Publ. Math. Debrecen
67/3-4 (2005), 471–480

On a class of Einstein space-time manifolds

By ADELA MIHAI (Bucharest) and RADU ROSCA (Paris)

Abstract. We deal with a general space-time (M, g) with usual differentiability conditions and hyperbolic metric g of index 1, which carries 3 skew-symmetric Killing vector fields X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g. It is shown that such a space-time (M, g) is an Einstein manifold of curvature -1, which is foliated by space-like hypersurfaces M_s normal to e and the immersion $x : M_s \to M$ is pseudo-umbilical. In addition, it is proved that the vector fields X, Y, Z and e are exterior concurrent vector fields and X, Y, Z define a commutative Killing triple, M admits a Lorentzian transformation which is in an orthocronous Lorentz group and the distinguished spatial 3-form of M is a relatively integral invariant of the vector fields X, Y and Z.

0. Introduction

Let (M, g) be a general space-time with usual differentiability conditions and hyperbolic metric g of index 1.

We assume in this paper that (M, g) carries 3 skew-symmetric Killing vector fields (abbr. SSK) X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g (see [R1], [MRV]. Therefore, if ∇ is the Levi–Civita connection and $\wedge$ means the wedge product of vector

---

Mathematics Subject Classification: 53C15, 53D15, 53C25.
Key words and phrases: space-time, skew-symmetric Killing vector field, exterior concurrent vector field, orthocronous Lorentz group.
This paper was written while the first author has visited Yamagata University, Faculty of Education, supported by a JSPS postdoctoral research fellowship. She would like to express her hearty thanks for the hospitality she received during this visit.

fields, one has

$$\nabla X = X \wedge e, \quad \nabla Y = Y \wedge e, \quad \nabla Z = Z \wedge e. \tag{0.1}$$

Setting $\alpha = X^\flat$, $\beta = Y^\flat$, $\gamma = Z^\flat$ ($\flat : TM \to T^*M$, $X \mapsto X^\flat$, denotes the musical isomorphism defined by g), one derives by (0.1)

$$d\alpha = 2\omega \wedge \alpha, \quad \omega = e^\flat, \tag{0.2}$$

and clearly similar equations for β and γ.

Since one finds that ω is an exact form, equations (0.2) may be written in terms of d^ω-cohomology as $d^{-2\omega}\alpha = 0$ and say that α, β, γ are $d^{-2\omega}$-exact forms. From (0.2) it also follows that the existence of such a space-time is determined by an exterior differential system in involution [C2].

The following theorem is proved:

Any space-time (M, g) satisfying (0.1) is an Einstein manifold of curvature -1, which is foliated by space-like hypersurfaces M_s normal to e and the immersion $x : M_s \to M$ is pseudo-umbilical [Ch].

The following additional properties are also obtained.

i) the vector fields X, Y, Z and e are exterior concurrent vector fields (abbr. EC) and X, Y, Z define a commutative Killing triple;

ii) M admits a Lorentzian transformation which is in an orthocronous Lorentz group [CWD];

iii) the 3-form $\varphi = \alpha \wedge \beta \wedge \gamma$ is a relatively integral invariant of the vector fields X, Y and Z.

1. Preliminaries

Let (M, g) be a pseudo-Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume that M is oriented and that ∇ is the Levi–Civita connection. Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ the classical musical isomorphisms defined by g.

Following [P], we set $A^q(M, TM) = \Gamma\,\mathrm{Hom}(\Lambda^q TM, TM)$ and notice that elements of $A^q(M, TM)$ are vector valued q-forms, $q \leq \dim M$.

Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$ the *exterior covariant derivative operator* with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$).

If $p \in M$, then the vector valued 1-form $dp \in A^1(M,TM)$ is called *the soldering form* of M (dp is the canonical vector valued 1-form of M and one has $d^\nabla(dp) = 0$). The operator $d^\omega = d + e(\omega)$ acting on ΛM is called *the cohomology operator*, where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, i.e. $d^\omega u = du + \omega \wedge u$, for any $u \in \Lambda M$. We have $d^\omega \circ d^\omega = 0$ and if $d^\omega u = 0$, u is said to be d^ω-*closed* [GL].

