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Inhomogeneous Cauchy exponential functional equations

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Abstract. We show that equations of the form $f(x)f(y)-f(x+y) = \Gamma(x,y)$, termed here inhomogeneous Cauchy exponential functional equations, can be solved quite easily. Furthermore, their solutions are almost always unique. Both of these results contrast starkly with the situation for the inhomogeneous Cauchy additive functional equation $f(x) + f(y) - f(x + y) = \Gamma(x, y)$.

1. Introduction

We consider functional equations of the form

$$f(x)f(y) - f(x + y) = \Gamma(x, y), \tag{1}$$

where Γ is a given function and f is an unknown function. Generally we shall assume that the domain is a commutative semigroup S or group G and the co-domain is a (commutative) field F. So (1) is supposed to hold for all $x, y \in S$, where $\Gamma : S \times S \to F$ and $f : S \to F$. When $\Gamma = 0$, equation (1) reduces to the Cauchy exponential functional equation, so we may call (1) the *inhomogeneous Cauchy exponential functional equation.*

Equation (1) resembles the inhomogeneous Cauchy additive functional equation

$$f(x) + f(y) - f(x + y) = \Gamma(x, y), \tag{2}$$

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which has been studied by several authors [2], [4]. Necessary and sufficient conditions on Γ for the existence of solutions are known [3]. On the other hand it is rather difficult in general to get one's hands on a solution. Solutions of (2) can be constructed provided Γ satisfies certain growth conditions as $x \to \infty$ or as $x \to 0$. Since such growth conditions are not necessary for the existence of solutions, it may be said that no general method of constructing solutions of (2) is known.

In contrast to this, we will show that it is easy to construct solutions of (1). Although we do not have a satisfactory set of necessary and sufficient conditions on Γ for the existence of such solutions, our method provides very specific forms of solution candidates. Then one only needs to check these candidates to determine whether they are solutions. If they are not, then (1) has no solutions.

We also present a complete answer to the question of uniqueness of solutions of (1). It is almost always the case that solutions of (1) are unique, and the few exceptional cases are given explicitly.

Another way to think about this is as follows. If one starts with an arbitrary specified function f_0 and calculates the corresponding Γ through (1) with a specified group operation in the domain, then it is almost always the case that $f = f_0$ is the unique solution of the functional equation so generated. One might say that this functional equation is robust. Moreover, most of the (already few) exceptional non-unique cases for a given group operation may be covered by changing to a different group or semigroup operation in the domain. For example, we show how to characterize any given nonconstant real function as the unique solution of a functional equation of the form (1).

2. Existence results

First we observe that the left side of (1) is symmetric in x and y.

Remark 1. A necessary condition for the existence of solutions of (1) is that Γ must be a symmetric function.

If $\Gamma = 0$, then the solutions of (1) are the *Cauchy exponentials*, which are the functions $E : S \to F$ satisfying the Cauchy exponential equation

$$E(x)E(y) = E(x+y). \tag{3}$$

Henceforth let us assume $\Gamma \neq 0$.

If 0 is in the domain of (1), then one simple attempt to solve (1) would be to set $y = 0$, resulting in $f(x)[f(0) - 1] = \Gamma(x, 0)$. If $f(0) - 1 \neq 0$, then $f(x) = a\Gamma(x, 0)$ for some constant a. The serious drawback of this method is that it will never produce any of the solutions for which $f(0) = 1$. (As we will see, such solutions exist frequently.) A better method is the following.

Since $\Gamma \neq 0$, there exists some pair (x_0, y_0) for which $\Gamma(x_0, y_0) \neq 0$. From (1) we get

$$\begin{aligned}
&\Gamma(x,y)f(z) + \Gamma(x+y,z) \\
&\qquad = [f(x)f(y) - f(x+y)]f(z) + [f(x+y)f(z) - f(x+y+z)] \\
&\qquad = f(x)f(y)f(z) - f(x+y+z) \\
&\qquad = f(x)[f(y)f(z) - f(y+z)] + [f(x)f(y+z) - f(x+y+z)] \\
&\qquad = f(x)\Gamma(y,z) + \Gamma(x,y+z).
\end{aligned}$$

Putting $(x, y) = (x_0, y_0)$, we find that

$$f(z) = \Gamma(x_0, y_0)^{-1}\{f(x_0)\Gamma(y_0, z) + \Gamma(x_0, y_0 + z) - \Gamma(x_0 + y_0, z)\}.$$

Replacing $f(x_0)$ by an arbitrary constant, we have proved the following.

Theorem 2. *Suppose S is a semigroup, F is a field, and $\Gamma : S \times S \to F$. If $\Gamma = 0$, then every solution $f : S \to F$ of* (1) *is a* (3) *Cauchy exponential. Otherwise there exist $x_0, y_0 \in S$ such that $\Gamma(x_0, y_0) \neq 0$ and every solution of* (1) *is of the form*

$$f(z) = \Gamma(x_0, y_0)^{-1}\{a\Gamma(y_0, z) + \Gamma(x_0, y_0 + z) - \Gamma(x_0 + y_0, z)\} \tag{4}$$

for some constant $a \in F$.

