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# On problem of multiple lattice circle arrangements

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*Dedicated to Professor J. Horváth on his 70th birthday*

**Abstract.** We deal with lattice circle arrangements which form a $k$-fold covering (with closed circles) and a $(k+1)$-fold packing (of open circles) for some suitable values of $k$. The centres of these special circle arrangements give the points of regular plane patterns yielding the least limiting variance in case of a special plane stochastic process.

## 1. Introduction

A system of congruent open circles forms a $k$-fold packing if each point of the plane belongs to at most $k$ circles of the system. Analogously, a system of congruent closed circles is a $k$-fold covering if each point of the plane belongs to at least $k$ circles of the system. The notion of multiple packing and covering was introduced by L. Fejes Tóth. A circle arrangement is called a lattice arrangement if the centers of the circles form a plane lattice.

The density of a multiple circle packing and covering can be defined analogously to the simple packing and covering. The densest simple lattice packing, the thinnest simple lattice covering, the densest 2-fold lattice packing and one of the densest 4-fold lattice packings (see Figure 1) have

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an interesting property. The circles in these circle arrangements form a $k$-fold covering (with closed circles) and a $(k+1)$-fold packing (of open circles) for some suitable values of $k$.

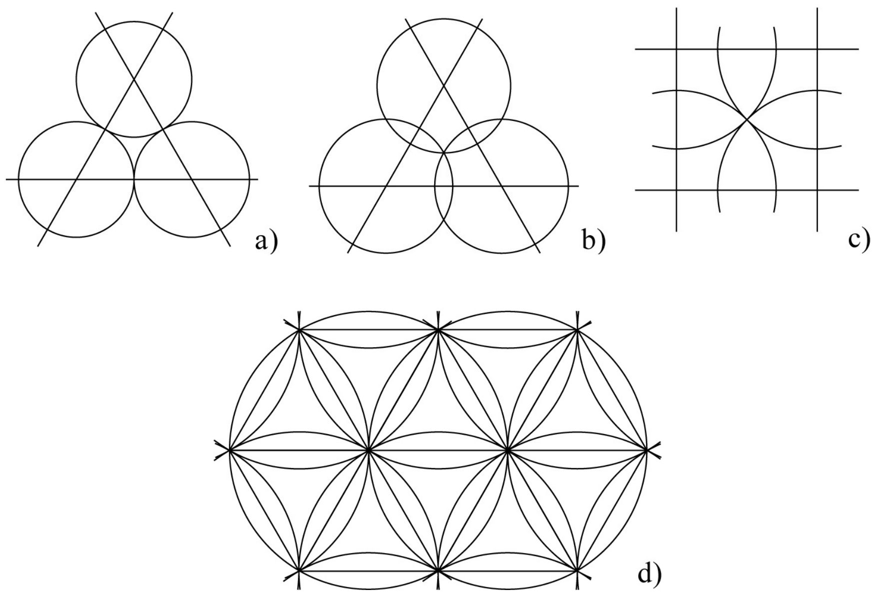


*Figure 1*

The centres of these special circle arrangements give the points of regular plane patterns yielding the least limiting variance in case of a special plane stochastic process [4].

In this paper we give the possible lattice circle arrangements with the above property for $0 \leq k \leq 8$ and $k \geq 2 \cdot 10^5 - 1$. We also give some non-lattice circle arrangements of the above type.

## 2. Notations, definitions

The origin of the plane lattice $\Gamma$ will be denoted by $O$ (Figure 2.a). Denote by $X$ the vector $\vec{OX}$ and its endpoint. The basis $\vec{OA}$ and $\vec{OB}$ of

$\Gamma$ is reduced by Minkowski, i.e.

$$|A| \leq |B| \leq |B-A|, \quad 0 \leq \angle(AOB) \leq \frac{\pi}{2}. \tag{1}$$

With the notations $a = |A|$ $b = |B|$ $x = \frac{a}{b}$ and $\alpha = \angle(AOB)$ (1) goes over into

$$0 < x \leq 1, \quad 0 \leq \cos\alpha \leq \frac{x}{2}. \tag{2}$$

We consider a rectangular cartesian coordinate system $x$, $y = \cos\alpha$. A point of the triangle $OPQ$ with $O(0,0)$, $P(1,0)$, $Q\left(1, \frac{1}{2}\right)$ (Figure 2.b) can be ordered to the lattice $\Gamma$ because of the inequalities (2), and a lattice belongs to a point $(x,y) \neq (0,0)$ of the triangle $OPQ$ (up to a similarity).