Any vector field $X \in \Gamma TM$ such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M,TM)$$

is said to be an *exterior concurrent* vector field.

The 1-form π, which is called the *concurrence form*, is given by

$$\pi = fX^\flat, \quad f \in C^\infty M.$$

If $\mathcal{R}$ denotes the *Ricci tensor* of ∇, we have

$$\mathcal{R}(X,Y) = -(n-1)fg(X,Z), \quad Z \in \Gamma TM,\ n = \dim M,$$

and consequently

$$f = -\frac{1}{n-1}\operatorname{Ric}(X),$$

where $\operatorname{Ric}(X)$ means the *Ricci curvature* of M with respect to X.

Let $\mathcal{O} = \{e_A \mid A \in \{1,\dots,n\}\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe.

With respect to $\mathcal{O}$ and $\mathcal{O}^*$, the soldering form dp and E. Cartan's structure equations in indexless form are

$$dp = \omega^A \otimes e_A \in A^1(M,TM), \tag{1.1}$$

$$\nabla e = \theta \otimes e \in A^1(M,TM), \tag{1.2}$$

$$d\omega = -\theta \wedge \omega, \tag{1.3}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.4}$$

In the above equations, θ, respectively Θ, are the *local connection forms* in the bundle $\mathcal{O}(M)$, respectively the *curvature forms* on M.

2. Main result

Let (M,g) be a general space-time with usual differentiability conditions and normal *hyperbolic* metric g, i.e. of index 1. Let $\mathcal{O}=\{e_A \mid A\in\{1,\dots,n\}\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe. We agree to denote by e_a, $a,b\in\{1,2,3\}$ and by e_4 the space-like vector basis and the time-like vector basis, respectively, w.r.t. g. Then, by reference to [C1] (see also [MRV]), one has

$$dp = -\omega^a \otimes e_a + \omega^4 \otimes e_4 \Longrightarrow \langle dp, dp\rangle = (\omega^4)^2 - \sum_{a=1}^{3}(\omega^a)^2, \tag{2.1}$$

and Cartan's structure equations are expressed by

$$\begin{cases} \nabla e_a = -\theta_a^b e_b + \theta_a^4 \otimes e_4, \\ \nabla e_4 = -\theta_4^a \otimes e_a, \end{cases} \tag{2.2}$$

$$\begin{cases} d\omega^a = -\omega^b \wedge \theta_b^a + \omega^4 \otimes \theta_4^a, \\ d\omega^4 = -\omega^a \wedge \theta_a^4, \end{cases} \tag{2.3}$$

$$\begin{cases} d\theta_a^b = \Theta_a^b - \theta_a^c \wedge \theta_c^b + \theta_a^4 \wedge \theta_4^b, \\ d\theta_4^a = \Theta_4^a - \theta_4^c \wedge \theta_c^a. \end{cases} \tag{2.4}$$

In the following, in order to simplify, we set $\omega^4 = \omega$ and $e_4 = e$.

In this paper we assume that the manifold M under consideration carries 3 space-like vector fields X, Y, Z which are skew-symmetric Killing vector fields (abbr. SSK) [R1], [MRV] having as generative the unit time-like vector field e. In order to simplify, we also set

$$X^\flat = \alpha, \quad Y^\flat = \beta, \quad Z^\flat = \gamma. \tag{2.5}$$

Under these conditions, by reference to [R1], [MRV], one has

$$\begin{cases} \nabla X = X \wedge e = \omega \otimes X - \alpha \otimes e, \\ \nabla Y = Y \wedge e = \omega \otimes Y - \beta \otimes e, \\ \nabla Z = Z \wedge e = \omega \otimes Z - \gamma \otimes e, \end{cases} \tag{2.6}$$

and since X, Y, Z are space-like, we set

$$X = X^a e_a, \quad Y = Y^a e_a, \quad Z = Z^a e_a, \tag{2.7}$$

i.e.