Example 3. Let $S = F = \mathbb{R}$ (the real numbers), and let

$$\Gamma(x, y) = b\sin(x + y) + c\sin x \sin y$$

for some constants b, c not both zero. Then (1) takes the form

$$f(x)f(y) - f(x+y) = b\sin(x+y) + c\sin x \sin y. \tag{5}$$

If $b = 0$, then $\Gamma(\pi/2, \pi/2) = c \neq 0$. Choosing $x_0 = y_0 = \pi/2$ in (4), we get

$$f(z) = c^{-1}\{a(c \sin z) + c \cos z - 0\} = a \sin z + \cos z$$

for some constant a. Inserting this form for f back into (5) and simplifying, we find that

$$(a^2 + 1) \sin x \sin y = c \sin x \sin y$$

must hold for all x, y in $\mathbb{R}$. So we must have $c \geq 1$ and
$f(x) = \pm\sqrt{c-1} \sin x + \cos x$.

On the other hand, if $b \neq 0$, then $\Gamma(0, \pi/2) = b \neq 0$, so we choose $x_0 = 0$, $y_0 = \pi/2$ in (4) and get

$$f(z) = a \cos z + d \sin z,$$

where $d = b^{-1}(a-1)c$. Substituting this back into (5) with $y = 0$, we find that

$$(a \cos x + d \sin x)(a - 1) = b \sin x$$

for all real x. Thus $a(a-1) = 0$. Moreover, for $x = \pi/2$ we have $d(a-1) = b$. So $a - 1$ cannot be zero since $b \neq 0$. Hence $a = 0$, $d = -b$, and

$$f(x) = -b \sin x.$$

Checking this in (5), we see that $c = b^2$ is necessary.

In conclusion: (5) with $(b, c) \neq (0, 0)$ has solutions if and only if either $b = 0$ and $c \geq 1$, or $c = b^2 \neq 0$. In the former case $f(x) = \pm\sqrt{c-1} \sin x + \cos x$, while in the latter case $f(x) = -b \sin x$.

The example illustrates that equation (1) admits non-unique solutions for certain functions Γ but not for others. In the next section we show that there are very few forms of Γ that admit non-unique solutions, and we exhibit all such functional forms.

3. Uniqueness results

Suppose we can find a particular solution $f = f_0$ of the inhomogeneous Cauchy exponential functional equation (1). Under what conditions is this the unique solution to the equation? We shall see that it is almost always the case that f_0 is the only solution.

Lemma 4. *Let S be a set closed under the binary operation $+$, let R be a ring in which 2^{-1}exists and commutes with every element of R, and let $\Gamma : S \times S \to R$. Suppose* (1) *has at least one solution f_0, and let $f : S \to R$ be an arbitrary solution of* (1)*. Then the maps $g, h : S \to R$ defined by*

$$g(x) := f(x) - f_0(x), \quad h(x) := 2^{-1}[f(x) + f_0(x)] \tag{6}$$

satisfy the functional equation

$$g(x + y) = g(x)h(y) + h(x)g(y) \tag{7}$$

for all x, y in S.

Proof. Since both f_0 and f satisfy (1) with the same Γ, we have

$$f_0(x)f_0(y) - f_0(x + y) = f(x)f(y) - f(x + y),$$

which by rearrangement gives

$$f(x + y) - f_0(x + y) = f(x)f(y) - f_0(x)f_0(y).$$

Defining maps g, h by (6) and using our hypotheses about R, we compute that

$$\begin{aligned} g(x)h(y) &+ h(x)g(y) \\ &= [f(x) - f_0(x)]2^{-1}[f(y) + f_0(y)] + 2^{-1}[f(x) + f_0(x)][f(y) - f_0(y)] \\ &= 2^{-1}\{2f(x)f(y) - 2f_0(x)f_0(y)\} \\ &= f(x + y) - f_0(x + y) \\ &= g(x + y). \end{aligned}$$ $\square$

Note that equation (6) shows that the uniqueness question for solutions of (1) reduces to the question of whether $g = 0$. Next we record here a well-known result (see for example [1], p. 212) concerning solutions of (7) on groups.

Proposition 5. *Let G be a group and F a quadratically closed (commutative) field with characteristic different from 2. The general solutions $g, h : G \to F$ of (7) are given by*

$$g = 0, \quad h \text{ arbitrary}; \tag{8}$$

or

$$g(x) = (2k)^{-1}[E_1(x) - E_2(x)], \quad h(x) = 2^{-1}[E_1(x) + E_2(x)]; \tag{9}$$

or

$$g(x) = A(x)E_1(x),\ h(x) = E_1(x); \tag{10}$$

where k is an arbitrary element of $F\backslash\{0\}$; $E_1, E_2 : G \to F$ are arbitrary solutions of (3); and $A : G \to F$ is an arbitrary solution of Cauchy's additive functional equation

$$A(x + y) = A(x) + A(y) \tag{11}$$

for all x, y in G.