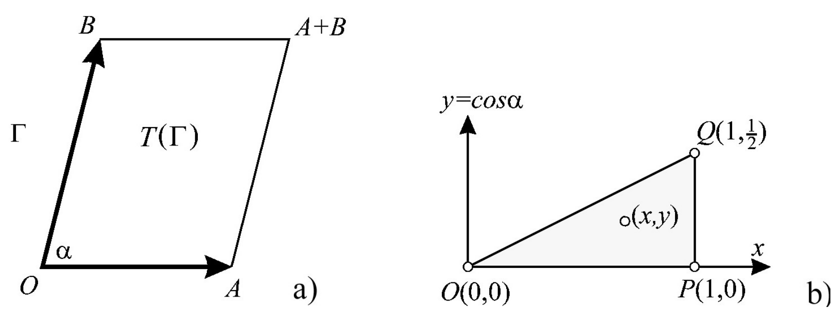


*Figure 2*

Let $T(\Gamma)$ be the area of the fundamental parallelogram and $h$ its height perpendicular to $OA$.

Let $x$, $y$, $z$ be the sides of a triangle and $T$ its area. Then the radius $r$ of the circumcircle is

$$r = \frac{xyz}{4T}. \tag{3}$$

In what follows we denote by $L(\Gamma, R)$ the circle arrangement, where $\Gamma$ is the lattice of the centers of the circles, and $R$ is the radius of the circles. The lattice circle arrangement $L(\Gamma, R)$ is of type $L_k^{k+1}(\Gamma, R)$ if the circles form a $k$-fold covering (with closed circles) and a $(k+1)$-fold packing (of open circles).

We denote by $\delta^p(B_2)$ and $\delta^p_\Gamma(B_2)$ the densities of the densest $p$-fold packings of open congruent circles in the general case and in the lattice case. We use the notations $\vartheta^q(B_2)$ and $\vartheta^q_\Gamma(B_2)$ for the densities of the thinnest $q$-fold coverings with closed congruent circles in the general case and in the lattice case.

Let $k[XYZ]$ be the circumcircle of the triangle $XYZ$ and $r[XYZ]$ its radius. We denote by $k[XY]$ the circle with the diameter $XY$. The radius of $k[XY]$ is $r[XY]$.

## 3. Results

**3.1.** The lattice $\Gamma_1$ is reduced by Minkowski and

$$2 \leq |A|\,.$$

The circle arrangements of unit circles $L(\Gamma_1, 1)$ are of type $L_0^1(\Gamma_1, 1)$ (Figure 3.a).

**3.2.** The lattice $\Gamma_2$ is reduced by Minkowski and

$$1 \leq |A| \leq \sqrt{3} \quad \text{and} \quad r[OAB] = 1.$$

The circle arrangements of unit circles $L(\Gamma_2, 1)$ are of type $L_1^2(\Gamma_2, 1)$ (Figure 3.b).

**3.3.** The lattice $\Gamma_3$ is reduced by Minkowski and

$$|A| = |B| = |B - A| = 1.$$

The circle arrangement of unit circles $L(\Gamma_3, 1)$ is of type $L_3^4(\Gamma_3, 1)$ (Figure 3.c).

