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Conformal changes of special Finsler spaces with a generalized Cartan connection

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Abstract. The purpose of the present paper is to investigate conformal changes of special Finsler spaces and to treat the Finsler spaces which are conformal to each other. Moreover we prove that a recurrent Finsler space (resp. a C^h-recurrent Finsler space) is closed by any conformal change.

1. Introduction

In the paper [4], M. Hashiguchi and Y. Ichijyō treated the conformal theory of generalized Berwald spaces and obtained the result:
A generalized Berwald space (esp. a Wagner space) remains to be a generalized Berwald space (esp. a Wagner space) by any conformal change. Therefore, we can say that the set of a generalized Berwald space is closed by any conformal change. So the notion of closed Finsler space by any conformal change causes the following problem: Are there any other Finsler spaces which are conformal to each other? In Section 4, new classes of Finsler spaces are given, which are also closed by any conformal change. Especially S. Bácsó [3] studied a special geodesic mapping Landsberg space into a P^*-Finsler space.

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An interesting example of a Finsler space which is characterized by some simple conditions imposed upon the curvature and torsion has been in the theory of Finsler geometry. These conditions are expressed by ([1], [2])

$$R_h{}^i{}_{jk} = 0, P_h{}^i{}_{jk} = 0, \quad \nabla^{h)}_m C_j{}^i{}_k = 0. \tag{1.1}$$

In the present paper, we consider special Finsler spaces (namely a recurrent space, C^h-recurrent space) having weaker condition than the above (1.1). A recurrent Finsler space was first introduced by A. Moór [8] as a generalized conception of Riemannian cases. On the other hand, M. Matsumoto [6] has studied a C^h-recurrent Finsler space in relation to a h-isotropic Finsler space. According to these papers, two dimensional Finsler space is always of recurrent curvature, and of C^h-recurrent.

2. Generalized Cartan connection and conformal change

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with the Finsler connection $F\Gamma = (F_j{}^i{}_k, N^i{}_k, V_j{}^i{}_k)$. Let $L(x, y)$ be a Finsler metric function, whose Finsler metric tensor g is given by $g_{ij} = \dot{\partial}_i \dot{\partial}_j (L^2/2)$. Then the h-covariant and v-covariant derivative of any tensor field $X^i(x, y)$ are defined as respectively

$$\begin{aligned} \nabla^{h)}_j X^i &= \delta_j X^i + X^r F_r{}^i{}_j, \\ \nabla^{v)}_j X^i &= \dot{\partial}_j X^i + X^r V_r{}^i{}_j, \end{aligned} \tag{2.1}$$

where $\delta_k = \partial_k - N^r{}_k \dot{\partial}_r$, $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_r = \partial/\partial y^r$.

Definition 2.1 ([2]). Suppose that a skew-symmetric tensor field $T_j{}^i{}_k$ is given in F^n. A Finsler connection is uniquely determined in F^n which satisfies the following five axioms:

$(C1)$ h-metrical: $\nabla^{h)} g = 0$,

$(C2)$ $(h)h$-torsion $T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j$,

$(C3)$ deflection $D = 0$,

$(C4)$ v-metrical: $\nabla^{v)}g = 0$,

$(C5)$ v-symmetric: $S^1 = 0$.

This connection is called *a generalized Cartan connection* (with respect to T) and denoted by $C\Gamma(T) = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$.

If the $(h)h$-torsion T is to vanish identically, then $C\Gamma(T)$ is just the Cartan connection $C\Gamma = (\Gamma_j{}^{*i}{}_k, G^i_j, C_j{}^i{}_k)$. The v-connection $C_j{}^i{}_k$ is common to $C\Gamma(T)$ and $C\Gamma$. Here we suppose that T is $(0)p$-homogenous as usual. A Finsler space with a skew-symmetric tensor T of $(1,2)$-type is called a *generalized Berwald space* (with respect to T), if the connection coefficients $F_j{}^i{}_k$ of $C\Gamma(T)$ are functions of position alone.

