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Randers spaces with reversible geodesics

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Abstract. A Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. The conditions for a Randers space to have reversible geodesics are obtained; this leads to a new simple proof of a well-known theorem giving necessary and sufficient conditions for a Randers space to be Berwald.

A geodesic in a Finsler space (where the Finsler function is *positively* homogeneous) should be thought of as an oriented path, that is, an imbedded one-dimensional submanifold with a sense of direction, or an equivalence class of curves determined up to reparametrization with positive derivative. There is in general no reason why a path which coincides with a geodesic as a point set but is traversed in the opposite direction should be a geodesic. If a Finsler space has the property that all of its geodesics remain geodesics when their orientation is reversed I shall say that the space has reversible geodesics. If the space is such that when $t \mapsto x^i(t)$ is a geodesic with constant Finslerian speed then $t \mapsto x^i(-t)$ is also a geodesic with constant Finslerian speed then I shall say that the space has strictly reversible geodesics.

A Riemannian space has strictly reversible geodesics; more generally, so has a Finsler space whose Finsler function is absolutely homogeneous. However, these examples do not by any means exhaust the possibilities for

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Finsler spaces with reversible geodesics. Consider a Randers space, with Finsler function

$$F(x,y) = \alpha + \beta = \sqrt{a_{ij}y^i y^j} + b_i y^i$$

where $a_{ij}b^i b^j < 1$. The equation for its geodesics with constant Finslerian speed is [1]

$$\begin{aligned} \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + a^{ij}(b_{j|k} - b_{k|j})\dot{x}^k\alpha(\dot{x}) \\ = \frac{1}{F}\dot{x}^i\big(a^{jl}b_l(b_{j|k} - b_{k|j})\dot{x}^k\alpha(\dot{x}) - b_{j|k}\dot{x}^j\dot{x}^k\big) \end{aligned}$$

where Γ^i_{jk} are the connection coefficients of the Levi–Civita connection of the Riemannian metric a_{ij}, and $b_{i|j}$ are the components of the covariant differential of b_i with respect to the same connection. It is clear that if $b_{j|i} = b_{i|j}$ then the Finslerian geodesics are projectively equivalent to the Riemannian ones, and so the Randers space is reversible, while if $b_{i|j} = 0$ the Finslerian geodesics are identical with the Riemannian ones, and the Randers space is strictly reversible. It is useful to recall that a Finsler space whose geodesics are projectively affine, as in the first case, is called a Douglas space, while one whose geodesics with constant Finslerian speed are affine, as in the second case, is called a Berwald space.

One aim of this note is to prove the converse to these results, namely that if a Randers space has reversible geodesics then $b_{j|i} = b_{i|j}$, and if it has strictly reversible geodesics then $b_{i|j} = 0$. These results generalize in some small way the well-known theorem that the vanishing of $b_{i|j}$ is the necessary and sufficient condition for a Randers space to be Berwald, and enable one to view that result from a new perspective, as well as providing a simple proof of it, different from the one in [3], that requires practically no calculation (which cannot be said of the derivation of the explicit geodesic spray coefficients of a Randers space quoted above).

I shall discuss the reversibility of geodesics in some generality. In fact the definitions of reversibility, and the corresponding conditions, can be formulated for any spray. Consider a spray

$$\Gamma = y^i \frac{\partial}{\partial x^i} + f^i(x,y)\frac{\partial}{\partial y^i}$$

where the f^i are positively homogeneous of degree 2 in the y^i. A curve $t \mapsto x^i(t)$ is a base integral curve of the spray if and only if it satisfies the equations $\ddot{x}^i = f^i(x, \dot{x})$. The curve $t \mapsto x^i(-t)$ is a base integral curve, up to reparametrization, if for some function $\varphi(t)$, $\ddot{x}^i = f^i(x, -\dot{x}) + \varphi\dot{x}^i$. Thus the spray is reversible, in the sense that the paths defined by its base integral curves remain so when their orientation is reversed, if and only if

$$f^i(x, -y) = f^i(x, y) + \lambda(x, y)y^i$$

for all $y^i \neq 0$, for some function λ, which must clearly be absolutely homogeneous of degree 1 in y^i. The condition for the spray to be strictly reversible, in the sense that for every base integral curve $t \mapsto x^i(t)$, the curve $t \mapsto x^i(-t)$ is also a base integral curve (without reparametrization) is that $f^i(x, -y) = f^i(x, y)$.