**Theorem 1.** *If the lattice arrangement of unit circles $L(\Gamma, 1)$ is of type $L_k^{k+1}(\Gamma, 1)$ and $k \leq 8$, then $\Gamma \equiv \Gamma_1$ or $\Gamma \equiv \Gamma_2$ or $\Gamma \equiv \Gamma_3$.*

Proof. The densenst $k$-fold lattice packings of congruent open circles are known for $k \leq 10$ [5], [7], [2], [3], [18], [16], [17] and the thinnest $k$-fold lattice coverings with congruent closed circles were determined for $k \leq 8$ [10], [1], [11]–[13], [3], [6], [14], [15]. We have the inequality $\delta_\Gamma^{k+1}(B_2) < \vartheta_\Gamma^k(B_2)$ for $6 \leq k \leq 8$ which shows that lattice arrangements of unit circles $L(\Gamma, 1)$ of type $L_k^{k+1}(\Gamma, 1)$ can exist for $k \leq 5$.

*1.* $k = 0$. The statement is trivial.

*2.* $k = 1$. The lattice $\Gamma$ of the circle arrangement is reduced by Minkowski. The lattice triangle $OAB$ is not obtuse angled and the open lattice circle $k[OAB]$ does not contain lattice points. If $1 > |A|$ holds, then for example the neighbourhood of the lattice point $A$ is covered at least 3 times. In case of $r[OAB] < 1$ the neighbourhood of the centre of $k[OAB]$ would be covered at least 3 times. If $r[OAB] > 1$, then the neighbourhood of the centre of $k[OAB]$ is not covered. Our circle arrangement $L(\Gamma, 1)$ is of type $L_1^2(\Gamma, 1)$. Therefore we have $1 \leq |A|$ and $r[OAB] = 1$. The regular triangle inscribed in the circle $k[OAB]$ with unit radius has maximal perimeter among the triangles inscribed in $k[OAB]$, namely the perimeter $3\sqrt{3}$. It follows $|A| \leq \sqrt{3}$. Then we have $\Gamma \equiv \Gamma_2$ which was to be proved.

*3.* $k = 2$. The lattice triangle $O(2A)(A + B)$ is not obtuse angled and the open lattice circle $k[O(2A)(A + B)]$ contains only one lattice point [8]. Our circle arrangement $L(\Gamma, 1)$ is of type $L_2^3(\Gamma, 1)$. Therefore we have $r[O(2A)(A + B)] = 1$ and $2 \leq |3A|$. Let $F$ be the midpoint of $O(A + B)$. The neighbourhood of $F$ is covered at least 4 times if $|A + B| < 2$. If $|A + B| \geq 2$ holds, then we have $r[O(2A)(A + B)] > 1$. This contradiction means that the circle arrangements $L(\Gamma, 1)$ can not be of type $L_2^3(\Gamma, 1)$.

*4.* $k = 3$. We consider the circle arrangement $L(\Gamma, 1)$. The lattice $\Gamma$ is reduced by Minkowski. Therefore the lattice triangle $O(2A)(B)$ is not obtuse angled and the open lattice circle $k[O(2A)(A + B)]$ contains only two lattice points namely $A$ and $A + B$. Our circle arrangement $L(\Gamma, 1)$ is of type $L_3^4(\Gamma, 1)$. Therefore we have $r[O(2A)(B)] = 1$, $1 \leq |2A|$ and $1 \leq |B|$.

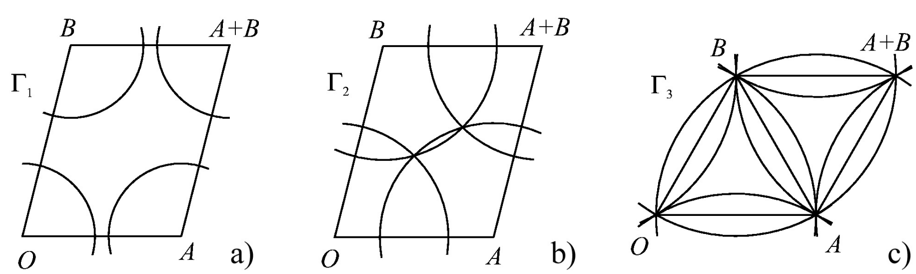


*Figure 3*

The circle arrangement $L(\Gamma, 1)$ is a 3-fold covering with closed unit circles. We distinguish three cases according to the lattice circles which contain two lattice points of $\Gamma$ in their interiors and are circumcircles of not obtuse angled lattice triangles [8].