Combining the previous lemma and proposition, we see that solutions of (1) are unique unless g, h have the forms given in (9) or (10). The next theorem provides the details of those exceptional cases.

Theorem 6. *Let G be a group and F a quadratically closed (commutative) field with characteristic different from 2. Suppose $\Gamma : G \times G \to F$ is not the zero function, and suppose (1) has at least one solution $f_0 : G \to F$. Then f_0 is the unique solution of (1) among all maps $f : G \to F$ unless Γ has one of two functional forms given below. In each of these two exceptional cases there are exactly two solutions f_1 and f_0 of (1). The two exceptional forms of Γ, together with their corresponding f_1 and f_0, are as follows.*

$$\begin{aligned} \Gamma(x, y) &= (b^2 - 4^{-1})[E_1(x) - E_2(x)][E_1(y) - E_2(y)], \\ f_1(x) &= (2^{-1} + b)E_1(x) + (2^{-1} - b)E_2(x), \\ f_0(x) &= (2^{-1} - b)E_1(x) + (2^{-1} + b)E_2(x); \end{aligned} \tag{12}$$

or

$$\Gamma(x, y) = A(x)A(y)E(x + y), \tag{13}$$

$$f_1(x) = [1 - A(x)]E(x),$$

$$f_0(x) = [1 + A(x)]E(x); \tag{14}$$

where b is an arbitrary constant in $F\backslash\{0, \pm 2^{-1}\}$; maps $E, E_1, E_2 : G \to F$ are Cauchy exponentials with $E \neq 0$, $E_1 \neq E_2$ but otherwise arbitrary; and $A : G \to F$ is an arbitrary nonzero (11) additive function.

PROOF. Let $f : S \to F$ be an arbitrary solution of (1). By the preceding lemma and proposition, we know that g, h defined by (6) are given by one of the three forms (8), (9), or (10). In the first case, we have $g = 0$ and f_0 is the unique solution of (1). Now let us consider the second and third cases. Note that we can solve (6) for f and f_0, obtaining

$$f(x) = h(x) + 2^{-1}g(x), \quad f_0(x) = h(x) - 2^{-1}g(x).$$

In the case g, h are given by (9) we have

$$f(x) = [2^{-1} + (4k)^{-1}]E_1(x) + [2^{-1} - (4k)^{-1}]E_2(x),$$

$$f_0(x) = [2^{-1} - (4k)^{-1}]E_1(x) + [2^{-1} + (4k)^{-1}]E_2(x).$$

Letting $b = (4k)^{-1}$, this is (12). The corresponding value of Γ can be computed by substituting the form of f (or f_0) into (1). Finally, in case g, h are given by (10) we have

$$f(x) = [1 + 2^{-1}A_1(x)]E_1(x), \quad f_0(x) = [1 - 2^{-1}A_1(x)]E_1(x)$$

for some additive function $A_1 : G \to F$. Defining $A := 2^{-1}A_1$, $E := E_1$, and again computing Γ by substituting the form of f into (1), we arrive at solution (13). This completes the proof. $\square$

Note that (12) can be extended to contain the solution $\Gamma = 0$, $f_1(x) = E_1(x)$, $f_0(x) = E_2(x)$, by permitting $b = \pm 2^{-1}$. However in this case we cannot conclude that there are exactly two solutions. Since E_1 and E_2 are arbitrary Cauchy exponentials, there are infinitely many solutions in this case.

This theorem explains the duplicity of solutions of (5) seen in Example 3. There the function Γ has the form

$$\Gamma(x, y) = c \sin x \sin y$$

for $c \geq 1$. In order to obtain this function, take $F = \mathbb{C}$, $E_1(x) = e^{ix}$, $E_2(x) = e^{-ix}$, and $b = \pm\frac{i}{2}\sqrt{c-1}$ in (12).

We also illustrate how one can obtain the real solutions of (1) from the complex solutions. But in order to do that, we need the following lemma.

Lemma 7. *Suppose $a : \mathbb{R} \to \mathbb{R}$ is an* (11) *additive function, $\theta_1, \theta_2 : \mathbb{R} \to \mathbb{R}$ are arbitrary solutions of the congruence*

$$\theta(x+y) \equiv \theta(x) + \theta(y) \pmod{2\pi}, \tag{15}$$

and

$$\exp a(x) \sin\theta_1(x) + \sin\theta_2(x) = 0, \tag{16}$$

for all x in $\mathbb{R}$. Then either

$$\theta_1(x) \equiv \theta_2(x) \equiv 0 \pmod{2\pi}$$

for all x, or

$$a(x) = 0 \quad \textit{and} \quad \theta_1(x) \equiv -\theta_2(x) \pmod{2\pi}$$

for all x.