We can express the condition for reversibility in a rather more elegant form, as follows. Denote by ρ the 'reflection map' $(x, y) \mapsto (x, -y)$, and for any spray Γ set $\bar{\Gamma} = -\rho_*\Gamma$ (note the necessity of the minus sign: $\rho_*\Gamma$ is not a spray). Then

$$\bar{\Gamma} = y^i \frac{\partial}{\partial x^i} + f^i(x, -y)\frac{\partial}{\partial y^i},$$

so it is natural to call $\bar{\Gamma}$ the reverse of Γ. Then Γ is reversible if and only if it is projectively equivalent to its reverse, and strictly reversible if and only if the two are equal.

The concept of reversibility is a projective one; that is to say, if a spray is reversible so are all sprays projectively equivalent to it. In fact a spray is reversible if and only if its projective equivalence class is invariant under the map which takes a spray to its reverse.

Since we have to deal with projectively equivalent sprays, the following simple observations about the geodesic sprays of Finsler spaces will prove very useful. Let F be a Finsler function – by assumption, positively homogeneous, and strongly convex, so that its fundamental tensor g_{ij} is positive-definite, and in particular non-singular. The geodesics of F are the solutions $x^i(t)$, $y^i = \dot{x}^i$, of the Euler–Lagrange equation with Lagrangian F,

$$\frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0.$$

Because of the homogeneity of F these equations do not determine the curves $t \mapsto x^i(t)$ completely, but only up to sense-preserving reparametrization. This is a consequence of the fact that a vector (v^i) satisfies

$$v^j \frac{\partial^2 F}{\partial y^i \partial y^j} = 0$$

if and only if $v^i = ky^i$ for some scalar k; the fact that this quantity vanishes if $v^i = ky^i$ is due to the assumed homogeneity of F, while the fact that it vanishes only if $v^i = ky^i$ follows, via the non-singularity of g_{ij}, from the assumption of strong convexity. These observations may be presented in a different light. The Euler–Lagrange equation may be regarded as an equation for geodesic sprays Γ, in the form

$$\Gamma\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0.$$

Assuming, as before, that F is homogeneous and strongly convex we see that two sprays Γ, $\tilde{\Gamma}$ satisfy the equation if and only if $\tilde{\Gamma} = \Gamma + \lambda\Delta$ where Δ is the Liouville vector field, $\Delta = y^i\partial/\partial y^i$, and λ is homogeneous of degree 1 in y^i. That is to say, the geodesic sprays of F form a projective equivalence class of sprays, and a spray Γ belongs to this class if and only if it satisfies the Euler–Lagrange equation as written above.

The geodesic spray Γ with constant Finslerian speed is singled out from amongst all those satisifying the Euler–Lagrange equation by the additional condition that $\Gamma(F) = 0$. I shall speak of 'a geodesic spray' when I mean any spray of the projective class of solutions of the Euler–Lagrange equation for F, and 'the geodesic spray' when I mean the one with constant Finslerian speed. With this choice, if $\tilde{\Gamma} = \Gamma + \lambda\Delta$ is a geodesic spray of F, and therefore projectively equivalent to the geodesic spray Γ, then

$$\tilde{\Gamma}(F) = \Gamma(F) + \lambda\Delta(F) = \lambda F,$$

so we have an explicit expression for λ, namely

$$\lambda = \frac{\tilde{\Gamma}(F)}{F}.$$

(Though it may not be immediately obvious, these results are essentially equivalent to those given by Shen in [5], Theorem 12.2.6. See also [6] for an intrinsic formulation of this and equivalent conditions, originally due to RAPCSÁK [4].)

It follows that a Finsler space has reversible geodesics if and only if

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0,$$

where $\bar{\Gamma}$ is the reverse of a geodesic spray Γ; it will be enough to check reversibility when Γ is the geodesic spray. The Finsler space has strictly reversible geodesics if and only if $\bar{\Gamma} = \Gamma$, where Γ is the geodesic spray.

