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On co-hopfian groups

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Abstract. A group is called *co-hopfian* if it is not isomorphic with a proper subgroup. The aim of this paper is to obtain sufficient conditions for a group to be co-hopfian or non-co-hopfian. For example, it is shown that a reduced soluble minimax group which is abelian-by-nilpotent-by-finite, but not nilpotent-by-finite, cannot be co-hopfian. This leads to the construction of many finitely generated soluble coherent groups which are not polycyclic. On the other hand, examples of co-hopfian polycyclic groups which are not nilpotent-by-finite are given. In addition it is shown that a soluble-by-finite group satisfying the minimal condition on normal subgroups is co-hopfian.

1. Introduction

A group G is said to be *co-hopfian* if it is not isomorphic with a proper subgroup, i.e., if every injective endomorphism $\varphi : G \to G$ is an automorphism. This is the dual of the well known hopfian property: a group is *hopfian* if it is not isomorphic with a proper quotient group.

One reason to be interested in co-hopfian groups is the connection with *coherent groups*, i.e., groups in which every finitely generated subgroup is finitely presented. For by a result of Bieri and Strebel [6] and

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Groves [9], a finitely generated soluble group is coherent if and only if it is either polycyclic or an ascending HNN-extension $\langle H, t \mid t^{-1}ht = h^{\varphi}, h \in H\rangle$, where H is a polycyclic group and φ is an injective endomorphism of H which is not an automorphism. So H is not co-hopfian. Therefore, from each non-co-hopfian polycyclic group we can construct a non-polycyclic, finitely generated soluble group which is coherent, and all such groups arise in this way.

The determination of the polycyclic groups which are not co-hopfian seems to be a difficult problem. Even in the case of a finitely generated infinite nilpotent group G, the situation is not simple. If the nilpotency class of G is at most 2, it is not hard to see that such groups are not co-hopfian – see Smith [14]. However, Smith in the paper just cited, and more recently Belegradek [3], have constructed examples of co-hopfian finitely generated nilpotent groups of class > 2. In his paper Belegradek also gives a characterization of finitely generated torsion-free nilpotent groups which are co-hopfian in terms of their Lie algebra automorphisms.

Here our first goal is to produce a large class of polycyclic groups which are not co-hopfian. This is achieved by showing that a polycyclic abelian-by-nilpotent group which is co-hopfian is necessarily nilpotent-by–finite – see Corollary 2.1. Our proof is cohomological. On the other hand, we also show, on the basis of a construction due to Belegradek [3] and Bryant and Groves [7], how to construct co-hopfian polycyclic groups which are not nilpotent-by-finite (Theorem 3.2).

It is very easy to see that a group satisfying the minimal condition on subgroups is co-hopfian. Our second main goal is to extend this result to larger classes of groups, for example to certain locally finite groups satisfying min-p, the minimal condition on p-subgroups, for all primes p. We also prove that soluble-by-finite groups satisfying the minimal condition on normal subgroups are co-hopfian (Theorem 4.2).

In the last part of this paper we find classes of groups satisfying a weak form of co-hopficity, namely groups G in which every endomorphism $\varphi : G \to G$ with finite kernel has its image G^{φ} of finite index in G.

2. Co-hopficity and soluble minimax groups

Recall that a *minimax group* is a group which has a series of finite length whose factors satisfy max or min, (i.e., the maximal or minimal condition on subgroups). A minimax group is said to be *reduced* if it has no quasicyclic subgroups. It is known that a soluble minimax group G is reduced if and only if it is residually finite. (For this and other properties of soluble minimax groups see [12, Chapter 5] or [13, Part 2, 10.3].)

Our first result provides a necessary condition for a soluble minimax group to be co-hopfian.

Theorem 2.1. *Let G be a reduced minimax group which has a normal abelian-by-nilpotent subgroup with finite index m. If G is co-hopfian, then it has a finite-by-nilpotent normal subgroup of index dividing a power of m.*

It is convenient to deduce this from the following more technical result.