In the following, the symbol (j/k) denotes the interchange of indices j and k in the previous terms. The h-curvature tensor R^2 and the hv-curvature tensor P^2 of F^n with $C\Gamma(T)$ are given by

$$\begin{aligned} R_h{}^i{}_{jk} &= K_h{}^i{}_{jk} + C_h{}^i{}_r R^r{}_{jk}, \\ P_h{}^i{}_{jk} &= \dot{\partial}_k F_h{}^i{}_j + \nabla^{h)}_j C_h{}^i{}_k - C_h{}^i{}_r P^r{}_{jk}, \end{aligned} \tag{2.2}$$

where $R^i{}_{jk} = \delta_k N^i{}_j - \delta_j N^i{}_k$, $P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j$ and

$$K_h{}^i{}_{jk} = \delta_k F_h{}^i{}_j + F_h{}^r{}_j F_r{}^i{}_k - (j/k).$$

For later use, we introduce the following Proposition.

Proposition 2.1 ([4], [5]). *Let $C\Gamma(T) = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ be given in a space with a Finsler metric L. If for a conformal change $\alpha : L \mapsto \bar{L} = e^{\alpha(x)} L$ we put*

$$\begin{aligned} \bar{F}_j{}^i{}_k &= F_j{}^i{}_k + \delta^i_j \alpha_k, \\ \bar{N}^i{}_k &= N^i{}_k + y^i \alpha_k, \\ \bar{C}_j{}^i{}_k &= C_j{}^i{}_k, \end{aligned} \tag{2.3}$$

the coefficients $(\bar{F}_j{}^i{}_k, \bar{N}^i{}_k, \bar{C}_j{}^i{}_k)$ define a generalized Cartan connection in the space with the metric $\bar{L}$, where $\alpha_k = \partial_k \alpha$.

Throughout the following we shall denote quantities of $\bar{F}^n$ by putting bar. Moreover, we assume that $\alpha_k \neq 0$, because if $\alpha_k = 0$ the conformal change (2.3) is trivial. For a given $C\Gamma(T)$, we can see that a changed connection by any conformal change is also a generalized Cartan connection, which is denoted by $C\bar{\Gamma}(\bar{T})$.

It is well known that the C-tensor $C_j{}^i{}_k$ is conformally invariant. For $\bar{F}^n$ and its generalized Cartan connection $C\bar{\Gamma}(\bar{T})$, from (2.1) and (2.3) we have

$$\begin{aligned}\bar{\nabla}^{h)}_m \bar{C}_j{}^i{}_k &= \bar{\delta}_m \bar{C}_j{}^i{}_k - \bar{C}_r{}^i{}_k \bar{F}_j{}^r{}_m + \bar{C}_j{}^r{}_k \bar{F}_r{}^i{}_m - \bar{C}_j{}^i{}_r \bar{F}_k{}^r{}_m \\ &= \nabla^{h)}_m C_j{}^i{}_k - \alpha_m y^r \dot{\partial}_r C_j{}^i{}_k - \alpha_m C_j{}^i{}_k,\end{aligned}$$

where $\bar{\delta}_m = \partial_m - \bar{N}^r{}_m \dot{\partial}_r$. Using the homogeneity of C, from the above we get

$$\bar{\nabla}^{h)}_m \bar{C}_j{}^i{}_k = \nabla^{h)}_m C_j{}^i{}_k. \tag{2.4}$$

According to the paper [5], the h-curvature tensor R^2 and the hv-curvature tensor P^2 are invariant by any conformal change α, that is,

$$\bar{R}^2 = R^2, \qquad \bar{P}^2 = P^2. \tag{2.5}$$

Summarizing the above, we have

Proposition 2.2. *By any conformal change α, the h-curvature tensor R^2, the hv-curvature tensor P^2 and $\nabla^{h)}_m C_j{}^i{}_k$ are all invariant.*

3. Conformal changes of symmetric Finsler spaces

We consider a Finsler space F^n with a generalized Cartan connection $C\Gamma(T)$. In [7] R. B. Misra discussed a symmetric Finsler space and the existence of projective motion of the space. First we state following definition.