Now if F is any Finsler function, and $\bar{F}$ is defined by $\bar{F}(x, y) = F(x, -y)$ then $\bar{F}$ is also a Finsler function; it is certainly positively homogeneous in y^i, and its fundamental tensor $\bar{g}_{ij}$ is given by $\bar{g}_{ij}(x, y) = g_{ij}(x, -y)$ (where g_{ij} is the fundamental tensor of F), so $\bar{g}_{ij}$, like g_{ij}, is everywhere positive definite. The geodesic spray $\bar{\Gamma}$ of $\bar{F}$ is just the reverse of the geodesic spray of F.

We can now apply these observations to a Randers space, with

$$F = \alpha + \beta = \sqrt{a_{ij} y^i y^j} + b_i y^i,$$

to show that the necessary and sufficient condition for the space to have reversible geodesics is that $b_{k|j} = b_{j|k}$, and the necessary and sufficient condition for the space to have strictly reversible geodesics is that $b_{j|k} = 0$. Of course, another way of saying that $b_{k|j} = b_{j|k}$ is that the 1-form $b = b_i dx^i$ is closed. Given that b is closed, another way of saying that $b_{j|k} = 0$ is that the function $\beta = b_i y^i$ is a first integral of the geodesic flow of the Riemannian metric a_{ij}. So we may equivalently say that the necessary and sufficient condition for the Randers space to have reversible geodesics is that b is closed, and the necessary and sufficient condition for its geodesics to be strictly reversible is that b is closed and $\beta = b_i y^i$ is a first integral of the Riemannian geodesic flow.

These results about reversibility of geodesics in a Randers space are in fact particular cases (though probably the most interesting ones) of more general, but similar, results concerning Randers changes. Let F_0 be a Finsler function, and $b = b_i dx^i$ a 1-form on the base manifold such that

$$\sup_{F_0(y)=1} |b_i y^i| < 1;$$

then $F(x,y) = F_0(x,y) + b_i(x)y^i$ is again a Finsler function, and the process of transforming F_0 to F is called a Randers change (see, for example, [5] and [6]). Suppose that F_0 is *absolutely* homogeneous; then the necessary and sufficient condition for F to have reversible geodesics is that b is closed, and the necessary and sufficient condition for the geodesics to be strictly reversible is that b is closed and $\beta = b_i y^i$ is a first integral of the geodesic flow of F_0. I shall devote the rest of this note to proving these assertions.

The necessary and sufficient condition for F to have reversible geodesics is that

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$$

where $\bar{\Gamma}$ is the reverse of Γ, the geodesic spray of F; moreover, $\bar{\Gamma}$ is the geodesic spray of $\bar{F}$. Now $F = F_0 + \beta$ where F_0 is absolutely homogeneous. Then $\bar{F} = F_0 - \beta$, so $F = \bar{F} + 2\beta$. Since

$$\bar{\Gamma}\left(\frac{\partial \bar{F}}{\partial y^i}\right) - \frac{\partial \bar{F}}{\partial x^i} = 0,$$

we have

$$\begin{aligned}\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} &= 2\left(\bar{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right)\\ &= 2\left(\bar{\Gamma}(b_i) - \frac{\partial b_j}{\partial x^i} y^j\right)\\ &= 2\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j.\end{aligned}$$

It follows that if F is obtained by a Randers change from an absolutely homogeneous Finsler function then it is geodesically reversible if and only if the 1-form defining the Randers change is closed.

