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Minimality and harmonicity for vector fields on the frame bundle

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Abstract. We introduce a natural metric on the frame bundle of a Riemannian manifold and show that the canonical vector fields on the frame bundle are geodesic. For a constant curved space, we show that the canonical vector fields are both minimal and harmonic and determine harmonic maps.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold and (T_1M, g_S) be its unit tangent bundle equipped with the corresponding Sasaki metric g_S. Furthermore, let $\Re^1(M)$ denote the set of smooth unit vector fields on M which is supposed to be non-empty. A unit vector field $V \in \Re^1(M)$ determines a mapping between M and T_1M embedding M into T_1M, and the mapping is also denoted by V. If M is compact and orientable, the consideration of unit vector fields leads to the introduction of two functionals on $\Re^1(M)$: the energy of V which is the energy of the corresponding map (see [EeSa64]) and the volume of V which is the volume of the immersion. A unit vector field V is said to be harmonic if it is critical for the energy functional and it is said to be minimal if it is critical for the volume functional. A minimal unit vector field corresponds to a minimal

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submanifold $V(M)$, but a harmonic unit vector fields does not necessarily yield a harmonic map. We refer to [Wi95] and [Gi01] for a general treatment of this and related problems. The harmonicity and minimality of a unit vector field and the harmonicity of its corresponding map have been considered in [BoVa00]–[BoVa01], and [GiLi01]–[Wo00] where a lot of examples are provided.

The main purpose of this paper is to consider another natural class of manifolds equipped with nonvanishing vector fields. let $F(M)$ be the orthonormal frame bundle of M

$$F(M) = \{(p; v_1, v_2, \dots v_n) : p \in M,\ v_i \in M_p,\ \langle v_i, v_j \rangle = \delta_{ij}\}.$$

It is known that the tangent bundle of $F(M)$ is trivial and there exist globally defined vector fields $\{E_i, E_{kl}, 1 \leq i \leq n,\ 1 \leq k < l \leq n\}$ that form a basis of the tangent space everywhere. These vector fields are called canonical vector fields. In this paper, we propose to investigate the properties of these canonical vector fields. Endowed with the natural metric on $F(M)$ to make these canonical vector fields orthonormal, we first establish explicitly the expression of the Levi–Civita connection under the canonical vector fields (see Proposition 4.2), which directly implies

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold and $F(M)$ be the orthogonal frame bundle of M endowed with the natural metric such that the canonical vector fields $\{E_i, E_{kl}, 1 \leq i \leq n,\ 1 \leq k < l \leq n\}$ are orthonormal. Then they are geodesic, i.e. the integral curves of them are geodesics.*

Naturally, one may consider the questions whether the canonical vector fields are harmonical and minimal and whether the induced maps are harmonic. The answers turn out to be positive when M is of constant curvature:

Theorem 1.2. *Let M be compact constantly curved Riemannian manifold and $F(M)$ be the frame bundle endowed with the natural metric such that the canonical vector fields are orthonormal, then the canonical vector fields are minimal and harmonic, and they induce harmonic maps between $F(M)$ and $T_1(F(M))$.*

Since there exist plenty of constant curved compact Riemannian manifolds, especially hyperbolic manifolds, Theorem 1.2 gives another class of

Riemannian manifolds that admit minimal and harmonic unit vector fields whose induced maps are harmonic.

2. Preliminaries

In this section we briefly recall some basic facts about minimal and harmonic vector fields. See [GoVa02] and the references there for details.

