(u) Q_7(u) du = O(x \log x)$.

Consider the case $j = 3$. We divide this function as

$$\begin{aligned}
 \int_1^x \psi(u) Q_3(u) du &= 32 \sum_{n_2 \leq x^{1/2}} \sum_{n_1 \leq n_2} \int_{n_2^2}^x \psi(u) \sum_{m_1 \leq u/n_1} \chi(m_1) \sum_{m_2 \leq u/n_2} \chi(m_2) du \\
 &\quad - 16 \sum_{n \leq x^{1/2}} \int_{n^2}^x \psi(u) \left(\sum_{m \leq u/n} \chi(m) \right)^2 du \\
 &= 32Q_{31} - 16Q_{32},
 \end{aligned}$$

say. By interchanging the summations and integration and by using the Pólya–Vinogradov inequality, we can see easily that $Q_{32} = O(x \log x)$.

As for Q_{31} , we have

$$\begin{aligned} Q_{31} &= \sum_{n_2 \leq x^{1/2}} \sum_{n_1 \leq n_2} \sum_{n_2 \leq m_2 \leq x/n_2} \chi(m_2) \int_{m_2 n_2}^x \psi(u) \sum_{m_1 \leq u/n_1} \chi(m_1) du \\ &= \psi_1(x) \sum_{n_2 \leq x^{1/2}} \sum_{n_1 \leq n_2} \left(\sum_{n_2 \leq m_2 \leq x/n_2} \chi(m_2) \right) \left(\sum_{m_1 \leq x/n_1} \chi(m_1) \right) \\ &= O(x), \end{aligned}$$

since the third and the fourth sums are independent of each other and both of them can be estimated by $O(1)$. By collecting these two estimates we obtain $\int_1^x \psi(u) Q_3(u) du = O(x \log x)$.

Consider the case $j = 4$. We have

$$\begin{aligned} &\int_1^x \psi(u) Q_4(u) du \\ &= -32 \int_1^x \psi(u) \left\{ \sum_{m \leq n_1 \leq u^{1/2}} \sum_{n_1 \leq m \leq u^{1/2}} + \sum_{n_1 \leq m \leq u^{1/2}} \sum_{n_1 \leq m \leq u^{1/2}} \right\} \chi(m) \psi\left(\frac{u}{m}\right) \sum_{n_2 \leq u/n_1} \chi(n_2) du \\ &\quad + 32 \int_1^x \psi(u) \sum_{m \leq u^{1/2}} \chi(m) \psi\left(\frac{u}{m}\right) \sum_{n_2 \leq u/m} \chi(n_2) du \\ &= -32 \{ Q_{41} + Q_{42} - Q_{43} \}, \end{aligned}$$

say. We have by Lemma 2, partial summation and the Pólya–Vinogradov inequality that $Q_{43} = O(x \log x)$. As for Q_{41} , we have

$$\begin{aligned} Q_{41} &= \sum_{n_1 \leq x^{1/2}} \sum_{m \leq n_1} \chi(m) \sum_{n_2 \leq n_1} \int_{n_1^2}^x \psi(u) \psi\left(\frac{u}{m}\right) du \\ &\quad + \sum_{n_1 \leq x^{1/2}} \sum_{m \leq n_1} \chi(m) \sum_{n_1 < n_2 \leq x/n_1} \chi(n_2) \int_{n_1 n_2}^x \psi(u) \psi\left(\frac{u}{m}\right) du \\ &= Q_{41}^{(1)} + Q_{41}^{(2)}, \end{aligned}$$

say. In $Q_{41}^{(1)}$, substituting the explicit formula in Lemma 2, we have

$$Q_{41}^{(1)} = \frac{1}{12} x \sum_{n_1 \leq x^{1/2}} \left(\sum_{m \leq n_1} \chi(m) m^{-1} \right) \left(\sum_{n_2 \leq n_1} \chi(n_2) \right)$$

$$-\frac{1}{12} \sum_{n_1 \leq x^{1/2}} n_1^2 \left(\sum_{m \leq n_1} \chi(m)m^{-1} \right) \left(\sum_{n_2 \leq n_1} \chi(n_2) \right) + O(x).$$