*4.1.* Let $G$ be the midpoint of $A(2A)$. If $|A+B-G| \geq \frac{3}{2}|A|$, then the lattice triangle $O(3A)(A+B)$ is not obtuse angled and the open circle $k[O(3A)(A+B)]$ contains the lattice points $A$ and $2A$ (Figure 4.a). We have the inequality $r[O(3A)(A+B)] > r[O(2A)(B)] = 1$. Thus the neighbourhood of the centre of $k[O(3A)(A+B)]$ is covered at least five times, that means $L(\Gamma, 1)$ is not of type $L_3^4(\Gamma, 1)$.

*4.2.* In case of $|A+B-G| \leq \frac{3}{2}|A|$ we consider the lattice triangle $O(2A-B)(A+B)$ (Figure 4.b). This triangle is not obtuse angled and $A$, $2A \in k[O(2A-B)(A+B)]$. We assume that $A-B$ does not belong to $k[O(2A-B)(A+B)]$. Then $k[O(2A-B)(A+B)] = r[O(2A)(B)] = 1$ holds ($L(\Gamma, 1)$ is of type $L_3^4(\Gamma, 1)$). Using (3) we have

$$\begin{aligned} &\frac{4a^2b^2(4a^2+b^2-4aby)}{16a^2b^2(1-y^2)} \\ &= \frac{(a^2+b^2+2aby)(a^2+4b^2-4aby)(4a^2+b^2-4aby)}{36a^2b^2(1-y^2)}. \end{aligned} \tag{4}$$

A simple calculation shows that (4) holds only for $x = 1$ and $y = \frac{1}{2}$. Then $\Gamma \equiv \Gamma_3$ (Figure 3.c).

*4.3.* If $|A+B-G| \leq \frac{3}{2}|A|$ and $A-B \in k[O(2A-B)(A+B)]$, then we consider the lattice triangle $OA(2B)$ (Figure 4.c). This triangle is not obtuse angled and $B$, $A+B \in k[OA(2B)]$. The equality $r[OA(2B)] = 1 = r[O(2A)(B)]$ holds ($L(\Gamma, 1)$ is of type $L_3^4(\Gamma, 1)$). From this equality we get $\Gamma \equiv \Gamma_3$.

*5.* $k = 4$. We consider the circle arrangement $L(\Gamma, 1)$. The lattice $\Gamma$ is reduced by Minkowski. The circle arrangement $L(\Gamma, 1)$ is of type $L_4^5(\Gamma, 1)$. Therefore we have $1 \leq \left|\frac{5}{2}A\right|$, $\left|\frac{3}{2}A\right| \leq 1$,

$$1 \leq |2B| \quad \text{and} \quad 2 \leq |2A+B|. \tag{5}$$

The following closed lattice circles contain six lattice points and are circumcircles of not obtuse angled lattice triangles [8]:

