*Publ. Math. Debrecen*
**43 / 1–2** (1993), 129–134

# Congruence lattices of modular lattices

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## 1. Introduction

In 1974 I have proved the following (see [1]):

**Theorem.** *Every finite distributive lattice is the congruence lattice of some modular lattice.*

In this note we give a short, new proof of this result. We will use two well-known lattice constructions.

## 2. Preliminaries

Let $L$ be a lattice and let $P$ be a partially ordered set, $L^P$ denotes the lattice of all order preserving maps of $P$ to $L$, partially ordered by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. Then $L^P$ is a special subdirect power of $L$. If $L$ is the two element lattice 2 then $2^P$ is a distributive lattice.

Every finite distributive lattice $D$ can be represented in this form, where $P$ is the dual of the poset of all nonzero join-irreducible element of $D$. Let $R$ be a well-ordered chain, $R^d$ denotes the dual of $R$. Let $R + 1$ denote the lattice obtained from $R$ by adjoining a new unit element.

**Lemma 1.** *Let $R$ be a well-ordered chain, then $2^{R^d} \cong R + 1$.*

If $a \in L$ then $\bar{a}$ denotes the corresponding constant mapping: $\bar{a}(x) = a$ for all $x \in P$. The elements $\bar{a}$ $(a \in L)$ form a sublattice of $L^P$, which is obviously isomorphic to $L$; we identify $L$ with this sublattice. Let $[a, b]$ be a prime-interval of $L$, then the correspondig interval $[\bar{a}, \bar{b}]$ of $L^P$ is isomorphic to $2^P$.

$\mathcal{M}_3$ denotes the five-element modular nondistributive lattice. The elements of $\mathcal{M}_3$ are $o < a, b, c < i$. In [2] it was proved the following:

**Lemma 2.** *Every congruence relation of $\mathcal{M}_3^P$ is determined by its restriction to the ideal $(\bar{a}]$, and conversely every congruence relation of this ideal can be extended to $\mathcal{M}_3^P$.*

If $P$ is a cain, then $\mathcal{M}_3^P$ can be easily visualized. Consider the following three sublattices of $\mathcal{M}_3$: $E = \{o, i\}$, $F = \{o, c, i\}$ and $G = \{o, a, b, i\}$. Then $E^P$, $F^P$, $G^P$ are sublattices of $\mathcal{M}_3^P$ and $E^P = F^P \cap G^P$ holds. Moreover, it is easy to see that $E^P \cong 2^P$, $G^P \cong 2^P \times 2^P$, while $F^P$ is isomorphic to the following lattice $\{(x, y) \in 2^P \times 2^P : x \leq y\}$. $G^P$ is a "square" and $F^P$ is a "half square". $F^P$ is called a flap. If $P$ is three element chain that $\mathcal{M}_3^P$ is illustrated by *Figure* 1. (The elements of $F^P$ are the black circles.) It is clear, that $\mathcal{M}_3^P = F^P \cup G^P$.

Figure 1.

Let $C_1$ and $C_2$ be two chains. The direct product $C_1 \times C_2$ we shall call the "grid"; its elements are the "grid elements". We augment the grid as follows: let $a, b \in C_1$, $a < b$, $c, d \in C_2$, $c < d$ and assume that the intervals are isomorphic. Then we add a flap to $[a, b] \times [c, d] = [(a, c), (b, d)]$ such that this flap with the direct product $[a, b] \times [c, d]$ is a lattice isomorphic to $\mathcal{M}_3^P$ where $P$ denotes the dual of the poset $\{x : a \leq x < b\}$. If $[a, b]$ and $[c, d]$ are prime-intervals then in the lattice $C_1 \times C_2$ the interval $[(a, c), (b, d)]$ is a prime square. In this case we add to $C_1 \times C_2$ only one new element $m$, and $(a, c), (a, d), (b, c), (b, d), m$ form a sublattice isomorphic to $\mathcal{M}_3$ (the "flap" contains three elements: $(a, c), (b, d)$ and $m$). If we have a family of disjunct squares then we can apply this augumentation simultaneously.

Te second construction is the Hall-Dilworth gluing: if a nonempty filter $\mathcal{F}$ of a lattice $L_0$ is isomorphic to an ideal $\mathcal{I}$ of a lattice $L_1$, let $L$ be the union of $L_0$ and $L_1$ with the elements of $\mathcal{F}$ and $\mathcal{I}$ identified via the isomorphism. $L$ can be ordered with the transitive closure of the union of the orders on $L_0$ and $L_1$. Then under this order $L$ is a lattice; $L_0$ is an ideal of $L$ and $L_1$ is a filter of $L$. If $L_0$ and $L_1$ are both modular then $L$ is a modular lattice. $A$ congruence relation $\Theta$ of $\mathcal{F} = \mathcal{I}$ can be extended to $L$ if and only if $\Theta$ can be extended to $L_0$ and $L_1$.

