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On central Frattini extensions of finite groups

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To Professor A. Bovdi for his seventieth birthday

Abstract. An extension of a group A by a group G is thought of here simply as a group H containing A as a normal subgroup with quotient H/A isomorphic to G. It is called a central Frattini extension if A is contained in the intersection of the centre and the Frattini subgroup of H. The result of the paper is that, given a finite abelian A and finite G, there exists a central Frattini extension of A by G if and only if A can be written as a direct product $A = U \times V$ such that U is a homomorphic image of the Schur multiplicator of G and the Frattini quotient of V is a homomorphic image of G.

1. Discussion

Given a finite abelian group A and an arbitrary finite group G, consider all extensions

$$1 \to A \to H \to G \to 1 \tag{1}$$

such that (the embedded copy of) A lies in the intersection of the centre and the commutator subgroup:

$$A \leq Z(H) \cap H'. \tag{2}$$

It is well known (for an elementary exposition and a bibliography, see Wiegold [5]) that such extensions exist if and only if A is a homomorphic

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image of the (Schur) multiplicator of G. The aim of this note is to put on record a similar result, with a similarly elementary proof, concerning *central Frattini extensions.* These are extensions (1) satisfying

$$A \leq Z(H) \cap \Phi(H) \tag{3}$$

instead of (2), with $\Phi(G)$ denoting the Frattini subgroup of G. Suzuki's book [4] discussed them at length under the name of irreducible central extensions, but stopped short of the following.

Theorem. *Given a finite group G and a finite abelian group A, there exists a central Frattini extension of A by G if and only if A can be written as a direct product $A = U \times V$ such that U is a homomorphic image of the multiplicator of G and the Frattini quotient of V is a homomorphic image of G.*

Since the proof requires no new ideas, it is somewhat surprising that the result still does not seem to be in the literature.

Let M denote the multiplicator of G; for each prime p, let $p^{f(p)}$ denote the order of the largest elementary abelian p-quotient of G; and write the exponent of A as $\prod_p p^{e(p)}$. Let B be the direct product of M and $\sum_p f(p)$ cyclic groups, $f(p)$ of which have order $p^{e(p)}$. An alternative form of the condition in the theorem is that A should be a homomorphic image of B.

A related question in the literature concerns Frattini extensions which are not necessarily central. Given G and a prime p, consider extensions (1) with elementary abelian p-groups A such that $A \leq \Phi(H)$, and view these A as $(\mathbb{Z}/p\mathbb{Z})G$-modules, the action of G coming from conjugation in H. Gaschütz [2] had shown that the A of maximal order are all isomorphic to the second Heller translate of the 1-dimensional trivial $(\mathbb{Z}/p\mathbb{Z})G$-module, which Griess and Schmid [3] then called the *Frattini module of G with respect to $\mathbb{Z}/p\mathbb{Z}$.* Gaschütz [2] also showed that the other A which occur in such extensions are precisely the homomorphic images of this Frattini module. It follows from our theorem that *the largest G-trivial quotient of the Frattini module of G with respect to $\mathbb{Z}/p\mathbb{Z}$ is the largest exponent-p quotient of $M \times (G/G')$.* (This was implicit in an aside in [3], immediately before Theorem 2, which invoked the Universal Coefficient Theorem.)

The results quoted here from [2] were extended in [1] to Frattini extensions with kernels that are not elementary abelian (and even to profinite

Frattini extensions). A special case of that work shows that if we consider Frattini extensions by G with abelian kernels A of exponent dividing some given positive integer $e = \prod_p p^{e(p)}$, the kernels of maximal order are all isomorphic as G-modules, and as groups they are just direct products of cyclic groups of order e. Thus one might speak of the *Frattini module of G with respect to $\mathbb{Z}/e\mathbb{Z}$,* and note that the theorem also determines the largest G-trivial quotient of this module.

Corollary. *The largest G-trivial quotient of the Frattini module of G with respect to $\mathbb{Z}/e\mathbb{Z}$ is the direct product of M/M^e and $\sum_p f(p)$ cyclic groups, $f(p)$ of which have order $p^{e(p)}$.* $\square$

(Here M/M^e denotes the largest exponent e quotient of M.)

2. Proofs

We shall need some preparatory results about finite abelian groups.