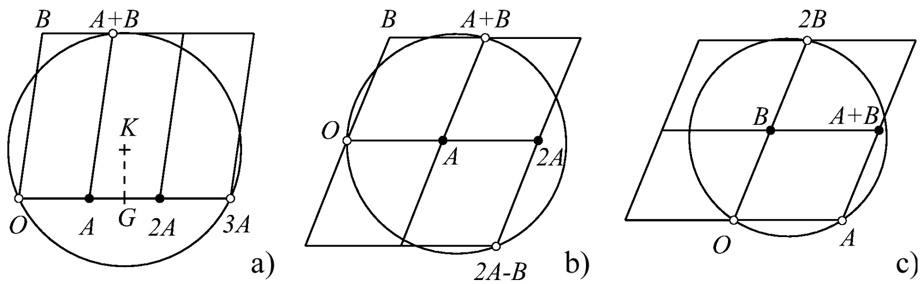


*Figure 4*

$k_1 = k[O(4A)(2A+B)]$ for $x \le \frac{1}{2}$ and $0 \le y \le \frac{x}{2}$; $k_2 = k[O(3A)(2A+B)]$ for $\frac{1}{2} \le x \le \sqrt{\frac{1}{2}}$ and $0 \le y \le \frac{x}{2}$; $k_3 = k[O(2A+B)(2A-B)]$ for $\sqrt{\frac{1}{2}} \le x \le 1$ and $\frac{2x^2-1}{4x} \le y \le \frac{1}{4x}$; $k_4 = k[O(2A)(2B)]$ for $\sqrt{\frac{1}{2}} \le x \le 1$ and $\frac{1}{4x} \le y \le \frac{x}{2}$; $k_5 = k[O(2A+B)(A+2B)]$ for $\sqrt{\frac{1}{2}} \le x \le 1$ and $0 \le y \le \frac{2x^2-1}{4x}$. Let $H_i$ $(i = 1,2,3,4,5)$ denote the set of the points in the triangle $OPQ$ corresponding to the lattices where the open circle $(i = 1,2,3,4,5)$ contains three lattice points (Figure 5).

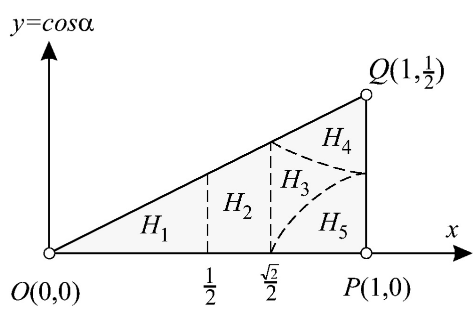


*Figure 5*

The circle arrangement $L(\Gamma,1)$ is of type $L_4^5(\Gamma,1)$. Therefore the radius $r_i$ of $k_i$ is the unit. It can be proved that $2 > |2A+B|$ for $i = 1,2,3,5$ and $1 > |2B|$ for $i = 4$ ($\Gamma$ is reduced by Minkowski ). This is in contradiction to (5) which means that the circle arrangements $L(\Gamma,1)$ cannot be of type $L_4^5(\Gamma,1)$.

*6.* $k = 5$. Let the circle arrangement $L(\Gamma, 1)$ be of type $L_5^6(\Gamma, 1)$. $L(\Gamma, 1)$ is a 5-fold covering. Therefore we consider the closed lattice circles which are circumcircles of not obtuse angled lattice triangles and contain 7 lattice points. In [12] one can find these lattice circles and the division of the triangle $OPQ$ in the set $H_i$, similarly as in 5. The circle arrangement $L(\Gamma, 1)$ is a 6-fold packing. In this case we have two types of open lattice circles. The first type is the open circle $k[\Delta j]$ which is the circumcircle of the not obtuse angled lattice triangle $\Delta j$ and contains 4 lattice points. In case of the second type we have the open circle $k[OZ_j]$ which with lattice points $O$, $Z_j$ where 5 lattice points belong to $k[OZ_j]$. In [9] one can find the necessary lattice circles and the division of the triangle $OPQ$ in the set $\tilde{H}_j$.

By using the results in [12] and [9] we have the following cases:

*6.1.* The lattice circle $k_8 = k[O(5A)(2A + B)]$ for $0 < x \leq \sqrt{\frac{2}{13}}$ and $0 \leq y \leq \frac{x}{2}$ or $\sqrt{\frac{2}{13}} \leq x \leq \sqrt{\frac{1}{6}}$ and $0 \leq y \leq \frac{1-6x^2}{x}$ satisfies the conditions in 6 in case of 5-fold covering. Clearly, the lattice circle $\tilde{k}_2 = k[O(3A)B]$ is proper for $0 \leq x \leq \sqrt{\frac{2}{3}}$ and $0 \leq y \leq \frac{x}{2}$ or $\sqrt{\frac{2}{3}} \leq x \leq 1$ and $0 \leq y \leq \frac{1}{3x}$ in case of a 6-fold packing. We have $r_8 = \tilde{r}_2 = 1$. (Otherwise the neighbourhood of the centres of $k_8$ and $\tilde{r}_2$ would be covered at least 7 times or at most 4 times.) It is clear that $\tilde{r}_2 < r_8$. This is a contradiction.