Definition 3.1. A Finsler space F^n is called a *symmetric space*, if its h-curvature tensor R^2 satisfies the relation

$$\nabla^{h)}_m R_h{}^i{}_{jk} = 0, \tag{3.1}$$

where $\nabla^{h)}$ denotes the h-covariant derivative with respect to $C\Gamma(T)$.

It is obvious that the class of symmetric Finsler spaces contain all Finsler spaces satisfying $R^2 = 0$. Now let us consider two different Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ with the same underlying manifold M^n. And we suppose that a conformal change $\alpha : L \mapsto \bar{L} = e^{\alpha(x)} L$ is given. First we shall prove

Theorem 3.1. *If a symmetric space F^n remains to be a symmetric space $\bar{F}^n$ by any conformal change ($\alpha_i(x) \neq 0$), then the h-curvature tensor R^2 vanishes.*

Proof. Let us assume that two Finsler metric L and $\bar{L}$ are given by any conformal change α. From (2.1), (2.3), and Proposition 2.2, the h-covariant derivative of $\bar{R}^2$ in $\bar{F}^n$ is given by

$$\begin{aligned}\bar{\nabla}^{h)}_m \bar{R}_h{}^i{}_{jk} &= \bar{\delta}_m \bar{R}_h{}^i{}_{jk} - \bar{R}_r{}^i{}_{jk} \bar{F}_h{}^r{}_m + \bar{R}_h{}^r{}_{jk} \bar{F}_r{}^i{}_m \\ &\quad - \bar{R}_h{}^i{}_{rk} \bar{F}_j{}^r{}_m - \bar{R}_h{}^i{}_{jr} \bar{F}_k{}^r{}_m \\ &= \nabla^{h)}_m R_h{}^i{}_{jk} - \alpha_m y^r \dot{\partial}_r R_h{}^i{}_{jk} - 2\alpha_m R_h{}^i{}_{jk}.\end{aligned} \tag{3.2}$$

Using Proposition 2.2 and $y^r \dot{\partial}_r R_h{}^i{}_{jk} = 0$, from (3.2) we get

$$\bar{\nabla}^{h)}_m \bar{R}_h{}^i{}_{jk} = \nabla^{h)}_m R_h{}^i{}_{jk} - 2\alpha_m R_h{}^i{}_{jk}. \tag{3.3}$$

Assuming that a symmetric space F^n remains to be a symmetric space $\bar{F}^n$ by any conformal change α, from (3.3) we obtain $R_h{}^i{}_{jk} = 0$ because $\alpha_m \neq 0$. This completes the proof. □

Definition 3.2. A Finsler space F^n is called *a P-symmetric space*, if its hv-curvature tensor P^2 satisfies the relation

$$\nabla^{h)}_m P_h{}^i{}_{jk} = 0. \tag{3.4}$$

In general every Finsler space with a vanishing hv-curvature tensor is a P-symmetric space, but the converse is not true. On a P-symmetric Finsler space, we will prove the following theorem.

Theorem 3.2. *If a P-symmetric space F^n remains to be a P-symmetric space $\bar{F}^n$ by any conformal change ($\alpha_i(x) \neq 0$), then the hv-curvature tensor P^2 vanishes.*

Proof. From (2.1), (2.3) and Proposition 2.2, we have directly

$$\bar{\nabla}^{h)}_m \bar{P}_h{}^i{}_{jk} = \nabla^{h)}_m P_h{}^i{}_{jk} - \alpha_m P_h{}^i{}_{jk}. \tag{3.5}$$

If a P-symmetric space F^n remains to be a P-symmetric space $\bar{F}^n$ by any conformal change α, from (3.5) we obtain $P_h{}^i{}_{jk} = 0$ because $\alpha_m \neq 0$. This completes the proof. □

From (2.4) we easily obtain the following theorem.

Theorem 3.3. *Let $F^n(C)$ be a Finsler space satisfying $\nabla^{h)}_m C_j{}^i{}_k = 0$. Then the space $F^n(C)$ is closed by any conformal change α.*

4. Conformal changes of recurrent Finsler spaces

In this section we shall deal with a recurrent Finsler space, which was first introduced by A. Moór [8]. By this paper F^2 is always of recurrent curvature, and F^3 is also of recurrent one if there exists an absolute parallelism of the line elements.