Let (M, g) be an n-dimensional smooth Riemannian manifold, ∇ its Levi–Civita connection and R the Riemannian curvature. Furthermore, let $\mathfrak{X}^1(M)$ denote the set of all smooth unit vector fields on M which we suppose to be non-empty. A unit vector field V can be regarded as an immersion of M into its unit tangent sphere bundle (T_1M, g_S), where g_S denotes the Sasaki metric. Then the induced metric V^*g_S is given by

$$(V^*g_S)(Y, Z) = g(Y, Z) + g(\nabla_Y V, \nabla_Z V).$$

We define two tensor fields of type $(1, 1)$, A_V and L_V, by

$$A_V = -\nabla V, \qquad L_V = I + A_V^t A_V,$$

and a function f by $f(V) = \sqrt{\det L_V}$. Then, for a closed oriented manifold M, the energy $E(V)$ and the volume $\mathrm{Vol}(V)$ of V are defined by

$$E(V) = \frac{1}{2}\int_M \mathrm{Tr}\, L_V dv = \frac{1}{2}m\,\mathrm{Vol}(M) + \frac{1}{2}\int_M |\nabla V|^2 dv,$$

$$\mathrm{Vol}(V) = \int_M f(V) dv,$$

where dv denotes the volume form on M. Note that $E(V)$ is, up to constants, equal to the quantity $\int_M |\nabla V|^2 dv$, known as the total bending of V [Wi95].

The critical point conditions for the functionals E and Vol on $\mathfrak{X}^1(M)$ have been established. To state these conditions, we introduce some tensor fields. The one-forms μ_V and $\bar{\mu}_V$ associated to the unit vector field V are defined by

$$\mu_V(X) = \mathrm{Tr}(Z \to (\nabla_Z A_V^t)X),$$

$$\tilde{\mu}_V(X) = \mathrm{Tr}(Z \to R(A_V Z, V)X).$$

Let $\{E_i, 1 \le i \le n\}$ be any local orthonormal frame field and define vector field $\triangle V$ by

$$\triangle V := \sum_{E_i} \nabla_{E_i}\nabla_{E_i} V - \nabla_{\triangle_{E_i} E_i} V.$$

Then V is a critical point for the energy functional E if and only if μ_X vanishes on $V^{\perp}$ or equivalently $\triangle V$ vanishes on $V^{\perp}$. Here $V^{\perp}$ denotes the distribution determined by tangent vectors orthogonal to V. A unit vector field X on M is said to be a harmonic vector field if it is such a critical point for the energy functional E. A harmonic field V does not always give rise to a harmonic map of M into T_1M. As was shown in [Gi01], V determines a harmonic map if and only if V is harmonic and moreover, $\tilde{\mu}_V$ vanishes on the whole tangent bundle TM.

Next, we define a tensor field K_V and a one-form ω_V, associated to V, by

$$K_V = -f(V) L_V^{-1} A_V^t,$$

$$\omega_V(X) = \mathrm{Tr}(Z \to (\nabla_Z K_V)X).$$

Then V is a critical point for the volume functional Vol if and only if ω_V vanishes on $V^{\perp}$. A field V is minimal if and only if the submanifold $V(M)$ is a minimal submanifold of (T_1M, g_S).

3. Canonical vector fields on the frame bundle

In this section, we collect some known facts about the frame bundle for the reader's convenience. See [ChCh78] for reference.

Let M be an n-dimensional Riemannian manifold, and let ∇ be the Riemannian connection over its tangent bundle. Let $e = (e_1, e_2, \dots e_n)$ be a local orthonormal frame field and $\omega := (\omega^1, \omega^2, \dots \omega^n)^T$ be the dual frame field of e. We shall make use of the following conventions about indices: $1 \le i, j, k, \dots \le n$, and shall agree that repeated indices are summed over their range. Under the local orthonormal frame field e, the connection ∇ can be expressed as $\nabla e = e\omega_c$, where $\omega_c = (\omega^i_j)$ is the connection matrix. If $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \cdots \tilde{e}_n)$ is another local orthonormal frame field with its dual frame field $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots \tilde{\omega}_n)^T$, and the transformation between them is given by $\tilde{e} = eA, \tilde{\omega} = B\omega$, where $B = A^{-1}$. Then $\tilde{w}_c = A^{-1}dA + A^{-1}\omega_c A$. Differentiating above equation, we have $\tilde{\Omega} = A^{-1}\Omega A$, where $\Omega = d\omega_c + \omega_c \wedge \omega_c$ is the curvature matrix. Assume $\Omega^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l$. We define the curvature operator as $R(X,Y)e_i = \langle X \wedge Y, \Omega^i_j \rangle e_j$. It is easy to check that $R(e_i, e_j)e_k = R^s_{kij}e_s$, and $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Set $R_{ijkl} = \langle R(e_i, e_j)e_l, e_k \rangle$. It is well-known that $R_{ijkl} = R_{klij}$. Hence we have