*6.2.* The lattice circle $k_7 = k[O(4A)(2A + B)]$ for $\sqrt{\frac{1}{6}} \leq x \leq \frac{1}{2}$ and $0 \leq y \leq \frac{6x^2-1}{4x}$ or $\frac{1}{2} \leq x \leq \sqrt{\frac{2}{7}}$ and $0 \leq y \leq \frac{2-7x^2}{2x}$ satisfies the conditions in 6 in case of a 5-fold covering. The lattice circle $\tilde{k}_2 = k[O(3A)B]$ is proper for the above $x$ and $y$ in case of a 6-fold packing. We have a contradiction as in 6.1.

*6.3.* For a 5-fold covering we have the lattice circle $k_6 = k[O(2A + B)(3A - B)]$ in case $\sqrt{\frac{2}{13}} \leq x \leq \sqrt{\frac{1}{6}}$ and $\frac{1-6x^2}{x} \leq y \leq \frac{x}{2}$ or $\sqrt{\frac{1}{6}} \leq x \leq \frac{1}{2}$ and $\frac{6x^2-1}{4x} \leq y \leq \frac{x}{2}$ or $\frac{1}{2} \leq x \leq \sqrt{\frac{2}{7}}$ and $\frac{2-7x^2}{6x} \leq y \leq \frac{1-x^2}{6x}$ or $\sqrt{\frac{2}{7}} \leq x \leq \sqrt{\frac{2}{5}}$ and $\frac{7x^2-2}{8x} \leq y \leq \frac{1-x^2}{6x}$. The lattice circle $\tilde{k}_2 = k[O(3A)B]$ is proper for the above $x$ and $y$ in case of a 6-fold packing (see 6.1). It follows $1 = r_6 > r[O(3A)(3A - B)] = \tilde{r}_2 = 1$ from $3A \in k_6$. This contradiction means that $L(\Gamma, 1)$ cannot be a circle arrangement of type $L_5^6(\Gamma, 1)$.

*6.4.* The lattice circle $k_2 = k[O(2A)(A+2B)]$ for $\sqrt{\frac{2}{7}} \le x \le \sqrt{\frac{2}{5}}$ and $0 \le y \le \frac{7x^2-2}{8x}$ or $\sqrt{\frac{2}{5}} \le x \le \sqrt{\frac{1}{2}}$ and $0 \le y \le \frac{x}{4}$ satisfies the conditions in 6 in case of a 5-fold covering. The lattice circle $\tilde{k}_2 = k[O(3A)B]$ is proper for $0 \le x \le \sqrt{\frac{2}{3}}$ and $0 \le y \le \frac{x}{2}$ or $\sqrt{\frac{2}{3}} \le x \le 1$ and $0 \le y \le \frac{1}{3x}$ in case of a 6-fold packing. We have $r_2 = \tilde{r}_2 = 1$. A simple calculation shows that $\tilde{r}_2 < r_2$. This is a contradiction.