Proof. First we show that if θ satisfies (15) and $\sin\theta(x) = 0$ for all x, then $\theta(x) \equiv 0 \pmod{2\pi}$. Indeed, $\sin\theta(x) = 0$ yields immediately $\theta(x) \equiv 0 \pmod{\pi}$. But then $\theta(2x) \equiv 2\theta(x) \equiv 0 \pmod{2\pi}$. Since R is divisible by 2, we have $\theta(u) \equiv 0 \pmod{2\pi}$.

If $a(x) = 0$, then (16) shows that

$$\sin\theta_1(x) = -\sin\theta_2(x) \tag{17}$$

for all x. This shows that for each x, either

$$\theta_1(x) \equiv -\theta_2(x) \pmod{2\pi} \quad \text{or} \quad \theta_1(x) \equiv \theta_2(x) + \pi \pmod{2\pi}. \tag{18}$$

Replacing x by $2x$ in (17) and using (15) with a double angle identity, we get $\sin\theta_1(x)\cos\theta_1(x) = -\sin\theta_2(x)\cos\theta_2(x)$. Therefore

$$\sin\theta_1(x)[\cos\theta_1(x) - \cos\theta_2(x)] = 0.$$

If $\sin\theta_1(x) = 0$ for all x, then as shown above $\theta_1(x) \equiv 0 \pmod{2\pi}$. Similarly $\theta_2(x) \equiv 0 \pmod{2\pi}$, so we have $\theta_1(x) \equiv 0 \equiv -\theta_2(x) \pmod{2\pi}$.

If on the other hand $\sin\theta_1(x_1) = 0$ for some x_1, then we have $\cos\theta_1(x_1) = \cos\theta_2(x_1)$ and so either $\theta_1(x_1) \equiv -\theta_2(x_1) \pmod{2\pi}$ or $\theta_1(x_1) \equiv \theta_2(x_1) \pmod{2\pi}$. Combining this alternative with (18), we have $\theta_1(x_1) \equiv -\theta_2(x_1) \pmod{2\pi}$.

Now suppose there exists a real number x_0 for which $a(x_0) \neq 0$. Observe that (15) implies $\theta(0) \equiv 0 \pmod{2\pi}$ and $\theta(-x) \equiv -\theta(x) \pmod{2\pi}$. Similarly, $a(-x) \equiv -a(x)$. Hence replacing x by $-x$ in (16) and comparing the result with (16), we find that $\exp a(x) \sin\theta_1(x) = \exp[-a(x)] \sin\theta_1(x)$, or

$$\sinh a(x) \sin\theta_1(x) = 0 \tag{19}$$

for all real x. Replacing x by $x + y$, expanding, and reducing by (19), we get

$$\sinh a(x) \cosh a(y) \cos\theta_1(x) \sin\theta_1(y)$$

$$+ \cosh a(x) \sinh a(y) \sin\theta_1(x) \cos\theta_1(y) = 0$$

for all x, y. Putting $x = x_0$ here, we deduce that

$$\cosh a(y) \cos\theta_1(x_0) \sin\theta_1(y) = 0, \tag{20}$$

since $\sinh a(x_0) \neq 0$ by hypothesis and thus $\sin\theta_1(x_0) = 0$ by (19). But then $\cos\theta_1(x_0) \neq 0$ and (20) reduces to

$$\sin\theta_1(y) = 0,$$

for all real y. Hence $\theta_1(y) \equiv 0 \pmod{2\pi}$ for all y, and now (16) yields also $\theta_2(x) \equiv 0 \pmod{2\pi}$ for all x. $\square$

Now we are ready to find the real solutions of (1).

Theorem 8. *Let G be the additive group of the reals, and let $F = \mathbb{R}$. Suppose (1) has at least one solution $f_0 : \mathbb{R} \to \mathbb{R}$ for some given $\Gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is not the zero function. Then f_0 is the unique solution of (1) among all functions $f : \mathbb{R} \to \mathbb{R}$, unless Γ has one of four functional forms. In each of these exceptional cases there are exactly two solutions f_1 and f_0 of (1). The exceptional (nonzero) forms of Γ, together with their corresponding f_1 and f_0, are as follows.*

$$\Gamma(x, y) = \frac{1}{4}(c^2 - 1) \exp A_1(x + y),$$

$$f_1(x) = \frac{1}{2}(1-c)\exp A_1(x), \qquad (21)$$
$$f_0(x) = \frac{1}{2}(1+c)\exp A_1(x);$$

or

$$\Gamma(x,y) = \frac{1}{4}(c^2-1)[\exp A_1(x) - \exp A_3(x)][\exp A_1(y) - \exp A_3(y)],$$
$$f_1(x) = \frac{1}{2}(1+c)\exp A_1(x) + \frac{1}{2}(1-c)\exp A_3(x), \qquad (22)$$
$$f_0(x) = \frac{1}{2}(1-c)\exp A_1(x) + \frac{1}{2}(1+c)\exp A_3(x);$$

or

$$\Gamma(x,y) = (d^2+1)\exp A_1(x+y)\sin\theta(x)\sin\theta(y),$$
$$f_1(x) = \exp A_1(x)[\cos\theta(x) + d\sin\theta(x)], \qquad (23)$$
$$f_0(x) = \exp A_1(x)[\cos\theta(x) - d\sin\theta(x)];$$

or

$$\Gamma(x,y) = A_2(x)A_2(y)\exp A_1(x+y),$$
$$f_1(x) = [1 - A_2(x)]\exp A_1(x), \qquad (24)$$
$$f_0(x) = [1 + A_2(x)]\exp A_1(x).$$