Proposition 2.1. *Let A and N be characteristic subgroups of a group G with A abelian and N/A nilpotent. Assume that A is a reduced minimax group and that $[A, {}_iN]$ is infinite for all $i > 0$. Then G is not co-hopfian.*

Proof. Since A is a minimax group, there is an integer $d > 0$ such that

$$[A, {}_dN]/[A, {}_{d+1}N]$$

is finite. Put $B = [A, {}_dN]$. Then $B/[B, N]$ is finite, while N/B is nilpotent. Now apply Theorem H in [11] to show that $H^n(N/B, B)$ has finite exponent for all $n \geq 0$. It follows, on applying the Lyndon–Hochschild–Serre spectral sequence, that each $H^n(\overline{G}, B)$ has finite exponent, where $\overline{G} = G/B$. Let e denote the exponent of $H^2(\overline{G}, B)$.

Since B has no quasicyclic subgroups, its torsion-subgroup has finite order, say t. Put $k = 1 + etq > 1$, where q is the product of all the primes in the spectrum of B, with $q = 1$ should this spectrum be empty. Then $b \mapsto b^k$ determines an injective endomorphism φ of B. We show that φ extends to an endomorphism of G.

Form the push-out diagram

$$\begin{array}{cccccc} & B & \rightarrowtail & G & \twoheadrightarrow & \overline{G} \\ \varphi & \downarrow & & \downarrow & & \| \\ & B & \rightarrowtail & G_1 & \twoheadrightarrow & \overline{G} \end{array}$$

If Δ is the cohomology class of the upper extension in the diagram, the lower extension has cohomology class $\Delta\varphi_* = k\Delta = \Delta$, since $eH^2(\overline{G}, B) = 0$. It follows that the lower extension is equivalent to the upper one and so we obtain an exact diagram

$$\begin{array}{ccccc} B & \rightarrowtail & G & \twoheadrightarrow & \overline{G} \\ \varphi \downarrow & & \psi \downarrow & & \| \\ B & \rightarrowtail & G & \twoheadrightarrow & \overline{G} \end{array}$$

Clearly ψ is an injective endomorphism of G: suppose that it is an automorphism. Then it follows that $B = B^k$ since B is characteristic in G. By [13, Lemma 10.31], B contains a finitely generated subgroup X such that B/X is a direct product of finitely many quasicyclic p-groups, where p is in the spectrum of B. Since no prime in the spectrum of B can divide k, it follows that $X \cap B^k = X^k$, so that $X = X^k$ and X is finite. Hence B is finite, which is a contradiction. Therefore ψ is not an automorphism and G is not co-hopfian. $\square$

Proof of Theorem 2.1. Assume that G is co-hopfian. By hypothesis there is a normal subgroup M such that $|G : M| = m$ and M is abelian-by-nilpotent. Put $N = G^m$ and note that $|G : N|$ is finite and divides some power of m. Also $A = \gamma_{c+1}(N)$ is abelian for some $c \geq 0$. Now apply Proposition 2.1 to show that some $[A, {}_iN]$ is finite. Hence N is finite-by-nilpotent. $\square$

Corollary 2.1. *Let G be a reduced minimax group which is abelian-by-nilpotent-by-finite. If G is co-hopfian, then it is nilpotent-by-finite.*

For G is finite-by-nilpotent-by-finite and hence is nilpotent-by-finite. Thus co-hopficity forces the group to be virtually nilpotent.

Corollary 2.2. *Let G be an infinite reduced minimax group which is abelian-by-nilpotent and has trivial centre. Then G is not co-hopfian.*

For, if G were co-hopfian, Theorem 2.1 (with $m = 1$) would imply that G is finite-by-nilpotent. Hence, by a theorem of P. Hall – see [13, Part 1, 4.25] – some term of the upper central series of G has finite index. Since the centre of G is trivial, we reach the contradiction that G is finite.

We record one further corollary of Proposition 2.1.

Corollary 2.3. *Let G be a reduced soluble minimax group with derived length $d \geq 2$. If G is co-hopfian, then G is finite-by-nilpotent-by-(soluble of derived length at most $d-2$).*

Proof. Let $A = G^{(d-1)}$, the least non-trivial term of the derived series of G. Apply Proposition 2.1 with $N = G^{(d-2)}$ to get $[A, {}_iN]$ finite for some i. The result now follows. □

In particular, if a reduced metabelian minimax group is co-hopfian, it must be finite-by-nilpotent. This permits the construction of many finitely generated coherent soluble groups of derived length 3 which are not polycyclic.