Definition 4.1. A Finsler space F^n is called *recurrent*, if the h-curvature tensor R^2 satisfies the relation

$$\nabla^{h)}_m R_h{}^i{}_{jk} = \phi_m R_h{}^i{}_{jk},$$

where $\phi_m = \phi_m(x, y)$ is a covariant vector field.

Now if we assume that $\bar{F}^n$ is recurrent with a recurrence vector $\bar{\phi}_m$, where F^n is arbitrary, that is, $\bar{\nabla}^{h)}_m \bar{R}_h{}^i{}_{jk} = \bar{\phi}_m \bar{R}_h{}^i{}_{jk}$. From (2.5) and (3.3) we have

$$\nabla^{h)}_m R_h{}^i{}_{jk} = (\bar{\phi}_m + 2\alpha_m) R_h{}^i{}_{jk}. \tag{4.1}$$

From this equation, we obtain $\nabla^{h)}_m R_h{}^i{}_{jk} = \phi_m R_h{}^i{}_{jk}$ by putting

$$\phi_m = \bar{\phi}_m + 2\alpha_m,$$

which means that the set of the recurrent Finsler space is closed by any conformal change. Thus we have

Theorem 4.1. *A recurrent Finsler space with a recurrent vector ϕ_m remains to be a recurrent Finsler space with a recurrent vector $\phi_m - 2\alpha_m$ by any conformal change α.*

Definition 4.2. A Finsler space F^n is called *P^h-recurrent*, if the *hv*-curvature tensor P^2 satisfies the relation

$$\nabla^{h)}_m P_h{}^i{}_{jk} = \phi_m P_h{}^i{}_{jk},$$

where $\phi_m = \phi_m(x, y)$ is a covariant vector field.

Next, we are concerned with a conformal change α, where F^n is an arbitrary but $\bar{F}^n$ is a P^h-recurrent Finsler space, that is, $\bar{\nabla}^{h)}_m \bar{P}_h{}^i{}_{jk} = \bar{\phi}_m \bar{P}_h{}^i{}_{jk}$. From (3.5) we obtain

$$\nabla^{h)}_m P_h{}^i{}_{jk} = (\bar{\phi}_m + \alpha_m) P_h{}^i{}_{jk}, \tag{4.2}$$

which easily leads to $\nabla^{h)}_m P_h{}^i{}_{jk} = \phi_m P_h{}^i{}_{jk}$ by putting

$$\phi_m = \bar{\phi}_m + \alpha_m.$$

Consequently we have

Theorem 4.2. *A P^h-recurrent Finsler space with a recurrent vector ϕ_m remains to be a P^h-recurrent Finsler space with a recurrent vector $\phi_m - \alpha_m$ by any conformal change α.*

Definition 4.3. A Finsler space F^n is called C^h-it recurrent, if the C-tensor satisfies the relation

$$\nabla^{h)}_m C_j{}^i{}_k = \phi_m C_j{}^i{}_k,$$

where $\phi_m = \phi_m(x, y)$ is a covariant vector field.

In the previous paper [6] M. Matsumoto treated a C^h-recurrent Finsler space. According to this paper, the C^h-recurrent space is reasonable only, because if a Finsler space F^n is C^v-recurrent (or C^0-recurrent), then F^n is essentially Riemannian. The author gave two notes. One is that any essentially Riemannian F^n is C^h-recurrent, other is that F^2 is also C^h-recurrent.

Now we assume that $\bar{F}^n$ is C^h-recurrent with a recurrent vector $\bar{\phi}_m$ by any conformal change α, where F^n is arbitrary. Since C-tensor is conformally invariant, from (2.4) we have

$$\nabla^{h)}_m C_j{}^i{}_k = \bar{\phi}_m C_j{}^i{}_k,$$

so that we get $\nabla^{h)}_m C_j{}^i{}_k = \phi_m C_j{}^i{}_k$ by putting $\phi_m = \bar{\phi}_m$. Thus we have

Theorem 4.3. *A C^h-recurrent Finsler space remains to be a C^h-recurrent Finsler space by any conformal change α.*

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