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l = -\omega^i_k \wedge \omega^k_j + \frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l. \quad (1)$$

Now assume that $F(M)$ is the orthonormal frame bundle of M:

$$F(M) = \{(p; v_1, v_2, \dots v_n) : p \in M,\ v_i \in M_p,\ \langle v_i, v_j \rangle = \delta_{ij}\},$$

with the natural projection $\pi : F(M) \to M$. The triple $(F(M), \pi, M)$ defines a $O(n)$-principal bundle, where $O(n)$ is the standard orthogonal group. Locally one can express elements in $F(M)$ as $v = eX$ with $X = (x^i_j) \in O(n)$. Set $Y = X^{-1}$ and define $n + \frac{n(n-1)}{2}$ forms on $F(M)$ as follows:

$$\theta^i = y^i_k \omega^k \quad \text{or} \quad \theta = (\theta^i) = Y\omega, \quad (2)$$

$$\theta^i_j = y^i_k dx^k_j + y^i_k \omega^k_l x^l_j \quad \text{or} \quad (\theta^i_j) = YdX + Y(\omega^i_j)X. \quad (3)$$

To verify that above definition is not dependent on the choice of local basis e, assume $v = \tilde{e}\tilde{X}$=eX, then $\tilde{X} = BX$ and $\tilde{Y} = YA$. Therefore $\tilde{\theta} = \tilde{Y}\tilde{\omega} = YAB\omega = Y\omega = \theta$, and

$$\begin{aligned}
\widetilde{(\theta^i_j)} &= \tilde{Y}d\tilde{X} + \tilde{Y}\widetilde{(\omega^i_j)}\tilde{X} \\
&= YA(dB \cdot X + BdX) + YA(BdA + B(\omega^i_j)A)BX \\
&= YdX + Y(\omega^i_j)X = (\theta^i_j).
\end{aligned}$$

It is easy to see that $(\theta^i, \theta^i_j; i < j)$ generate (du_i, dx^i_j) and hence, by counting the dimension, they form a global frame for the cotangent bundle on $F(M)$. Differentiate equations (2) and equation (3), we obtain the structure equations:

$$d\theta^i = -\theta^i_k \wedge \theta^k, \quad (4)$$

and

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2} R(x^k_i e_k, x^l_j e_l, x^m_p e_m, x^n_q e_n)\theta^p \wedge \theta^q. \tag{5}$$

Set $\theta^{ij} = \theta^i_j$ and define the natural Riemannian metric on $F(M)$ as follows

$$ds^2 = \sum_i (\theta^i)^2 + \sum_{1\le k<l\le n} (\theta^{kl})^2.$$

Let $(E_i, E_{kl} : k < l)$ be the dual vector fields to $(\theta^i, \theta^{kl} : k < l)$.

Definition 3.1. Above vector fields $\{E_i, E_{kl} : k < l\}$ on $F(M)$ will be called the *canonical vector fields* and the orthonormal frame field they form will be called the *canonical frame field*.