3. The role of the Fitting subgroup

The Fitting subgroup of a soluble minimax group plays a crucial part in the question of whether the group is co-hopfian. This is made clear by the following result.

Theorem 3.1. *Let G be a soluble-by-finite minimax group and let F denote its Fitting subgroup. If F is co-hopfian, then so is G.*

To prove this we need a number of auxiliary results. In the first of these it should be kept in mind that the Fitting subgroup of a group need not be fully invariant.

Lemma 3.1. *Let G be a soluble-by-finite minimax group and let φ be an injective endomorphism of G. Then $(\mathrm{Fit}(G))^\varphi \leq \mathrm{Fit}(G)$.*

Proof. First of all recall that $F = \mathrm{Fit}(G)$ is nilpotent. Thus it suffices to show that N^φ is subnormal in G whenever N is a normal nilpotent subgroup of G. We argue by induction on the minimality $m(G)$. (Recall that the minimality of a soluble minimax group is the number of infinite factors in a series with cyclic or quasicyclic factors). Then $m(G)$ can be assumed to be positive since otherwise G is finite and φ is an automorphism.

In the first place, *there is a non-trivial, fully invariant, abelian subgroup B which is either torsion-free or a radicable torsion group.* Indeed, if G is not reduced, take B to be the maximum radicable abelian torsion subgroup of G. If G is reduced, there is a $k > 0$ for which G^k is torsion-free and soluble, with derived length $d > 0$ say, and we set $B = (G^k)^{(d-1)}$. Note that in both cases B is fully invariant in G.

Next define

$$A = \{x \in G \mid x^{\varphi^i} \in B \text{ for some } i > 0\}.$$

Observe that $B \leq A$ and A is the union of an ascending chain of isomorphic copies of subgroups of B. Thus A is abelian. Notice also that $A \lhd G$ and $A^\varphi \leq A$: in addition $G^\varphi \cap A = A^\varphi$.

From the definition of A we see that φ induces an injective endomorphism in G/A. Clearly $m(G/A) < m(G)$, so by induction we have $N^\varphi A/A \leq \mathrm{Fit}(G/A)$, and $N^\varphi A$ is subnormal in G. Since $G \simeq G^\varphi$, we have $m(G) = m(G^\varphi)$ and hence $|G : G^\varphi|$ is finite. It follows that $|A : A^\varphi|$ is finite – of order m say – since $G^\varphi \cap A = A^\varphi$.

If B is a radicable torsion group, then A is torsion and so has min. Hence $A = A^\varphi$ and $N^\varphi A = (NA)^\varphi$ is nilpotent, so that N^φ is subnormal in G.

Now assume that B is torsion-free. Then A is torsion-free and, if $a \in A$, we have $a^m \in A^\varphi$ and

$$[a, {}_r N^\varphi]^m = [a^m, {}_r N^\varphi] \leq [A^\varphi, {}_r N^\varphi] = 1,$$

where r is the nilpotent class of NA. Thus $[A, {}_r N^\varphi] = 1$ and N^φ is subnormal in $N^\varphi A$ and hence in G. □

Next come some elementary results on HNN-extensions.

Lemma 3.2. *Let φ be an injective endomorphism of a group G and let $H = \langle t, G \mid x^t = x^\phi, x \in G\rangle$ be the corresponding HNN-extension. If t^G is locally nilpotent, then φ is an automorphism.*