$$\|X\|^2 = -\sum_{a=1}^{3}(X^a)^2, \quad \|Y\|^2 = -\sum_{a=1}^{3}(Y^a)^2, \quad \|Z\|^2 = -\sum_{a=1}^{3}(Z^a)^2. \tag{2.8}$$

From (2.2) and (2.6), one derives

$$dX^a - X^b\theta_b^a = X^a\omega \tag{2.9}$$

and

$$\alpha = -X^a\theta_a^4. \tag{2.10}$$

Hence, it follows from (2.10)

$$\theta_a^4 = -\omega^a \tag{2.11}$$

Setting $2\varphi_x = \sum_{a=1}^{3}(X^a)^2$, $2\varphi_y = \sum_{a=1}^{3}(Y^a)^2$, $2\varphi_z = \sum_{a=1}^{3}(Z^a)^2$, one gets at once from (2.9)

$$\frac{d\varphi_x}{2\varphi_x} = \omega, \quad \frac{d\varphi_y}{2\varphi_y} = \omega, \quad \frac{d\varphi_z}{2\varphi_z} = \omega, \tag{2.12}$$

which shows that the generative ω (i.e. the time-like covector) is exact. Further, taking the exterior differential of α, one finds by (2.9) and by the structure equations (2.3)

$$d\alpha = 2\omega \wedge \alpha, \tag{2.13}$$

which is the general equation of a SSK vector field [R1].

Now, by (2.11) and the second equation (2.4), one derives on behalf of the structure equations (2.3) regarding the space-like covectors ω^a

$$\Theta_4^a = -\omega^a \wedge \omega. \tag{2.14}$$

Further, taking the exterior differentials of (2.4)

$$\begin{cases} X^2(\Theta_1^2 - \omega^1 \wedge \omega^2) + X^3(\Theta_1^3 - \omega^1 \wedge \omega^3) = 0, \\ X^1(\Theta_2^1 - \omega^2 \wedge \omega^1) + X^3(\Theta_2^3 - \omega^2 \wedge \omega^3) = 0. \end{cases}$$

Since, clearly, similar equations hold for Y and Z, one infers

$$\Theta_b^a = \omega^b \wedge \omega^a. \tag{2.15}$$

So, by reference to a known formula regardind space forms, we conclude by (2.14) and (2.15) that the space-time manifold (M, g) under consideration is an *Einstein manifold* of curvature -1.

Since $d\omega = 0$, it is seen that (M, g) is foliated by space-like hypersurfaces M_s and by (2.11) and the equation (2.2), one has

$$\nabla e = \omega^a \otimes e_a = -dp_s, \tag{2.16}$$

where dp_s is the *spatial component* of the soldering form dp. Then the second fundamental form $II = -\langle dp_s, dp_s\rangle$ associated with the immersion $x : M_s \to M$ being conformal to the metric tensor g_s of M_s, it follows [Ch] that the immerssion x is *pseudo-umbilical*.

On the other hand, clearly, Y and Z enjoy the same properties as X and one may write

$$\begin{cases} d\beta = 2\omega \wedge \beta \Leftrightarrow d^{-2\omega}\beta = 0, \\ d\gamma = 2\omega \wedge \gamma \Leftrightarrow d^{-2\omega}\gamma = 0 \end{cases} \tag{2.17}$$

(since ω is an exact form one may say that, cohomologically, the dual forms α, β, γ of the SSK vector fields X, Y, Z are $d^{-2\omega}$-exact).

Finally, by (2.12), (2.13) and (2.17), it is seen that the existence of the considered space-time is defined by an exterior differential system Σ, whose characteristic numbers [C2] are $r = 4$, $s_0 = 1$, $s_1 = 3$. Consequently, we conclude that Σ is in *involution*, in the sense of Cartan [C2], and depends of 3 arbitrary functions of one argument.

Since the vector fields X, Y, Z are orthogonal to e, one easily finds by (2.6)

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which proves that the SSK vector fields X, Y, Z define a *commutative triple.*

Summing up, we state the following.