Here c is an arbitrary constant in $\mathbb{R}\backslash\{0,\pm 1\}$; d is an arbitrary constant in $\mathbb{R}\backslash\{0\}$; $A_1, A_2, A_3 : \mathbb{R} \to \mathbb{R}$ are arbitrary additive functions with $A_2 \neq 0$, $A_3 \neq A_1$; and $\theta : \mathbb{R} \to \mathbb{R}$ is an arbitrary (not congruent to zero) solution of the congruence (15).

Proof. Since $\mathbb{R}$ is a subfield of $\mathbb{C}$, we may apply the previous theorem with $f, f_0 : \mathbb{R} \to \mathbb{C}$ and $\Gamma : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$. The conclusion is that f_0 is the unique solution unless Γ and f_0 are of the forms given in (12) or (13), with $A, E_1, E_2 : \mathbb{R} \to \mathbb{C}$. Let us consider first (12). Now we require that f_0 and f_1 take real values. That is, we must have

$$f_1(x) = (1/2+b)E_1(x) + (1/2-b)E_2(x) \in \mathbb{R},$$
$$f_0(x) = (1/2-b)E_1(x) + (1/2+b)E_2(x) \in \mathbb{R},$$

for all x in $\mathbb{R}$. Taking the sum and difference of these equations, we find that

$$E_1(x) + E_2(x) \in \mathbb{R}, \quad b[E_1(x) - E_2(x)] \in \mathbb{R}, \tag{25}$$

for all x in $\mathbb{R}$. The general forms of Cauchy exponentials $E : \mathbb{R} \to \mathbb{C}$ are given by (see [1], p. 54)

$$E = 0 \quad \text{and} \quad E(x) = \exp[a(x) + i\theta(x)],$$

where $a : \mathbb{R} \to \mathbb{R}$ is additive and $\theta : \mathbb{R} \to \mathbb{R}$ is an arbitrary solution of the congruence (15). Since $\Gamma \neq 0$, we cannot have $E_1 = E_2 = 0$. We consider two cases.

First, suppose one of E_1 or E_2 is zero. Without loss of generality, let us say $E_2 = 0$. Then $E_1(x) = \exp[a(x) + i\theta(x)]$ and because of (25) we have $E_1(x) \in \mathbb{R}$, $bE_1(x) \in \mathbb{R}$. Hence b is real, and (since $b \neq 0$)

$$E_1(x) = \exp a(x)[\cos\theta(x) + i\sin\theta(x)] \in \mathbb{R},$$

for all x in $\mathbb{R}$. As we saw in the proof of the previous lemma, $\sin\theta(x) = 0$ for θ satisfying (15) means $\theta(x) \equiv 0 \pmod{2\pi}$ for all x. Therefore E_1 has the form

$$E_1(x) = \exp a(x),$$

and this case leads to solution (21) with $c := 2b$ and $A_1 := a$.

Next, suppose that neither E_1 nor E_2 is zero. That is we have

$$E_1(x) = \exp[a_1(x) + i\theta_1(x)], \quad E_2(x) = \exp[a_2(x) + i\theta_2(x)],$$

with a_1, a_2 additive and θ_1, θ_2 "additive modulo 2π". Because of (25) we must have

$$\begin{aligned} &e^{a_1(x)}[\cos\theta_1(x) + i\sin\theta_1(x)] + e^{a_2(x)}[\cos\theta_2(x) + i\sin\theta_2(x)] \in \mathbb{R}, \\ &b\big\{e^{a_1(x)}[\cos\theta_1(x) + i\sin\theta_1(x)] - e^{a_2(x)}[\cos\theta_2(x) + i\sin\theta_2(x)]\big\} \in \mathbb{R}, \end{aligned} \tag{26}$$

for all real x. Note that the first of these two inclusions implies

$$e^{a_1(x)}\sin\theta_1(x) + e^{a_2(x)}\sin\theta_2(x) = 0,$$

or

$$e^{a_1(x)-a_2(x)}\sin\theta_1(x) + \sin\theta_2(x) = 0.$$

Since $a_1 - a_2$ is additive whenever a_1, a_2 are, we conclude from the previous lemma that either

$$\theta_1(x) \equiv \theta_2(x) \equiv 0 \pmod{2\pi}$$

for all x, or

$$a_1(x) = a_2(x) \quad \text{and} \quad \theta_1(x) \equiv -\theta_2(x) \pmod{2\pi} \tag{27}$$

for all x. In case of the former equation, we deduce from (26) that b is real. Now (12) reduces to

$$\Gamma(x,y) = (b^2 - 1/4)[e^{a_1(x)} - e^{a_2(x)}][e^{a_1(y)} - e^{a_2(y)}],$$

$$f_1(x) = (1/2 + b)e^{a_1(x)} + (1/2 - b)e^{a_2(x)},$$

$$f_0(x) = (1/2 - b)e^{a_1(x)} + (1/2 + b)e^{a_2(x)}.$$

Defining $c := 2b$, $A_1 := a_1$, and $A_3 := a_2$, we have solution (22). Note that $A_3 \neq A_1$ since $\Gamma \neq 0$.