Proof. Let $x \in [G, t]$ and put $U = \langle x^t, x^{t^2}, \dots\rangle$. Then $U^\varphi = U^t \leq U$ and $t^G \geq \langle t, x\rangle = \langle t, U\rangle$. Now $\langle t, U\rangle$ is an HNN-extension of U. It is also a finitely generated nilpotent group, so it satisfies max. Therefore $U = U^t = U^\varphi$ and it follows that $[G, t]^\varphi = [G, t]$. For any $x \in G$ we have

$$x = x^\varphi (x^{-1}x^\varphi)^{-1} \equiv x^\varphi \mod [G, t],$$

from which it follows that $G = G^\varphi$. □

Lemma 3.3. *Let φ be an injective endomorphism of a group G which is not an automorphism. Let $H = \langle t, G\rangle$ be the corresponding HNN-extension. Then*

$$\mathrm{Fit}(H) \leq \bigcup_{i=1,2,\dots} G^{t^{-i}}.$$

Proof. Write $\bar{G} = \bigcup_{i=1,2,\dots} G^{t^{-i}}$. Suppose that $u \in \mathrm{Fit}(H)$, but $u \notin \bar{G}$: clearly we may assume that $u = t^i v$, where $i > 0$ and $v \in \bar{G}$. Replacing u by a suitable conjugate, we may also assume that $v \in G$. Notice that conjugation by u induces an injective endomorphism in G and u^G is nilpotent. Applying Lemma 3.2 to $\langle u, G\rangle$, we conclude that $G^u = G$ and hence $G^t = G$, i.e., $G^\varphi = G$, a contradiction. □

Lemma 3.4. *Let φ be an injective endomorphism of a group G and let $H = \langle t, G\rangle$ be the corresponding HNN-extension. If H is abelian-by-finite, then φ is an automorphism.*

Proof. Let $N \lhd H$, where N is abelian and H/N is finite. Then there is an integer $k > 0$ such that $t^k \in N$, so that $(G \cap N)^{t^{-k}} = G \cap N$, since N is abelian. Hence $G \cap N = (G \cap N)^t$ and $G \cap N \lhd H$. Put $\bar{G} = \bigcup_{i=1,2,\dots} G^{t^{-i}}$. Then $\bar{G}/G \cap N$ is the union of the ascending chain of $G^{t^{-i}}/G \cap N = (G/G \cap N)^{t^{-i}}$, $i = 1, 2, \dots$, and these have boundedly finite orders. Hence $\bar{G}/G \cap N$ is finite and $G = G^t = G^\varphi$. □

Proof of Theorem 3.1. Suppose that φ is an injective endomorphism of G which is not an automorphism. By Lemma 3.1 we have $F^\varphi \leq F$, where $F = \mathrm{Fit}(G)$, and hence $F^\varphi = F$ by co-hopficity of F. Form the HNN-extension $H = \langle t, G\rangle$, where $x^t = x^\varphi$ for $x \in G$. Then $F \lhd H$, so $F \leq F_0 = \mathrm{Fit}(H)$.

By Lemma 3.3 we have $F_0 \leq \bar{G}$, where $\bar{G} = \bigcup_{i=1,2,\dots} G^{t^{-i}}$, and hence

$$F_0 = \bigcup_{i=1,2,\dots} (F_0 \cap G^{t^{-i}}).$$

Now $F_0 \cap G^{t^{-i}} = (F_0 \cap G)^{t^{-i}} \leq F^{t^{-i}} = F$, from which it follows that $F = F_0 \lhd H$.

Since $\bar{G}$ is the union of a chain of isomorphic copies of subgroups of G, we see that it is a finite extension of a soluble group with finite abelian total rank (for this concept see [12, Chapter 5]), as must be H. By a well known result of Mal'cev – see [12, 5.2] – the group H is nilpotent-by-abelian-by-finite and so H/F is abelian-by-finite. But H/F is also an HNN-extension, so by Lemma 3.4 its associated endomorphism is an automorphism. Hence $G^\varphi = G$ and G is co-hopfian. □

Application to the construction of co-hopfian polycyclic groups

We now show how to apply Theorem 3.1 to construct examples of co-hopfian polycyclic groups which are not nilpotent-by-finite. This is in contrast to results in Section 2 such as Corollary 2.1.

The construction hinges on the following result of Belegradek [3, Corollary 3.1], which is based on earlier work of Bryant and Groves [7].

Let A be an arithmetic subgroup of $GL_n(\mathbb{Q})$. Then there is a finitely generated, torsion-free, co-hopfian nilpotent group G of derived length at most 3 such that G_{ab} is free abelian of rank n and A is commensurable with I, the image of the canonical map $\mathrm{Aut}(G) \to \mathrm{Aut}(G_{ab})$.