Notice that for any spray $\tilde{\Gamma}$,

$$\tilde{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i} = \left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j.$$

So for the geodesic spray Γ_0 of the 'reference' Finsler function F_0

$$\Gamma_0\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = \Gamma_0\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i} = \left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j,$$

from which it follows that b being closed is also the necessary and sufficient condition for Γ_0 to be projectively equivalent to Γ; if b is closed we have $\Gamma = \Gamma_0 - \lambda\Delta$ with $\lambda = \Gamma_0(F)/F = \Gamma_0(\beta)/(F_0 + \beta)$, and similarly $\bar{\Gamma} = \Gamma_0 - \mu\Delta$ with $\mu = -\Gamma_0(\beta)/(F_0 - \beta)$. Thus given that b is closed, the condition for F to have strictly reversible geodesics, so that $\Gamma = \bar{\Gamma}$, is that $\Gamma_0(\beta)/(F_0 + \beta) = -\Gamma_0(\beta)/(F_0 - \beta)$, or $\Gamma_0(\beta) = 0$; then $\Gamma = \Gamma_0 = \bar{\Gamma}$. In fact, when b is closed the geodesic sprays of both F and $\bar{F}$ are projectively equivalent to the (strictly reversible) geodesic spray of F_0; and when $\Gamma_0(\beta) = 0$ the two geodesic sprays coincide with the geodesic spray of F_0. (Projective equivalence under a Randers change is discussed in [5] and [6]. The condition on b was originally found by Hashiguchi and Ichijyō [2].)

The necessary and sufficient conditions for a Randers space to be Douglas or Berwald are simple corollaries of the results just obtained. Those results apply of course to a Randers space, with F_0 the Riemannian Finsler function. If a Randers space is a Douglas space, so that its geodesic spray is projectively equivalent to an affine spray, then the geodesics of the Randers space must be reversible, so b must be closed. If a Randers space is Berwald, so that its geodesic spray is affine, its geodesics must be strictly reversible, so β must be a first integral of the Riemannian geodesic flow. In each case, the affine spray is the Riemannian geodesic spray.

Finally, I shall point out how an example of Shen's [5] provides a memorable illustration of a Randers space with non-reversible geodesics. We start with the spray Γ on R^2 given by

$$\Gamma = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \alpha\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right), \quad \alpha = \sqrt{u^2 + v^2},$$

where now (x, y) are the base coordinates and (u, v) the fibre coordinates. This spray is manifestly non-reversible. Its base integral curves are in fact circles of constant radius 1, traversed in the anti-clockwise sense. To see this, note first that $\Gamma(u^2 + v^2) = 0$, which means that $\dot{x}^2 + \dot{y}^2$ is constant on any base integral curve. For a point describing the circle with centre (a, b) and radius 1, with constant speed α in the anti-clockwise sense, we have $(x-a)^2 + (y-b)^2 = 1$; $\dot{x}(x-a) + \dot{y}(y-b) = 0$; $\dot{x} = -\alpha(y-b)$, $\dot{y} = \alpha(x-a)$ with $\alpha = \sqrt{\dot{x}^2 + \dot{y}^2}$ constant – note that at $x = a + 1$, $y = b$ we have $\dot{x} = 0$, $\dot{y} = \alpha > 0$ as is required for the motion to be anti-clockwise; and finally $\ddot{x} = -\alpha\dot{y}$, $\ddot{y} = \alpha\dot{x}$, so the circle is indeed a base integral curve of Γ.

Consider now the function

$$F(x, y, u, v) = \sqrt{u^2 + v^2} + \tfrac{1}{2}(yu - xv) = \alpha + \beta.$$

I show that Γ is a geodesic spray of this function, by calculating the Euler–Lagrange expressions, using the fact that (due to rotational symmetry)

$$\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} \right)(\alpha) = 0;$$

we easily find that

$$\Gamma\left(\frac{\partial F}{\partial u}\right) - \frac{\partial F}{\partial x} = \Gamma\left(\frac{u}{\alpha} + \tfrac{1}{2}y\right) + \tfrac{1}{2}v = -v + \tfrac{1}{2}v + \tfrac{1}{2}v = 0$$

$$\Gamma\left(\frac{\partial F}{\partial v}\right) - \frac{\partial F}{\partial y} = \Gamma\left(\frac{v}{\alpha} - \tfrac{1}{2}x\right) - \tfrac{1}{2}u = u - \tfrac{1}{2}u - \tfrac{1}{2}u = 0.$$

Now F is a Finsler function on the open disc $x^2 + y^2 < 4$; so we have here an example of a Finsler function whose geodesics are unit circles – but always traversed in the anti-clockwise sense.

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