Also we have

$$\begin{aligned} Q_{41}^{(2)} &= \frac{1}{12} x \sum_{n_1 \leq x^{1/2}} \sum_{m \leq n_1} \chi(m)m^{-1} \sum_{n_1 < n_2 \leq x/n_1} \chi(n_2) \\ &\quad - \frac{1}{12} \sum_{n_1 \leq x^{1/2}} n_1 \sum_{m \leq n_1} \chi(m)m^{-1} \sum_{n_1 < n_2 \leq x/n_1} \chi(n_2)n_2 + O(x) \\ &= -Q_{41}^{(1)} + \frac{1}{12} \sum_{n_1 \leq x^{1/2}} n_1 \sum_{m \leq n_1} \chi(m)m^{-1} \int_{n_1}^{x/n_1} \sum_{n_2 \leq u} \chi(n_2)du + O(x). \end{aligned}$$

As for the innermost part of the final formula in the above, we get from (2.3) that

$$\int_{n_1}^{x/n_1} \sum_{n_2 \leq u} \chi(n_2)du = \frac{1}{2n_1}x - \frac{1}{2}n_1 + O(1).$$

By using this formula and

$$\sum_{m \leq y} \chi(m)m^{-1} = \frac{\pi}{4} + O(y^{-1}),$$

we have

$$\begin{aligned} Q_{41}^{(2)} &= -Q_{41}^{(1)} + \frac{\pi}{48} \sum_{n_1 \leq x^{1/2}} n_1 \left\{ \frac{1}{2n_1}x - \frac{1}{2}n_1 \right\} + O(x \log x) \\ &= -Q_{41}^{(1)} + \frac{\pi}{144}x^{3/2} + O(x \log x). \end{aligned}$$

Hence we obtain $Q_{41} = \pi x^{3/2}/144 + O(x \log x)$. Similarly, we have

$$\begin{aligned} Q_{42} &= \sum_{m \leq x^{1/2}} \chi(m) \sum_{n_1 \leq m} \sum_{n_2 \leq m^2/n_1} \chi(n_2) \int_{m^2}^x \psi(u)\psi\left(\frac{u}{m}\right) du \\ &\quad + \sum_{m \leq x^{1/2}} \chi(m) \sum_{n_1 \leq m} \sum_{m^2/n_1 < n_2 \leq x/n_1} \chi(n_2) \int_{n_1 n_2}^x \psi(u)\psi\left(\frac{u}{m}\right) du \\ &= -\frac{1}{12} \sum_{m \leq x^{1/2}} \chi(m)m^{-1} \sum_{n_1 \leq m} n_1 \int_{m^2/n_1}^{x/n_1} \sum_{n_2 \leq u} \chi(n_2)du + O(x) \end{aligned}$$

by partial summation. Thus by applying (2.3), we obtain $Q_{42} = O(x)$, and therefore $\int_1^x \psi(u)Q_4(u)du = -2\pi x^{3/2}/9 + O(x \log x)$.

Consider the case $j = 8$. We have

$$\begin{aligned} \int_1^x \psi(u)Q_8(u)du &= 16 \sum_{n \leq x^{1/2}} \chi(n) \int_{n^2}^x u^{1/2} \psi(u) \psi\left(\frac{u}{n}\right) du \\ &+ 128 \sum_{n \leq x^{1/2}} \chi(n) \int_{n^2}^x \psi(u) \psi\left(\frac{u}{n}\right) \left\{ \psi_1\left(\frac{u^{1/2}-1}{4}\right) - \psi_1\left(\frac{u^{1/2}-3}{4}\right) \right\} du \\ &+ O(x). \end{aligned} \tag{4.2}$$

By using Lemma 2 and partial summation, we can see that the first part on the right-hand side in (4.2) is equal to $2\pi x^{3/2}/9 + O(x)$. As for the second one, we have by applying the similar method used in the case $j = 9$ that this part can be estimated by $O(x)$. Therefore we have $\int_1^x \psi(u)Q_8(u)du = 2\pi x^{3/2}/9 + O(x)$.