Theorem. *Let (M, g) be a space-time manifold with normal hyperbolic metric g. Assume that M carries 3 skew-symmetric Killing vector* *fields X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g. Then M is an Einstein manifold of curvature -1. The existence of the triple $\{X,\ Y,\ Z\}$ is assured by an exterior differential system Σ in involution. Such a manifold (M, g) has also the following properties:*

i) *M is foliated by space-like hypersurfaces M_s tangent to $\{X, Y, Z\}$, normal to e and the immersion $x : M_s \to M$ is pseudo-umbilical;*

ii) *the vector fields $\{X,\ Y,\ Z\}$ define a commutative triple of Killing vector fields.*

3. Additional properties

In this section we shall make some additional considerations regarding the Einstein manifold defined in Section 2.

By using (2.16) and operating by d^∇, one may write

$$d^\nabla(dp_s) = \omega \wedge dp_s. \tag{3.1}$$

Because $dp = \omega \otimes e - dp_s$, one derives

$$\nabla^2 e = \omega \wedge dp. \tag{3.2}$$

By reference to [R1], [PRV], the above equation proves that e is an EC vector field. Hence, following the general theory [PRV], if W is any vector field on M, one may write $\mathcal{R}(e, W) = -3g(e, W)$ (we notice that for any space-like vector field Z_s one has $\mathcal{R}(e, Z_s) = 0$).

Recall that the *sectional curvature* $K_{U\wedge V}$ of any vector fields U, V is expressed by

$$K_{U\wedge V} = \frac{g(R(U,V)V,U)}{\|U\|^2\,\|V\|^2 - g(U,V)^2}. \tag{3.3}$$

Then by (2.6) and recalling that X, Y, Z are space-like vector fields, one finds $K_{X\wedge Y} = K_{Y\wedge Z} = K_{Z\wedge X} = -1$, which means that M is of curvature -1.

Next, operating by d^∇ on the vector fields X, Y, Z, one derives

$$\nabla^2 X = \alpha \wedge dp, \quad \nabla^2 Y = \beta \wedge dp, \quad \nabla^2 Z = \gamma \wedge dp. \tag{3.4}$$

From (3.4) it follows that the SSK vector fields $X,\ Y,\ Z$ are also exterior concurrent, as the time-like vector field e.

Next, setting $\alpha = \lambda U^1$, $\beta = \lambda U^2$, $\gamma = \lambda U^3$ as a space-like covector basis, one may define a subgroup of Lorentz by the group of space-like rotations $O(3)$ (*orthocronous transformations* [CWD]) preserving the time-like vector field e.

Also, one has

$$U^4 = \omega^4 = \omega, \quad \alpha^2 + \beta^2 + \gamma^2 = \lambda^2 \sum_{a=1}^{3} (\omega^a)^2, \tag{3.5}$$

where λ is a scalar field.

Equations (3.5) imply

$$\sum_{a=1}^{3} \left[(X^a)^2 + (Y^a)^2 + (Z^a)^2 \right] = \lambda^2 \tag{3.6}$$

and

$$\begin{cases} \sum_{a \neq b} [X^a X^b + Y^a Y^b + Z^a Z^b] = 0, \\ \sum_{a \neq b} [X^a X^c + Y^a Y^c + Z^a Z^c] = 0, \\ \sum_{a \neq b} [X^b X^c + Y^b Y^c + Z^b Z^c] = 0. \end{cases} \tag{3.7}$$

Making use of equations (2.9), one finds that the differentiation of (3.7) holds good and the differentiation of (3.6) gives

$$\frac{d\lambda}{\lambda} = \omega \tag{3.8}$$

(recall that ω is an exact form).

Finally, we shall outline a certain property of the Lie algebra induced by the space-like vector fields X, Y, Z. We agree to call the 3-form

$$\varphi = \alpha \wedge \beta \wedge \gamma \tag{3.9}$$

the *distinguished spatial form* of M.