4. Levi–Civita connection of the frame bundle

In this section, we will study the Levi–Civita connection of $F(M)$ with the natural Riemannian metric defined in the last section. For the convenience, we adapt *double indices* that are simply ordered pairs $\{ij, 1 \le i < j \le n\}$. To compare with double index, ordinary indices $1 \le i, j, k, \dots, \le n$ will be called single indices. Greek letters $\alpha, \beta, \gamma, \dots$ will be used to denote double indices and capital letters $A, B, C, \dots$ will be used to denote both single and double indices, which will be called general indices. Again, repeated indices will always mean taking the summations over each of their own ranges. So we have the index ranges for the summation of the single indices, double indices, and general indices are from $\{i \mid 1 \le i \le n\}$, $\{kl \mid 1 \le k < l \le n\}$, and $\{i, kl \mid 1 \le i \le n,\ 1 \le k < l \le n\}$ respectively. Although double indices should be ordered pairs like kl with $k < l$, we will still write kl for a double index without assuming $k < l$. The convention is to switch the order of a pair if needed to make it a double index by changing the sign. For example, $c^1_{21,2}$ stands for $-c^1_{12,2}$ and $c^{21}_{13,32} = c^{12}_{13,23}$. Since

$$\pi_* E_i(\omega^j) = E_i(\pi^* \omega^j) = E_i(x^j_k \theta^k) = x^j_i,$$

therefore

$$\pi_* E_i = x^j_i e_j, \qquad \langle \pi_* E_i, \pi_* E_j \rangle = \langle E_i, E_j \rangle = \delta_{ij}.$$

Set $s_{ijkl} = R(\pi_* E_i, \pi_* E_j, \pi_* E_k, \pi_* E_l)$, we rewrite equation (4) and equation (5) as follows

$$d\theta^i = -\theta^{ik} \wedge \theta^k, \tag{6}$$

$$d\theta^{ij} = -\theta^{ik} \wedge \theta^{kj} + \frac{1}{2} s_{ijpq} \theta^p \wedge \theta^q. \tag{7}$$

Let $\mathbf{D}$ stand for the Levi-Civita connection for $F(M)$, and let $c^A_{B,C}$ be the connection coefficients under E_A. Then we have

$$\mathbf{D}_{E_A} E_B = c^C_{B,A} E_C, \qquad c^B_{A,C} = -c^A_{B,C}. \tag{8}$$

The connection forms are defined by

$$\Phi^A_B = c^A_{B,C} \theta^C, \qquad \Phi^A_B = -\Phi^B_A. \tag{9}$$

The structure equations are

$$d\theta^i = -\Phi^i_j \wedge \theta^j - \Phi^i_\alpha \wedge \theta^\alpha, \tag{10}$$

$$d\theta^\alpha = -\Phi^\alpha_k \wedge \theta^k - \Phi^\alpha_\beta \wedge \theta^\beta, \tag{11}$$

and

$$d\Phi^A_B = -\Phi^A_C \wedge \Phi^C_B + \frac{1}{2} K_{ABCD} \theta^C \wedge \theta^D, \tag{12}$$

where K_{ABCD} denotes the curvature tensor on $F(M)$. By equation (6) and equation (10), we have

$$c^i_{j,k} \theta^k \wedge \theta^j = 0, \quad c^i_{\alpha,\beta} \theta^\alpha \wedge \theta^\beta = 0, \quad ((c^i_{j,\alpha} - c^i_{\alpha,j})\theta^\alpha - \theta^{ij}) \wedge \theta^j = 0.$$

So we obtain

$$c^i_{j,k} = c^i_{k,j}, \qquad c^i_{\alpha,\beta} = c^i_{\beta,\alpha}, \tag{13}$$

and

$$c^i_{j,ij} - c^i_{ij,j} = 1, \quad c^i_{j,\alpha} = c^i_{\alpha,j} \quad \text{if } \alpha \neq ij. \tag{14}$$