On the other hand, in case (27) holds, then it follows from (26) that b is purely imaginary, say

$$b = (d/2)i, \quad d \in \mathbb{R}\backslash\{0\}.$$

In this case (12) takes the form of (23), where we define $A_1 := a_1$, $\theta := \theta_2$.

Finally, we come to the exceptional case (13). Since f_0 and f_1 must take real values, we have

$$f_1(x) + f_0(x) = E(x) \in \mathbb{R},$$

$$f_1(x) - f_0(x) = A(x)E(x) \in \mathbb{R},$$

so both A and E are real-valued. (Note: A must be real-valued because $E \neq 0$.) With $A_2 := A$, $E := \exp A_1$, we have (24), and the proof is complete. $\square$

Note that (23) of the theorem contains the duplicitous solutions seen in Example 3.

The theorem above shows that almost every function $f_0 : \mathbb{R} \to \mathbb{R}$ is characterized by its Cauchy exponential difference $f_0(x)f_0(y) - f_0(x+y)$.

The only functions not characterized uniquely in this way are those from the short list in the theorem and the Cauchy exponential functions themselves, for which the difference is zero.

4. Extensions

It may be possible to obtain unique characterizations of the "exceptional" functions listed in the foregoing results by changing to a different group or semigroup operation in the domain. We start this section by illustrating how this can be done for real functions.

The functions $f_0 : \mathbb{R} \to \mathbb{R}$ that are not uniquely characterized in Theorem 8 are

$$
\begin{aligned}
f_0(x) &= E(x),\\
f_0(x) &= \frac{1}{2}(1+c)\exp A_1(x),\\
f_0(x) &= \frac{1}{2}(1-c)\exp A_1(x) + \frac{1}{2}(1+c)\exp A_3(x),\\
f_0(x) &= \exp A_1(x)[\cos\theta(x) - d\sin\theta(x)],\\
f_0(x) &= [1 + A_2(x)]\exp A_1(x);
\end{aligned}
$$

where $E : \mathbb{R} \to \mathbb{R}$ is an arbitrary Cauchy exponential; c is an arbitrary constant in $\mathbb{R}\backslash\{0, \pm 1\}$; d is an arbitrary constant in $\mathbb{R}\backslash\{0\}$; $A_1, A_2, A_3 : \mathbb{R} \to \mathbb{R}$ are arbitrary additive functions with $A_2 \neq 0$, $A_3 \neq A_1$; and $\theta : \mathbb{R} \to \mathbb{R}$ is an arbitrary (not congruent to zero) solution of the congruence (15). The first in the list arises from $\Gamma = 0$. We show how each of these listed functions except the constant ones can be characterized.

In order to set up our next theorem, we first establish the following.

Lemma 9. *Let $f_0 : \mathbb{R} \to \mathbb{R}$ be any nonconstant function that is not uniquely characterized in Theorem 8. Specifically, f_0 takes one of the forms*

$$f_0(x) = E(x), \tag{28}$$

$$f_0(x) = k\exp A_2(x), \tag{29}$$

$$f_0(x) = (1-k)\exp A_1(x) + k\exp A_3(x), \tag{30}$$

$$f_0(x) = \exp A_1(x)[\cos\theta(x) - d\sin\theta(x)], \tag{31}$$

$$f_0(x) = [1 + A_2(x)]\exp A_1(x), \tag{32}$$

where E is a Cauchy exponential with $E \neq 0$, $E \neq 1$; where A_1, A_2, A_3 are additive functions with $A_2 \neq 0, A_1 \neq A_3$; where k, d are real constants with $k \notin \{0, 1/2, 1\}, d \neq 0$; and where θ is any (not congruent to 0) solution of (15). Then $f_0(0) \neq 0$ and $f_0 \neq 1$.

Proof. In case f_0 is given by (28), we only have to prove that $E(0) \neq 0$. But if $E(0) = 0$, then $E = 0$ (just put $y = 0$ in (3)), which is excluded.

If f_0 is given by (29), then $f_0(0) = k \neq 0$, and $f_0 \neq 1$ since the only constant additive function is the zero function but $A_2 \neq 0$.