By (2.13) one gets at once

$$d\varphi = 6\omega \wedge \varphi \Longleftrightarrow d^{-6\omega}\varphi = 0 \Longleftrightarrow \mathcal{L}_e\varphi = 6\varphi, \tag{3.10}$$

which shows that φ is an $d^{-6\omega}$-exact form and e defines an *infinitesimal conformal transformation* of φ.

By (2.12) one has

$$\frac{dg(X,X)}{4g(X,X)} = \frac{dg(Y,Y)}{4g(Y,Y)} = \frac{dg(Z,Z)}{4g(Z,Z)} = \omega, \tag{3.11}$$

and one derives

$$i_X\varphi = g(X,X)\beta \wedge \gamma + g(X,Y)\gamma \wedge \alpha + g(X,Z)\alpha \wedge \beta$$

and similar relations for Y and Z.

So, by (2.13), one infers $d(i_X\varphi) = 8\omega \wedge i_X\varphi$, which gives $\mathcal{L}_X\varphi = 2\omega \wedge i_X\varphi$.

Finally, by exterior differentiation, one may write $d(\mathcal{L}_X\varphi) = 0$ and clearly similar relations for Y and Z hold.

Hence, following a known definition [A], one may state that the distinguished spatial 3-form φ is a *relatively integral invariant* of the SSK vector fields X, Y, Z.

Consequently, the following results were obtained.

Theorem. *Let (M, g) be the space-time manifold defined in the Section 2 and let X, Y, Z be the* 3 *skew-symmetric Killing vector fields which determine M and e the unit time-like vector field of the hyperbolic metric g. One has the following properties:*

i) *the vector fields X, Y, Z and e are exterior concurrent vector fields;*

ii) *M admits an orthogonal transformation of a space-like Lorentz subgroup $O(3)$;*

iii) *the* 3*-form $\varphi = \alpha \wedge \beta \wedge \gamma$ is a relatively integral invariant of the vector fields X, Y and Z.*

References

[A1] R. Abraham, Foundations of Mechanics, *W. A. Benjamin*, *New York*, 1967.

[C1] E. Cartan, Oeuvres, *Gauthier-Villars*, *Paris*, 1955.

[C2] E. Cartan, Systèmes Différrentiels Extérieurs et Leurs Applications Géométriques, *Hermann*, *Paris*, 1945.

[Ch] B. Y. Chen, Geometry of Submanifolds, *M. Dekker*, *New York*, 1973.

[CWD] Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics, *North Holland*, 1977.

[D] J. Dieudonné, Treatise on Alnalysis, Vol. 4, *Academic Press*, *New York*, 1974.

[GL] F. Guedira and L. Lichnerowicz, Géométrie des algèbres de Lie locales de Kirilov, *J. Math. Pures Appl.* **63** (1984), 407–484.

[MRV] I. Mihai, R. Rosca and L. Verstraelen, Some Aspects of the Differential Geometry of Vector Fields, Vol. 2, PADGE, *KU Leuven, KU Brussel*, 1996.

[PRV] M. Petrovic, R. Rosca and L. Verstraelen, Exterior concurrent vector fields on Riemannian manifolds, *Soochow J. Math.* **15** (1989), 179–187.

[P] W. A. Poor, Differential Geometric Structures, *Mc Graw Hill*, 1981.

[R1] R. Rosca, Exterior concurrent vector fields on a conformal cosymplectic manifold endowed with a Sasakian structure, *Libertas Math.* **6** (1986), 167–174.

[R2] R. Rosca, On exterior concurrent skew symmetric Killing vector field, *Rend. Sem. Mat. Messina* **2** (1993), 131–145.

ADELA MIHAI
FACULTY OF MATHEMATICS
UNIVERSITY OF BUCHAREST
STR. ACADEMIEI 14
010014 BUCHAREST
ROMANIA

E-mail: adela@math.math.unibuc.ro

RADU ROSCA
59 AV. EMILE ZOLA
75015 PARIS
FRANCE

(Received May 24, 2004)