Hence

$$c^i_{A,i} = -c^i_{i,A} = 0, \quad \text{for any } i \text{ and } A. \tag{15}$$

Similarly, from equation (7) and equation (11), we have

$$0 = \left(\frac{1}{2} s_{ijkl} - c^{ij}_{k,l} \right) \theta^k \wedge \theta^l, \tag{16}$$

$$0 = c^{ij}_{\beta,\gamma}\theta^\gamma \wedge \theta^\beta - \theta^{ik} \wedge \theta^{kj}, \tag{17}$$

$$0 = (c^\alpha_{\beta,i} - c^\alpha_{i,\beta})\theta^\beta \wedge \theta^i. \tag{18}$$

For convenience, we call (ij, jk, ik) a circle triple if i, j and k are different. By equation (16) to equation (18) we have

$$c^{ij}_{k,l} = c^{ij}_{l,k} + s_{ijkl}, \quad c^\alpha_{i,\beta} = c^\alpha_{\beta,i}, \quad c^{ij}_{kj,ik} = 1 + c^{ij}_{ik,kj}, \tag{19}$$

$$c^\alpha_{\beta,\gamma} = c^\alpha_{\gamma,\beta} \quad \text{if } \alpha,\ \beta,\ \gamma \text{ is not a circle triple.} \tag{20}$$

Combining equation (19) and equation (20), we have

$$c^\alpha_{A,\alpha} = -c^\alpha_{\alpha,A} = 0. \tag{21}$$

By equation (15) and equation (21), we have

$$c^A_{A,B} = c^A_{B,A} = c^B_{A,A} = 0 \qquad \forall A, B. \tag{22}$$

For further computations, the following algebraic observation is useful.

Lemma 4.1. *Let $S(3)$ be the 3-order permutation group and P be a set of three-fold indexed numbers*

$$P = \{P^{i_2}_{i_1,i_3} : (i_1, i_2, i_3) \in I\},$$

such that P is closed under the natural action of $S(3)$, namely

$$\tau \cdot p^{i_2}_{i_1,i_3} := p^{i_{\tau 2}}_{i_{\tau 1},i_{\tau 3}} \in P \quad \forall \tau \in S(3),\ \forall\, p^{i_2}_{i_1,i_2} \in P.$$

For $\mu = (12)$ and $\nu = (13)$, assume

$$\mu \cdot p = -p \qquad \nu \cdot p = p \quad \forall\, p \in P.$$

Then every element in P is 0.

Proof. The lemma follows from the observation that $(\nu \circ \mu)^3 = (123)^3 = \mathrm{id}$ and the assumption $(\nu \circ \mu)^3 \cdot p = -p$. □

According to above lemma, it is easy to see from equations (13), (19) and (20) that

$$c^i_{j,k} = c^i_{\alpha,\beta} = 0, \tag{23}$$

$$c^{\alpha}_{\beta,\gamma} = 0 \quad \text{if } (\alpha,\beta,\gamma) \text{ is not a circle triple.} \tag{24}$$

To handle a situation involved with a double index, there are two cases. First we compute $c^i_{j,ij}$.

$$\begin{aligned} c^i_{j,ij} &= -c^j_{i,ij} = 1 - c^j_{ij,i} = 1 + c^{ij}_{j,i} = 1 + c^{ij}_{i,j} + s_{ijji} \\ &= 1 - s_{ijij} - c^i_{ij,j} = 2 - s_{ijij} - c^i_{j,ij}. \end{aligned} \tag{25}$$

For the other double index $kl \neq ij$, we have

$$\begin{aligned} c^i_{j,kl} &= -c^j_{i,kl} = -c^j_{kl,i} = c^{kl}_{j,i} = c^{kl}_{i,j} + s_{klji} = -c^i_{kl,j} + s_{klji} \\ &= -c^i_{j,kl} + s_{klji}. \end{aligned} \tag{26}$$