*6.5.* We consider the lattice circle $k_3 = k[O(2A)(2B)]$ for $\sqrt{\frac{2}{5}} \le x \le \sqrt{\frac{1}{2}}$ and $\frac{x}{4} \le y \le \frac{3x^2-1}{2x}$ or $\sqrt{\frac{1}{2}} \le x \le \sqrt{\frac{6}{11}}$ and $\frac{x}{4} \le y \le \frac{3-5x^2}{2x}$. This circle satisfies the conditions in 6 in case of a 5-fold covering. For a 6-fold packing the open lattice circle $\tilde{k}_5 = k[(B-A)(A-B)]$ contains 5 lattice points for $0 < x \le 1$ and $0 \le y \le \frac{x}{2}$. $(2A)(2B)$ can be equal to the diameter of $k_3$ only for $x = \sqrt{\frac{1}{2}}$ and $y = \frac{1}{2}\sqrt{\frac{1}{2}}$. In this case the lattice circle $\tilde{k}_4 = k[O(2A+B)(2A-B)]$ satisfies the conditions for a 6-fold packing and $\tilde{r}_4 < r_3$ holds. Therefore $\tilde{r}_5 < r_3$ or $\tilde{r}_4 < r_3$ which is in contradiction to $\tilde{r}_5 = \tilde{r}_4 = r_3 = 1$.

*6.6.* We prove that the lattice circle $k_4 = k[O(A+B)(3A-B)]$ for $\frac{1}{2} \le x \le \sqrt{\frac{2}{5}}$ and $\frac{1-x^2}{6x} \le y \le \frac{x}{2}$ or $\sqrt{\frac{2}{5}} \le x \le \sqrt{\frac{1}{2}}$ and $\frac{3x^2-1}{2x} \le y \le \frac{x}{2}$ satisfies the conditions in 6 in case of the 5-fold covering $\sqrt{\frac{1}{2}} \le x \le 1$ and $\frac{1-x^2}{2x} \le y \le \frac{x}{2}$. For a 6-fold packing we have the open lattice circle $\tilde{k}_5 = k[(B-A)(A-B)]$ as in 6.5. The side $(A+B)(3A-B)$ cannot be equal to the diameter of $k_4$ (the scalar product $(A+B).(3A-B)$ is positive). Then $1 = \tilde{r}_5 < r_4 = 1$ holds. This is a contradiction.

*6.7.* Let $\overline{H}_1 = \left\{(x,y) \mid \sqrt{\frac{1}{2}} \le x \le 1,\ 0 \le y \le \frac{1-x^2}{2x}\right\}$. The lattice circle $k_1 = k[O(3A)B]$ satisfies the conditions in 6 for a 5-fold covering in $\overline{H}_1$. Now let $\tilde{H}_{42} = \left\{(x,y) \mid \sqrt{\frac{1}{2}} \le x \le 1,\ 0 \le y \le \frac{2x^2-1}{4x}\right\}$, $\tilde{H}_5 = \left\{(x,y) \mid \sqrt{\frac{1}{2}} \le x \le \sqrt{\frac{7}{12}},\ \frac{3-4x^2}{4x} \le y \le \frac{x}{2} \text{ and } \sqrt{\frac{7}{12}} \le x \le 1,\ \frac{4x^2+1}{20x} \le y \le \frac{x}{2}\right\}$,
$\tilde{H}_6 = \left\{(x,y) \mid \sqrt{\frac{1}{2}} \le x \le \sqrt{\frac{7}{12}},\ \frac{2x^2-1}{x} \le y \le \frac{3-4x^2}{4x}\right\}$,
$\tilde{H}_7 = \left\{(x,y) \mid \sqrt{\frac{1}{2}} \le x \le \sqrt{\frac{7}{12}},\ \frac{2x^2-1}{4x} \le y \le \frac{2x^2-1}{x} \text{ and } \sqrt{\frac{7}{12}} \le x \le 1,\ \frac{2x^2-1}{4x} \le y \le \frac{4x^2+1}{20x}\right\}$. The following lattice circles are proper for a 6-fold

packing: $\tilde{k}_4 = k[O(2A+B)(2A-BO)]$ for $\tilde{H}_{42} \cap \overline{H}_1$, $\tilde{k}_5 = k[(B-A)(A-BO)]$ for $\tilde{H}_5 \cap \overline{H}_1$, $\tilde{k}_6 = k[O(3A)(2A+BO)]$ for $\tilde{H}_6 \cap \overline{H}_1$, $\tilde{k}_7 = k[O(2A+B)(A-BO)]$ for $\tilde{H}_7 \cap \overline{H}_1$. It holds $r_1 = \tilde{r}_i = 1\,(i = 4,5,6,7)$ for $\tilde{H}_i \cap \overline{H}_1$ $(i = 42,5,6,7)$ ($L(\Gamma,1)$ is of type $L_5^6(\Gamma,1)$). Calculations show that $\tilde{r}_i < r_1$ $(i = 4,5,6,7)$. (The calculations are somewhat lengthy and thus they will not be presented here.) This is a contradiction.