We apply this result with $A = \mathrm{SL}_n(\mathbb{Z})$, $n > 1$. Choose $\beta \in A$ such that no power of β is unipotent. By commensurability $\beta^k \in I$ for some $k > 0$. Let $\alpha \in \mathrm{Aut}(G)$ map to β^k under the map $\mathrm{Aut}(G) \to \mathrm{Aut}(G_{ab})$ and note that α has infinite order.

Now form the semidirect product H of $\langle\alpha\rangle$ and G. Then H is a nilpotent-by-cyclic polycyclic group. Clearly $G \leq \mathrm{Fit}(H)$: if this containment were proper, $\alpha^l \in \mathrm{Fit}(H)$ for some $l > 0$ and β^{kl} would be unipotent, a contradiction which shows that $\mathrm{Fit}(H) = G$. Since G is co-hopfian, so is H by Theorem 3.1. Also H cannot be nilpotent-by-finite. Thus we have proved:

Theorem 3.2. *There is a torsion-free nilpotent-by-cyclic polycyclic group of derived length at most 4 which is co-hopfian, but not nilpotent-by-finite.*

We are grateful to the referee for suggesting that it might be possible to use this method of construction for co-hopfian polycyclic groups.

4. Co-hopficity and locally finite groups with min-p for all p

Since any group satisfying min is co-hopfian, a natural class of groups to test for co-hopficity is the class of locally finite groups with min-p for all primes p. However there are countable locally finite groups with min-p for all p which are not co-hopfian (see for instance [8, 5.4.10]). In fact, Belyaev [5] – see also [8, 5.5.13] – has proved that a countable locally finite group with min-p for all primes p which is co-hopfian is necessarily hyperfinite. Belyaev mentions that the converse of this is true. Since there seems to be no proof of this fact in the literature, we shall provide one here.

Theorem 4.1. *Let G be a hyperfinite group with min-p for all primes p. Then G is co-hopfian.*

Proof. Let φ be an injective endomorphism of G.

(i) *We may assume that G has finite Sylow subgroups.*
Let D be the subgroup generated by all the quasicyclic subgroups of G. Note that $D \trianglelefteq G$ and $D^\varphi \leq D$. Consider an ascending normal series of G with finite factors. Since D must centralize each factor of this series, it is contained in the hypercentre of G. By a theorem of Černikov (see [13, see Part 2, 9.23]), we deduce that D is abelian, and so D is the direct product of its Sylow p-subgroups. It follows from min-p that $D^\varphi = D$. If P_1/D is a Sylow p-subgroup of G/D, then P_1 is locally nilpotent and hence $P_1 = PD$, where P is the p-component of P_1. Since P is a Černikov p-group, it follows that PD/D is finite. Therefore, by replacing G by G/D, we may assume that all Sylow subgroups of G are finite.

(ii) *If π is a finite set of primes, then G has a unique largest finite normal π-subgroup F.*
Define F to be the subgroup generated by all the finite normal π-subgroups of G. If H is a finite π-subgroup of G, then $|H| \leq \prod_{p\in\pi} p^{m(p)}$, where $p^{m(p)}$ is the order of a Sylow p-subgroup of G. Hence H has boundedly finite order and so F is finite.

(iii) $F^\varphi = F$.
Let P be a Sylow p-subgroup of G, where $p \in \pi$. Then $P \simeq P^\varphi \leq P_1$ for some Sylow p-subgroup P_1 of G. But P and P_1 are conjugate since they are finite. Hence $P^\varphi = P_1$. It follows that G^φ contains a Sylow p-subgroup

of F for all $p \in \pi$. Consequently $F \leq G^{\varphi}$, since F is generated by its Sylow subgroups. Also F is a finite normal π-subgroup of G^{φ}, so $F \leq F^{\varphi}$ by maximality. Hence $F = F^{\varphi}$ since F is finite.