Solving equation (25) and equation (26) and using equation (14), we obtain

$$c^i_{j,kl} = \delta_{ijkl} - \frac{1}{2} s_{ijkl}, \tag{27}$$

$$c^i_{kl,j} = -\frac{1}{2} s_{ijkl}, \tag{28}$$

where $\delta_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. Similarly

$$c^{kl}_{il,ki} = 1 + c^{kl}_{ki,il} = 1 - c^{ki}_{kl,il}, = c^{ki}_{li,kl} = -c^{li}_{ki,kl} = 1 - c^{il}_{kl,ik} = 1 - c^{kl}_{il,ki}.$$

So we obtain

$$c^{kl}_{il,ki} = -c^{kl}_{ki,il} = \frac{1}{2}, \qquad c^{\alpha}_{\beta,\gamma} = 0 \quad \text{in the other case.}$$

In summary, we have proved

Proposition 4.2. *The Levi–Civita connection on $F(M)$ under the canonical frame field can be expressed as:*

$$\begin{aligned} &\mathbf{D}_{E_{ij}} E_{il} = -\frac{1}{2} E_{jl}, \quad \mathbf{D}_{E_\alpha} E_\beta = 0 \quad \textit{otherwise}, \\ &\mathbf{D}_{E_i} E_\alpha = \frac{1}{2} s_{ij\alpha} E_j, \quad \mathbf{D}_{E_i} E_j = -\frac{1}{2} s_{ij\alpha} E_\alpha, \\ &\mathbf{D}_{E_\alpha} E_i = -\delta_{ij\alpha} E_j + \frac{1}{2} s_{ij\alpha} E_j. \end{aligned} \tag{29}$$

As a corollary, one can see that geodesic vector fields $\{E_A\}$ are geodesic, which proves Theorem 1.1.

5. The proof of Theorem 1.2

The proof of Theorem 1.2 is broken down into several propositions and each of them involves some computations that might be lengthy, but never tricky. From now on, M denotes a constant curved Riemannian manifold with constant curvature c. The notation ∇, instead of $\mathbf{D}$, is used to denote the connection on $F(M)$. The expression of the connection under the canonical vector field frame is reduced to

$$\begin{aligned}
&\nabla_{E_{ij}} E_{il} = -\frac{1}{2} E_{jl}, \quad \nabla_{E_\alpha} E_\beta = 0 \quad \text{otherwise}, \\
&\nabla_{E_i} E_\alpha = \frac{c}{2} \delta_{ij\alpha} E_j, \quad \nabla_{E_\alpha} E_i = \left(\frac{c}{2} - 1\right) \delta_{ij\alpha} E_j, \\
&\nabla_{E_i} E_j = -\frac{c}{2} E_{ij}.
\end{aligned} \tag{30}$$

From the above equations, we have

$$(\nabla_{E_B} E_D, \nabla_{E_B} E_C) = 0 \quad \text{if } B \neq C. \tag{31}$$

Define $E_{ij \vee il} = E_{jl}$ and $E_{ij \vee kl} = 0$ if the set $\{i, j\}$ does not intersect with the set $\{k, l\}$. So it is easy to see that $E_{(ij \vee il) \vee ij} = E_{il}$. To simplify the notations, set $L_B = L_{E_B}$ and $A_B = A_{E_B}$. We now compute the expressions for the operators A_B^t explicitly. Since

$$(A_i^t E_j, E_l) = (E_j, A_i E_l) = (E_j, -\nabla_l E_i) = \left(E_j, \frac{c}{2} \delta_{li\beta} E_\beta\right) = 0,$$

and

$$(A_i^t E_j, E_\alpha) = (E_j, A_i E_\alpha) = -(E_j, \nabla_{E_\alpha} E_i) = \left(1 - \frac{c}{2}\right) \delta_{ij\alpha}.$$

So we have

$$A_i^t(E_j) = \left(1 - \frac{c}{2}\right) E_{ij}. \tag{32}$$

Similar calculations imply

$$A_i^t(E_\alpha) = -\frac{c}{2} \delta_{ij\alpha} E_j, \quad A_\alpha^t(E_i) = \frac{c}{2} \delta_{ij\alpha} E_j, \quad A_\alpha^t(E_\beta) = \frac{1}{2} E_{\alpha \vee \beta}. \tag{33}$$