The cases 6.1–6.7 show that the circle arrangements $L(\Gamma,1)$ cannot be of type $L_4^5(\Gamma,1)$. This completes the proof. □

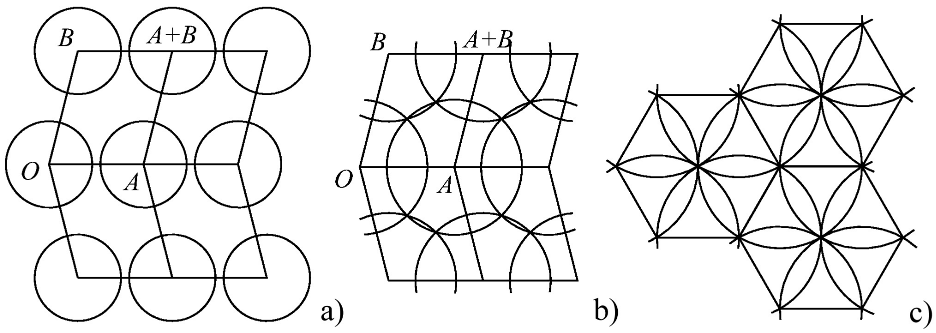


*Figure 6*

*Remark.* Using the lattice circle arrangements $L(\Gamma_1,1)$ and $L(\Gamma_2,1)$ we can construct an infinite number of non-lattice circle arrangements, for example using reflections on a lattice line which are of type $L_0^1(\Gamma,1)$ and $L_1^2(\Gamma,1)$ (Figure 6.a, 6.b). We place the centers of circles at the vertices of a regular $\{6,3\}$ tessellation, the radii of the circles being equal to the side of the hexagons. This circle arrangement is of type $L_1^2(\Gamma,1)$ (Figure 6.c).

**Theorem 2.** *Assume that* $1 < h < 2$ *for* $\Gamma$. *Then there are no lattice arrangements of unit circles* $L(\Gamma,1)$ *of type* $L_k^{k+1}(\Gamma,1)$ *for* $k > 3$.

Proof. Bolle proved in [3] the equalities $\delta_\Gamma^{k+1}(B_2) = (k+1)\delta_\Gamma^1(B_2) = (k+1)\frac{\pi}{\sqrt{12}}$ and $\vartheta_\Gamma^k(B_2) = k\vartheta_\Gamma^1(B_2) = k\frac{2\pi}{\sqrt{27}}$ for $\Gamma$ with $1 < h < 2$ and for $k \in N$. We obtain $\delta_\Gamma^{k+1}(B_2) < \vartheta_\Gamma^k(B_2)$ for $k > 3$ which proves the statement. □

**Theorem 3.** *There are no lattice arrangements of unit circles* $L(\Gamma,1)$ *of type* $L_k^{k+1}(\Gamma,1)$ *for* $k \geq 2 \cdot 10^5 - 1$.

PROOF. Using the results in [3] we have the inequalities $\frac{\delta_\Gamma^{k+1}}{k+1} < 1 - \frac{0.1}{(k+1)^{\frac{3}{4}}}$ for $k \geq 2 \cdot 10^5 - 1$ and $1 + \frac{0.179}{k^{\frac{3}{4}}} < \frac{\vartheta_\Gamma^k}{k}$ for $k \geq 12000$. A simple calculation shows that $\delta_\Gamma^{k+1}(B_2) < \vartheta_\Gamma^k(B_2)$ for $k \geq 2 \cdot 10^5 - 1$. □

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