(iv) *Conclusion.*
Consider an ascending normal series of G with finite factors and apply (ii) and (iii) to its successive terms, using transfinite induction. This yields an ascending normal series $\{F_\alpha\}$ with finite factors such that $F_\alpha^{\varphi} = F_\alpha$ for all α. Hence $G^{\varphi} = G$ and G is co-hopfian. □

We can apply 4.1 and Belyaev's result cited above [8, 5.5.13], together with the simple fact that a hyperfinite group with min-p for all p is countable, to conclude that the following classes of groups coincide:

(a) countable, locally finite, co-hopfian groups with min-p for all primes p;
(b) hyperfinite groups with min-p for all primes p.

Recall that a group is said to be of *finite rank in the sense of Prüfer* if there is a positive integer n such that every finitely generated subgroup can be generated by at most n elements. Then we have:

Corollary 4.1. *A locally finite group G of finite rank is co-hopfian.*

Proof. First note that G satisfies min-p for all primes p. By a deep theorem of Belyaev [4] – see also [8, 3.5.15] – the group G is locally soluble-by-finite: furthermore, if R is its Hirsch–Plotkin radical, G/R is abelian-by-finite with finite Sylow subgroups by a result of Kargapolov – see [8, 3.2.3]. Also the primary components of R are Černikov groups. Therefore G is hyperfinite and thus is co-hopfian. □

Soluble groups with min-n

As usual, let min-n denote the minimal condition for normal subgroups. By a well known theorem of Baer [1], a soluble group with min-n is locally finite. Thus it is natural to ask if such groups are co-hopfian. Notice that an insoluble group satisfying min-n need not be co-hopfian. For example, let A be the finitary alternating group on the set of positive integers. Then A is isomorphic with the stabilizer of the integer 1, so it is not co-hopfian. Of course A is simple, so it satisfies min-n.

The next result shows that the situation is entirely different for soluble groups.

Theorem 4.2. *A soluble-by-finite group satisfying min-n is co-hopfian.*

Proof. Let φ be an injective endomorphism of G. First assume that G is soluble with min-n. We must prove that φ is surjective. For this purpose we argue by induction on the derived length d of G. If $d \leq 1$, then G satisfies min and the result is known. Suppose that $d > 1$ and put $A = G^{(d-1)}$: thus $A^\varphi \leq A$. For each integer $i > 0$, we define a subgroup

$$K_i = \{x \in G \mid x^{\varphi^i} \in A\}.$$

Clearly each K_i is normal in G and contains A, and K_i is isomorphic to a subgroup of A; thus in particular K_i is abelian. Furthermore $K_i \leq K_{i+1}$ and $K_{i+1}^\varphi \leq K_i$. Hence $K = \bigcup_{i=0,1,2\dots} K_i$ is an abelian normal subgroup of G and $K^\varphi \leq K$. Then φ induces in G/K an endomorphism $\overline{\varphi}$ which is injective, since $x^\varphi \in K$ implies that $x \in K$.

By induction G/K is co-hopfian and thus $\overline{\varphi}^i$ is surjective for each integer $i > 0$; hence $G = KG^{\varphi^i}$. Since K is abelian and $A^{\varphi^i} \lhd G^{\varphi^i}$, it follows that $A^{\varphi^i} \lhd G$. Applying min-n to the descending chain $(A^{\varphi^i})_{i>0}$, we obtain $A^{\varphi^{i+1}} = A^{\varphi^i}$ for some $i > 0$, whence $A^\varphi = A$. Thus φ induces in G/A an injective endomorphism; by induction this endomorphism is surjective, so $G = AG^\varphi = A^\varphi G^\varphi = G^\varphi$ and φ is surjective.

Now assume that G is soluble-by-finite with min-n and let H be a soluble normal subgroup of G such that G/H is finite, of order m say. Then G/G^m is a soluble-by-finite group of finite exponent with min-n. Using the fact that min-n is inherited by subgroups of finite index – see [13, Part 1, 5.31] – we can easily prove that such a group is finite. Therefore G^m has finite index in G and thus it has min-n. Therefore G^m, being soluble, is co-hopfian. Since φ induces an injective endomorphism in G^m, this endomorphism is an automorphism and $(G^m)^\varphi = G^m$. This implies that the endomorphism induced by φ in G/G^m is injective and hence surjective, since G/G^m is finite. Therefore $G = G^\varphi(G^m) = G^\varphi(G^m)^\varphi = G^\varphi$, as required. □

5. A weak form of co-hopficity

In a sense co-hopficity is too strong a property. For example, even the infinite cyclic group is not co-hopfian; however, in this case the image of each injective endomorphism at least has finite index in the group, and it is not difficult to show that this result remains true in any finitely generated abelian group. On the other hand, in a finitely generated metabelian group the image of an injective endomorphism can have infinite index in the group.