From the equation (32) and equation (33), one can explicitly express the operators L_B as follow.

$$L_i(E_j) = \left(1 + \frac{c^2}{4}\right) E_j, \quad L_i(E_\alpha) = \left(1 + \left(1 - \frac{c}{2}\right) \epsilon_{i\alpha}\right) E_\alpha, \tag{34}$$

where $\epsilon_{i\alpha}$ is 1 if $i \in \alpha$ and 0 otherwise. and

$$L_\alpha(E_i) = \left(1 + \frac{c^2}{4}\epsilon_{i\alpha}\right) E_i, \quad L_\alpha(E_\beta) = \left(1 + \frac{1}{4}\epsilon_{\alpha\vee\beta}\right) E_\beta. \tag{35}$$

Where $\epsilon_{\alpha\vee\beta} = 1$ if $E_{\alpha\vee\beta} \neq 0$ and $\epsilon_{\alpha\vee\beta} = 0$ otherwise. Therefore L_B is diagonalized with constant eigenvalues and $\sqrt{\det(L_B)}$ is constant. By the definition, $(K_C(E_D), E_B)$ is constant for any B, C and D. Therefore we obtain

$$(\nabla_{E_B}(K_C(E_D)), E_B) = -(K_C(E_D), \nabla_{E_B} E_B) = 0, \tag{36}$$

and for any indices B, C and D such that $C \neq D$

$$\begin{aligned}(K_C(\nabla_{E_B} E_D), E_B) &= \mathrm{const}(A_C^t(\nabla_{E_B} E_D), E_B)\\ &= \mathrm{const}(\nabla_{E_B} E_D, A_C(E_B))\\ &= -\,\mathrm{const}(\nabla_{E_B} E_D, \nabla_{E_B}(E_C)) = 0.\end{aligned} \tag{37}$$

Combining equation (36) and equation (37), we have

$$\begin{aligned}\omega_{E_C}(E_D) &= (\nabla_{E_B} K_{E_C}) E_D, E_B)\\ &= (\nabla_{E_B}(K_{E_C} E_D), E_B) - K_{E_C}(\nabla_{E_B} E_D), E_B) = 0,\end{aligned}$$

for any indices B, C and D such that $C \neq D$. Therefore we have proved

Proposition 5.1. *Each of the canonical vector fields is a minimal unit vector field.*

Now we prove that $\{E_B\}$ are unit harmonic vector fields and they induce harmonic maps from $F(M)$ to the sphere bundle $T_1(F(M))$.

Proposition 5.2. *Each of the canonical vector field is a unit harmonic vector field.*

Proof. For any single index i, we have by equations (30)

$$\begin{aligned}\triangle E_i &= \nabla_{E_j}\nabla_{E_j} E_i + \nabla_{E_\alpha}\nabla_{E_\alpha} E_i\\ &= \frac{c}{2}\nabla_{E_j}(\delta_{ij\alpha} E_\alpha) + \left(\frac{c}{2} - 1\right)\nabla_{E_\alpha}\delta_{ij\alpha} E_j\\ &= \frac{c^2}{4}\delta_{ij\alpha}\delta_{jk\alpha} E_k + \left(\frac{c}{2} - 1\right)^2 \delta_{ij\alpha}\delta_{jk\alpha} E_k\end{aligned}$$

$$= -\left(\frac{c^2}{4} + \left(\frac{c}{2} - 1\right)^2\right) E_i.$$

Similarly, for any double index β, we have

$$\begin{aligned}\triangle E_\beta &= \nabla_{E_j}\nabla_{E_j}E_\beta + \nabla_{E_\alpha}\nabla_{E_\alpha}E_\beta = \frac{c}{2}\nabla_{E_j}(\delta_{ji\beta}E_i) - \frac{1}{2}\nabla_{E_\alpha}E_{\alpha\vee\beta}\\ &= -\frac{c^2}{4}\delta_{ij\beta}\delta_{ji\delta}E_\delta - \frac{1}{4}(2n-4)E_\beta = -\frac{c^2+n-2}{2}E_\beta.\end{aligned}$$