In case of (30), (31), or (32), it is clear that $f_0(0) = 1$. It only remains to be shown that $f_0 \neq 1$ in each of these three cases. In the first case, $f_0 = 1$ only if

$$(1-k)\exp A_1(x) + k\exp A_3(x) = 1 \tag{33}$$

for all x. Substituting $x + y$ for x and expanding, we get

$$(1-k)\exp A_1(x)\exp A_1(y) + k\exp A_3(x)\exp A_3(y) = 1.$$

By (33) this can be re-written as

$$k\exp A_3(x)[\exp A_3(y) - \exp A_1(y)] = 1 - \exp A_1(y).$$

Since $A_1 \neq A_3$, there exists a real number y_0 for which $A_1(y_0) \neq A_3(y_0)$. Thus we have

$$k\exp A_3(x) = \frac{1 - \exp A_1(y_0)}{\exp A_3(y_0) - \exp A_1(y_0)}$$

which means (since $k \neq 0$) that A_3 is constant. So $A_3 = 0$ and (33) reduces to

$$(1-k)\exp A_1(x) = 1 - k.$$

But this is impossible because $k \neq 1$ and $A_1 \neq A_3 = 0$.

In case of (31), $f_0 = 1$ only if

$$\exp A_1(x)[\cos\theta(x) - d\sin\theta(x)] = 1 \tag{34}$$

for all x. Replacing x by $-x$ and adding the result to (34), we find that

$$2\cos\theta(x) = 2\cosh A_1(x).$$

This is impossible since θ is not congruent to zero modulo 2π.

Finally we come to (32). In this case, $f_0 = 1$ only if

$$1 + A_2(x) = \exp(-A_1(x)) \tag{35}$$

for all x. Again, replacing x by $-x$ and adding the result to (35), we find that

$$2 = 2\cosh A_1(x).$$

Hence $A_1 = 0$. But now (35) is impossible, since $A_2 \neq 0$. This completes the proof of the lemma. □

The preparations are in place for our next theorem, in which the group operation of addition on $\mathbb{R}$ is replaced by the semigroup operation of multiplication in the domain.

Theorem 10. *Let $f_0 : \mathbb{R} \to \mathbb{R}$ be any nonconstant function that is not uniquely characterized in Theorem 8. That is, f_0 takes one of the five forms listed in the preceding lemma. Then $f = f_0$ is the unique solution of the functional equation*

$$f(x)f(y) - f(xy) = \Gamma(x, y), \tag{36}$$

where Γ is generated by

$$\Gamma(x, y) = f_0(x)f_0(y) - f_0(xy). \tag{37}$$

Proof. Comparing (36) and (37) with $y = 0$, we see that

$$[f(x) - 1]f(0) = [f_0(x) - 1]f_0(0)$$

for all real x. By the lemma we have $f_0(0) \neq 0$ and $f_0 \neq 1$. Thus $f(0) \neq 0$ and

$$f(x) = \lambda f_0(x) + 1 - \lambda \tag{38}$$

for some constant λ. Our goal is to prove that $\lambda = 1$, and therefore $f = f_0$.

Suppose to the contrary that $\lambda \neq 1$. Substituting (38) and (37) into (36), we obtain

$$[\lambda f_0(x) + 1 - \lambda][\lambda f_0(y) + 1 - \lambda] - [\lambda f_0(xy) + 1 - \lambda] = f_0(x)f_0(y) - f_0(xy),$$

which can be transformed into

$$(\lambda^2 - 1)f_0(x)f_0(y) + \lambda(1-\lambda)[f_0(x) + f_0(y)] + (1-\lambda)f_0(xy) - \lambda(1-\lambda) = 0.$$

Then division by $(1 - \lambda)$ yields

$$-(\lambda + 1)f_0(x)f_0(y) + \lambda[f_0(x) + f_0(y)] + f_0(xy) - \lambda = 0. \tag{39}$$

We consider two cases.

In case f_0 is given by (28), (30), (31), or (32), we have $f_0(0) = 1$. This is clear for (30), (31), or (32). For (28), it follows from (3) by putting $x = y = 0$ and using $E \neq 0$. Therefore in this case (39) with $y = 0$ reduces to

$$-f_0(x) + 1 = 0$$

which contradicts $f_0 \neq 1$.

In case f_0 is given by (29), equation (39) takes the form

$$\begin{aligned} -(\lambda+1)k^2 \exp A_2(x+y) + \lambda k[\exp A_2(x) \\ + \exp A_2(y)] + k \exp A_2(xy) - \lambda = 0, \end{aligned} \tag{40}$$

with $A_2 \neq 0$, $k \notin \{0, 1/2, 1\}$. Putting $y = 0$ here we find that

$$[\lambda - (\lambda+1)k][k \exp A_2(x) - 1] = 0.$$

Because $A_2 \neq 0$ it follows that

$$\lambda = \frac{k}{1-k}.$$

Now after some manipulations (40) reduces to

$$k \exp A_2(x+y) - k[\exp A_2(x) + \exp A_2(y)] + (k-1)\exp A_2(xy) + 1 = 0.$$

That is, defining $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := \exp A_2(x) - 1,$$