For example, let $\langle a\rangle$ be infinite cyclic and $F = \langle x, y\rangle$ a free abelian group of rank 2. Denote by G the wreath product $\langle a\rangle wr\ F$. Then G is a finitely generated metabelian group. The assignments $x \mapsto x$, $y \mapsto y$, $a \mapsto a^x a^y$ determine an injective endomorphism φ of G, but it is easy to see that G^φ has infinite index in G.

Our next result shows that this phenomenon cannot occur in a soluble group with finite abelian ranks. (For the definition see [12, Chapter 5]). Furthermore, it is not necessary to suppose that the endomorphism is injective, but merely that its kernel is finite.

Theorem 5.1. *Let G be a soluble-by-finite group with finite abelian ranks and let φ be an endomorphism. If $\ker\varphi$ is finite, then the index of G^φ in G is finite.*

A special case of this result appears as Lemma 10 in [2].

Proof. *(i) Case: G is abelian.*
In the torsion subgroup T of G, each primary component is a direct product of finitely many cyclic and quasicyclic groups. Thus T is co-hopfian and so $T^\varphi = T$. Since $\overline{G} = G/T$ is torsion-free of finite rank, φ induces an injective endomorphism $\overline{\varphi}$ in $\overline{G}$. Then $\overline{G}^{\overline{\varphi}}$ has finite index in $\overline{G}$ by a result of Fuchs – see [12, 6.1.3]. Hence $|G : TG^\varphi| < \infty$ and so $|G : G^\varphi| < \infty$, since $T = T^\varphi$.
(ii) Case: G is soluble.
Let G have derived length $d > 1$. Put $A = G^{(d-1)}$ and

$$K = \{x \in G \mid x^{\varphi^i} \in A, \text{ for some } i > 0\}.$$

Then K is abelian and $K^\varphi \leq K \triangleleft G$, so that $|K : K^\varphi| < \infty$ by (i). By considering the injective endomorphism induced by φ in G/K, we obtain $|G : KG^\varphi| < \infty$ by induction on d. Hence $|G : G^\varphi|$ is finite.

(iii) The general case.

The argument in the soluble-by-finite case is similar to that used to prove Theorem 4.2. □

We remark that the converse of Theorem 5.1 fails, namely the fact that $|G : G^\varphi|$ is finite does not imply that $\ker\varphi$ is finite. Indeed, consider a direct product G of quasicyclic p-groups, where p ranges over the set of primes, and the endomophism φ of G defined on the p-component by $x^\varphi = x^p$. Then G is an abelian group with finite abelian ranks and φ is surjective, but $\ker\varphi$ is infinite.

Nevertheless, the converse property holds for minimax groups. Indeed, if G is a soluble minimax group and φ is an endomorphism of G such that $|G : G^\varphi|$ is finite, it is easy to see that

$$m(G) = m(G^\varphi) = m(G) - m(\ker\varphi),$$

so that $\ker\varphi$ is finite. As in Theorem 5.1, this result may be extended to soluble-by-finite minimax groups without difficulty. Therefore we can state a partial converse of Theorem 5.1 in the following form.

Theorem 5.2. *Let G be a soluble-by-finite minimax group and let φ be an endomorphism of G. If $|G : G^\varphi|$ is finite, then $\ker\varphi$ is finite.*

(A similar result was proven by Hirshon in the case where G is a finitely generated residually finite group [10]). Theorems 5.1 and 5.2 combine to give the following result.

Corollary 5.1. *In a soluble-by-finite minimax group the kernel of an endomorphism is finite if and only if its image has finite index in the group.*

Note that a finitely generated soluble group of finite rank is a minimax group [13, Part 2, 10.38], so Corollary 5.1 applies to such groups.

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