Therefore $\triangle E_C \equiv 0 \bmod (E_C)$ for any C, which implies that E_C is harmonic. □

Proposition 5.3. *Each of the canonical vector field induces a harmonic map.*

Proof. First of all, by definition, the curvature operator R can be explicitly expressed as follow:

$$\begin{aligned}&R(E_{\alpha\vee\beta}, E_\alpha)E_\beta = 0,\\ &\quad R(E_k, E_j)E_\beta = -\frac{c^2}{4}\delta_{jl\beta}\delta_{kl\alpha}E_\alpha + \frac{c^2}{4}\delta_{kl\beta}\delta_{jl\alpha}E_\alpha - \frac{c}{2}E_{kj\vee\beta},\\ &\quad R(E_j, E_\alpha)E_i = \frac{c}{2}\left(1 - \frac{c}{2}\right)\delta_{ik\alpha}\delta_{jk\beta}E_\beta + \frac{c}{4}\delta_{ij\beta}E_{\alpha\vee\beta} - \frac{c}{2}\delta_{jk\alpha}\delta_{ik\beta}E_\beta.\end{aligned}$$

For any single index j, using the above expressions for the curvature operator R, we have

$$\begin{aligned}&R(A_j(E_i), E_j)E_i + R(A_j(E_\beta), E_j)E_\beta\\ &\quad = \frac{c}{2}\delta_{ij\beta}R(E_\beta, E_j)E_i - \left(\frac{c}{2} - 1\right)\delta_{jk\beta}R(E_k, E_j)E_\beta\\ &\quad = -\frac{c}{2}\delta_{ij\beta}R(E_j, E_\beta)E_i - \frac{c-2}{2}\delta_{jk\beta}R(E_k, E_j)E_\beta + \frac{c^2-2c}{4}\delta_{jk\beta}E_{jk\vee\beta}\\ &\quad = -\frac{c^2}{4}\left(1 - \frac{c}{2}\right)\delta_{ij\beta}\delta_{ik\beta}\delta_{jk\alpha}E_\alpha + \frac{c}{4}\delta_{ij\beta}\delta_{ij\alpha}E_{\beta\vee\alpha} - \frac{c}{2}\delta_{ij\beta}\delta_{jk\beta}\delta_{ik\alpha}E_\alpha = 0,\end{aligned}$$

which implies $\tilde{\mu}_{E_j} = 0$. Similarly for any double index α, we have

$$R(A_\alpha(E_i), E_\alpha)E_i + R(A_\alpha(E_\beta), E_\alpha)E_\beta$$

$$= -\frac{c}{2}\delta_{ij\alpha}R(E_j, E_\alpha)E_i + \frac{1}{2}R(E_{\beta\vee\alpha}, E_\alpha)E_\beta$$

$$= -\frac{c^2}{4}\left(1 - \frac{c}{2}\right)\delta_{ij\alpha}\delta_{ik\alpha}\delta_{jk\beta} + \frac{c}{4}\delta_{ij\alpha}\delta_{ij\beta}E_{\alpha\vee\beta} - \frac{c}{2}\delta_{ij\alpha}\delta_{jk\alpha}\delta_{ik\beta}E_\beta = 0.$$

Therefore $\tilde{\mu}_{E_\alpha} = 0$. Thus all the canonical vector fields induce harmonic maps between $F(M)$ and $T_1(F(M))$. □

We have completed the proof of Theorem 1.2 by Propositions 5.1, 5.2 and 5.3.

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