Lemma 1. *If A is a finite abelian group and A/E an elementary abelian quotient, then A has a direct decomposition $A = \prod_i C_i$ with cyclic factors C_i such that $E = \prod_i (C_i \cap E)$.*

Proof. Induction on the order of A, exploiting that in a finite abelian group the maximum of the orders of cyclic subgroups equals the maximum of the orders of cyclic quotients. Let A/D be a cyclic quotient of maximal order. As $|a| \le |A/D|$ for all $a \in A$, if a coset aD generates A/D then $A = C \times D$ with $C = \langle a \rangle$. If $DE = A$, such an a can be chosen from E, and then $E = C \times (D \cap E)$. Otherwise choose a direct complement F/E for DE/E in A/E (this is where we use that A/E is elementary): then $DF = A$ and so C can be chosen within F. In this case, $C \cap DE \le F \cap DE = E$ whence $C \cap DE = C \cap E$ and $DE = (C \cap E) \times D$, $E = (C \cap E) \times (D \cap E)$ follow. The last conclusion being available in either case, an application of the inductive hypothesis (with D and $D \cap E$ in place of A and E) completes the proof. $\square$

Corollary 1. *If A is a finite abelian group and A/E is an elementary abelian quotient, then A has a direct decomposition $A = U \times V$ such that $E = U \times \Phi(V)$.*

Proof. Denote by U the product of the C_i that lie in E, and by V the product of the other C_i. $\square$

Corollary 2. *If A is a finite abelian group and B is any subgroup of A, then A has a direct factor V such that $BV = A$ and $B \cap V \leq \Phi(V)$.*

Proof. Apply Corollary 1 with E defined by $E/B = \Phi(A/B)$. Then $A = UV = EV$ so $A/B = (E/B)(BV/B) = \Phi(A/B)(BV/B) = BV/B$ (because the Frattini subgroup is omissible), while $B \cap V \leq E \cap V = \Phi(V)$. $\square$

Lemma 2. *Given two finite abelian groups V, W, there exists an abelian Frattini extension of V by W if and only if $V/\Phi(V)$ is a homomorphic image of W.*

Proof. As usual, if X is any group, we write $\exp X$ for the exponent of X and $d(X)$ for the minimum of the cardinalities of the generating sets of X. We can assume that we are dealing with abelian p-groups. If there is an extension X of the kind envisaged, then $V \leq \Phi(X)$ and $X/V \cong W$ imply that $d(V) \leq d(X) = d(W)$. Conversely, if $d(V) \leq d(W)$ and W is written as P/Q with P a direct product of $d(W)$ cyclic groups of orders $(\exp V)(\exp W)$, then Q has a subgroup R with $Q/R \cong V$, and P/R can serve as X. $\square$

Proof of the Theorem. Suppose first that there is a group H such that $A \leq Z(H) \cap \Phi(H)$ and $H/A \cong G$. Set $B = A \cap H'$ and use Corollary 2 to obtain $A = U \times V = BV$ and $B \cap V \leq \Phi(V)$. Then $U \cong A/V \leq Z(H/V) \cap (H/V)'$ and $(H/V)/(A/V) \cong G$, so (by the property of the multiplicator mentioned at the beginning of this paper) U is a homomorphic image of the multiplicator of G. Further, H/H' is an abelian Frattini extension of $H'V/H'$ by G/G', $H'V/H' \cong V/(B \cap V)$, and V has the same Frattini quotient as $V/(B \cap V)$, so it follows from Lemma 2 that $V/\Phi(V)$ is a homomorphic image of G.

Conversely, suppose that $A = U \times V$ where U is a homomorphic image of the multiplicator of G and $V/\Phi(V)$ is a homomorphic image of G. By that property of the multiplicator, there is a group T with a normal subgroup U such that $U \leq Z(T) \cap T'$ and $T/U \cong G$, and then $T/T' \cong G/G'$. By Lemma 2, there is an abelian group X with a subgroup V such that $V \leq \Phi(X)$ and $X/V \cong G/G'$. The quotient of $T \times X$ over $T'V$ has a direct decomposition $TV/T'V \times T'X/T'V$ with both direct factors isomorphic to G/G'. Let $H/T'V$ be the diagonal subgroup formed along any isomorphism between the two direct factors, so it is a common direct complement to each of those: $TH = HX = TX$ and $TV \cap H = H \cap T'X = TV \cap T'X = T'V$. In particular, it follows that

$$T \cap H = (T \cap TV) \cap H = T \cap (TV \cap H) = T \cap T'V = T'.$$

Thus the projection of the direct product $T \times X$ onto its second direct factor maps H onto X with kernel T', and it maps $T'V$ onto V. As $V \leq \Phi(X)$, we conclude that every maximal subgroup of H/T' must contain $T'V/T'$.

As $TX = HX$ and X is central, we have $T' = (TX)' = (HX)' = H'$. Recall that $U \leq T'$, so $A = UV \leq T'V \leq H$. As U is central in T and X is abelian, A is obviously central in H. If a maximal subgroup K of H failed to contain A, we would have $H = AK$ and hence $K' = H' = T'$; but then K/T' would be a maximal subgroup of H/T' not containing $T'V/T'$. This proves that $A \leq Z(H) \cap \Phi(H)$. It remains to note that

$$H \cap UX = H \cap (T'X \cap UX) = (H \cap T'X) \cap UX = T'V \cap UX = UV = A,$$

so $H/A = H/(H \cap UX) \cong HUX/UX = TX/UX \cong T/U \cong G$. $\square$

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