Finally, we treat the case $j = 5$. We have

$$\begin{aligned} &\int_1^x \psi(u)Q_5(u)du \\ &= 32 \sum_{m \leq x^{1/2}} \chi(m) \int_{m^2}^x \psi(u) \psi(u^{1/2}) \left(\sum_{n_1 \leq m} + \sum_{m < n_1 \leq u^{1/2}} \right) \sum_{n_2 \leq u/n_1} \chi(n_2) du \\ &= 32 \sum_{m \leq x^{1/2}} \chi(m) \left\{ \sum_{n_1 \leq m} \sum_{n_2 \leq m^2/n_1} \chi(n_2) \int_{m^2}^x \psi(u) \psi(u^{1/2}) du \right. \\ &+ \sum_{n_1 \leq m} \sum_{m^2/n_1 < n_2 \leq x/n_1} \chi(n_2) \int_{n_1 n_2}^x \psi(u) \psi(u^{1/2}) du \\ &+ \sum_{m < n_1 \leq x^{1/2}} \sum_{n_2 \leq n_1} \chi(n_2) \int_{n_1^2}^x \psi(u) \psi(u^{1/2}) du \\ &\left. + \sum_{m < n_1 \leq x^{1/2}} \sum_{n_1 < n_2 \leq x/n_1} \chi(n_2) \int_{n_1 n_2}^x \psi(u) \psi(u^{1/2}) du \right\}. \end{aligned} \tag{4.3}$$

We substitute the asymptotic formula in Lemma 4 into the integrals in (4.3). Then we see that the contribution of the error term in Lemma 4 is

$O(x \log x)$, which is admissible. The part coming from the explicit terms of the formula in Lemma 4 in the above (4.3) is

$$\begin{aligned}
 &= -\frac{8}{3} \sum_{m \leq x^{1/2}} \chi(m) \left\{ m \sum_{n_1 \leq m} \sum_{n_2 \leq m^2/n_1} \chi(n_2) + \sum_{n_1 \leq m} n_1^{1/2} \sum_{m^2/n_1 < n_2 \leq x/n_1} \chi(n_2) n_2^{1/2} \right. \\
 &\quad \left. + \sum_{m < n_1 \leq x^{1/2}} n_1 \sum_{n_2 \leq n_1} \chi(n_2) + \sum_{m < n_1 \leq x^{1/2}} n_1^{1/2} \sum_{n_1 < n_2 \leq x/n_1} \chi(n_2) n_2^{1/2} \right\} + O(x) \\
 &= \frac{4}{3} \sum_{m \leq x^{1/2}} \chi(m) \left\{ \sum_{n_1 \leq m} n_1^{1/2} \int_{m^2/n_1}^{x/n_1} u^{-1/2} \sum_{n_2 \leq u} \chi(n_2) du \right. \\
 &\quad \left. + \sum_{m < n_1 \leq x^{1/2}} n_1^{1/2} \int_{n_1}^{x/n_1} u^{-1/2} \sum_{n_2 \leq u} \chi(n_2) du \right\} + O(x) \tag{4.4}
 \end{aligned}$$

by the Pólya-Vinogradov inequality and partial summation. As for the first integral on the right-hand side of (4.4), we apply (2.3) to obtain

$$\int_{m^2/n_1}^{x/n_1} u^{-1/2} \sum_{n_2 \leq u} \chi(n_2) du = \left(\frac{x}{n_1}\right)^{1/2} - \left(\frac{m^2}{n_1}\right)^{1/2} + O(m^{-1} n_1^{1/2}).$$

Also we have

$$\int_{n_1}^{x/n_1} u^{-1/2} \sum_{n_2 \leq u} \chi(n_2) du = \left(\frac{x}{n_1}\right)^{1/2} - n_1^{1/2} + O(n_1^{-1/2}).$$

Hence we obtain that the final member of (4.4) can be estimated by $O(x)$. Therefore we obtain $\int_1^x \psi(u) Q_5(u) du = O(x)$.

Hence, by combining all the above estimates of $\int_1^x \psi(u) Q_j(u) du$ we obtain

$$\int_1^x \psi(u) P(u)^2 du = O(x \log x).$$

This completes the proof of Theorem 2.

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