we have

$$\frac{k}{1-k}h(x)h(y) = h(xy) \tag{41}$$

for all x, y in $\mathbb{R}$. With $y = 1$ we deduce that

$$h(x)\left[\frac{k}{1-k}h(1) - 1\right] = 0.$$

Since $h \neq 0$ (because $A_2 \neq 0$) we see that $h(1) = (1-k)/k$. Then $x = y = -1$ in (41) yields

$$h(-1) = \pm h(1) = \pm\frac{1-k}{k}. \tag{42}$$

Next, $y = -1$ in (41) gives

$$\frac{k}{1-k}h(x)h(-1) = h(-x),$$

so, combining this with (42), we see that h is either even or odd. If h is even, then from its definition we get $A_2 = 0$, which is prohibited. On the other hand if h is odd, then by definition we find that $\cosh A_2(x) = 1$ which again contradicts $A_2 \neq 0$. Since there are no other cases, we have shown that $\lambda \neq 1$ cannot hold.

Therefore $\lambda = 1$, $f = f_0$, and the proof is finished. □

Here it should be emphasized that, although our uniqueness theorem requires the domain to be a group and the co-domain to be a quadratically closed field, for our existence theorem we need only that the domain is a semigroup and the co-domain is a field.

To illustrate another way that Theorem 8 can be extended, we conclude with the following.

Theorem 11. *Let G be the multiplicative group of $\mathbb{R}_+$ (the positive reals), and let $F = \mathbb{R}$. Suppose* (36) *has at least one solution $f_0 : \mathbb{R}_+ \to \mathbb{R}$ for some given $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that is not the zero function. Then f_0 is the unique solution of* (36) *among all functions $f : \mathbb{R}_+ \to \mathbb{R}$, unless Γ has one of the forms listed below. In each of these exceptional cases there are exactly two solutions f_1 and f_0 of* (36)*. The exceptional (nonzero) forms of Γ, together with their corresponding f_1 and f_0, are as follows.*

$$\Gamma(x, y) = \frac{1}{4}(c^2 - 1)\exp A_1(\log xy),$$

$$f_1(x) = \frac{1}{2}(1-c)\exp A_1(\log x),$$

$$f_0(x) = \frac{1}{2}(1+c)\exp A_1(\log x);$$

or

$$\Gamma(x,y) = \frac{1}{4}(c^2-1)[\exp A_1(x) - \exp A_3(x)][\exp A_1(y) - \exp A_3(y)],$$

$$f_1(x) = \frac{1}{2}(1+c)\exp A_1(\log x) + \frac{1}{2}(1-c)\exp A_3(\log x),$$

$$f_0(x) = \frac{1}{2}(1-c)\exp A_1(\log x) + \frac{1}{2}(1+c)\exp A_3(\log x);$$

or

$$\Gamma(x,y) = (d^2+1)\exp A_1(\log xy)\sin\theta(\log x)\sin\theta(\log y),$$

$$f_1(x) = \exp A_1(\log x)[\cos\theta(\log x) + d\sin\theta(\log x)],$$

$$f_0(x) = \exp A_1(\log x)[\cos\theta(\log x) - d\sin\theta(\log x)];$$

or

$$\Gamma(x,y) = A_2(\log x)A_2(\log y)\exp A_1(\log xy),$$

$$f_1(x) = [1 - A_2(\log x)]\exp A_1(\log x),$$

$$f_0(x) = [1 + A_2(\log x)]\exp A_1(\log x).$$

Here c is an arbitrary constant in $\mathbb{R}\backslash\{0,\pm 1\}$; d is an arbitrary constant in $\mathbb{R}\backslash\{0\}$; $A_1, A_2, A_2 : \mathbb{R} \to \mathbb{R}$ are arbitrary additive functions with $A_2 \neq 0$, $A_3 \neq A_1$; and $\theta : \mathbb{R} \to \mathbb{R}$ is an arbitrary (not congruent to zero) solution of the congruence (15).

Proof. Equation (36)

$$f(x)f(y) - f(xy) = \Gamma(x,y)$$

for x, y in $\mathbb{R}_+$ can be transformed immediately into equation (1)

$$\widetilde{f}(u)\widetilde{f}(v) - \widetilde{f}(u+v) = \widetilde{\Gamma}(u,v)$$

for u, v in $\mathbb{R}$ by setting $x = e^u$, $y = e^v$, and defining $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, $\widetilde{\Gamma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(u) := f(e^u), \quad \widetilde{\Gamma}(u, v) := \Gamma(e^u, e^v).$$

Now we may apply the results of Theorem 8. $\square$

Note that the same method can be used to treat the functional equation

$$f(x)f(y) - f(\phi^{-1}[\phi(x) + \phi(y)]) = \Gamma(x, y)$$

for x, y in some set I, where $\phi : I \to \mathbb{R}$ is a bijection of I onto $\mathbb{R}$. Just define

$$\widetilde{f} := f \circ \phi^{-1}, \quad \widetilde{\Gamma}(u, v) := \Gamma(\phi^{-1}(u), \phi^{-1}(v))$$

and apply the theorem to $\widetilde{f}$, $\widetilde{\Gamma}$.

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