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A remark on a paper of L. Molnár

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Abstract. The purpose of this note is to prove the following result: Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping, such that $T(xyx) = T(x)yx$ holds for all pairs $x, y \in R$. In this case T is a left centralizer.

Throughout this note R will represent an associative ring. A ring R is 2-torsion free in case $2x = 0$ implies $x = 0$ for any $x \in R$. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. An additive mapping $T : R \to R$ is called left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of a right centralizer and a right Jordan centralizer should be self-explanatory. Obviously, any left centralizer is a left Jordan cenrtralizer. Zalar [2] has proved that any left Jordan centralizer on a 2-torsion free semiprime ring is a left centralizer. Molnar [1] has proved the following result: Let R be a 2-torsion free prime ring and let $T : R \to R$ be an additive mapping. If $T(xyx) = T(x)yx$ holds for every $x, y \in R$, then T is a left centralizer.

It is our aim in this note to prove the result below, which generalizes Molnar's result we have just mentioned above.

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Theorem. *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. If $T(xyx) = T(x)yx$ holds for every $x, y \in R$, then T is a left centralizer.*

Proof. The linearization of the relation

$$T(xyx) = T(x)yx, \quad x, y \in R \tag{1}$$

gives

$$T(xyz + zyx) = T(x)yz + T(z)yx, \quad x, y, z \in R.$$

For $z = x^2$ the relation above gives

$$T(xyx^2 + x^2yx) = T(x)yx^2 + T(x^2)yx, \quad x, y \in R. \tag{2}$$

On the other hand the substitution $xy + yx$ for y in the relation (1) gives

$$T(x^2yx + xyx^2) = T(x)yx^2 + T(x)xyx, \quad x, y \in R. \tag{3}$$

Subtracting (3) from (2), we arrive at

$$A(x)yx = 0, \quad x, y \in R, \tag{4}$$

where $A(x)$ stands for $T(x^2) - T(x)x$. It is our aim to prove that

$$A(x) = 0, \quad x \in R. \tag{5}$$

For this purpose we write in the relation (4) $xyA(x)$ for y, which gives $A(x)xyA(x)x = 0$, $x, y \in R$, whence it follows

$$A(x)x = 0, \quad x \in R, \tag{6}$$

by semiprimeness of R. Multiplying the relation (4) from the left side by x and from the right side by $A(x)$, we obtain $xA(x)yxA(x) = 0$, $x, y \in R$, which leads to

$$xA(x) = 0, \quad x \in R. \tag{7}$$

The linearization of the relation (6) gives

$$A(x)y + B(x, y)x + A(y)x + B(x, y)y = 0, \quad x, y \in R,$$

where $B(x, y)$ denotes $T(xy + yx) - T(x)y - T(y)x$. Putting in the above relation $-x$ for x and comparing the relation so obtained with the above relation we arrive at

$$A(x)y + B(x, y)x = 0, \quad x, y \in R.$$

Right multiplication of the above relation by $A(x)$ gives because of (7) $A(x)yA(x) = 0$, $x, y \in R$, whence it follows (5). We have therefore proved that $T(x^2) = T(x)x$ holds for all $x \in R$. In other words, T is a left Jordan centralizer. Now Proposition 1.4 in [2] completes the proof of the theorem. □

References

[1] L. Molnár, On centralizers of an H^*-algebra, *Publ. Math. Debrecen* **46**, 1–2 (1995), 89–95.

[2] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae* **32** (1991), 609–614.

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