Te ordinary sum of the lattices $K_0$ and $K_1$ will be denoted by $K_0 \oplus K_1$, we place $K_1$ on the top of $K_0$ and identify the unit element of $K_0$ with the zero of $K_1$. (This is obviously a special Hall-Dilworth gluing, where $\mathcal{I}$ is the zero element of $K_0$ and $\mathcal{F}$ is the unit element of $K_1$).

It is easy to see that the augmented grid can be defined as repeated gluing of lattices which are either isomorphic to the direct product of two chains or they are isomorphic to $\mathcal{M}_3^R$ for some chain $R$.

## 3. Proof

For every finite poset $P$ we have to construct a modular lattice $L_P$ such that $\operatorname{Con} L_P \cong 2^P$. We use induction on the size of $P$. If $|P| = 1$, i.e. $2^P \cong 2$, then $L_P$ is the two element chain, 2. We construct $L_P$ having the following two properties:

(1) $L_P$ has an element $a_P$ with a complement $a'_P$, and the filter $[a_P)$ is a well-ordered chain.

(2) $L_P$ contains a subchain $a_P = b_0 < b_1 < \cdots < b_n = 1_p$, where $n = |P|$ and the irreducible congruences of $L_P$ are exactly the congruences in the form $\Theta(b_{i-1}, b_i)$ $(i = 1, 2, \ldots, n)$.

It is clear, that for $|P| = 1$ the lattice $L_1 \cong 2$ satisfies these properties. Let $p$ be a minimal element of $P$, where $|P| = n > 1$. Then by our assumption for the poset $Q = P \setminus \{p\}$ there exists a modular lattice $L_Q$ satisfying (1), (2) and $\operatorname{Con} L_Q \cong 2^Q$. The element $a_Q$ is given in (1). By condition (2) $L_Q$ contains a chain $a_Q = b_0 < b_1 < \cdots < b_{n-1} = 1_Q$, and the join-irreducible congruences of $L_Q$ are the principal congruences $\Theta(b_{k-1}, b_k)$ $(k = 1, 2, \ldots, n-1)$, consequently we have a bijection $\varphi([b_{k-1}, b_k]) = p_k \in Q$, the map $\varphi$ is called a coloring, $p_k$ is the color of the interval $[b_{k-1}, b_k]$. Assume that $p_{k_1}, p_{k_2}, \ldots, p_{k_r}$, are the covers of $p$ in the poset $P$, i.e. $p \prec p_{k_j}$ $(j = 1, 2, \ldots, r)$. Let $C$ be the chain $[a_Q) \subseteq L_Q$. Then $a_Q = b_0 < b_1 < \cdots < b_{n-1} = 1_Q$ is a subchain of $C$. For every natural number $i$ let $C_i$ be a chain isomorphic to $[b_{k_1-1}, b_{k_1}] \oplus \cdots \oplus [b_{k_r-1}, b_{k_r}]$ and $b_k^i$ denotes the image of $b_k$ under this isomorphism. Finally, we consider the ordinary sum of these chains with a new unit element $1^*$ adjointed, i.e. $\mathcal{C} = \{C_0 \oplus C_1 \oplus C_2 \oplus \ldots\} \cup \{1^*\}$.

We extend $\varphi$ to $\mathcal{C}$ as follows: $\varphi([b_{k-1}^i, b_k^i]) = p_k \in Q$ for $i = 0, 1, 2 \ldots$.

Consider the grid $\mathcal{C} \times \mathcal{C}$ and for every $i$ and $k$ the square $[(b^i_{k-1}, b_{k-1}), (b^i_k, b_k)] = [b^i_{k-1}, b^i_k] \times [b_{k-1}, b_k]$. By the definition of $\varphi$, $\varphi([b^i_{k-1}, b^i_k]) = \varphi([b_{k-1}, b_k]) = p_k$, i.e. this is a monochromatic square. By Lemma 1 there exists a poset $P_k$ such that $[b_{k-1}, b_k] \cong 2^{P_k}$ (indeed $P_k$ is the dual of the chain $\{x \in [b_{k-1}, b_k] : x < b_k\}$). We extend all monochromatic squares to $\mathcal{M}_3^{P_k}$ as described in paragraph 2 for all $k \in \{k_1, k_2, \dots , k_r\}$ (don't forget that $p_{k_1}, p_{k_2}, \dots , p_{k_r}$ are the covers of $p$), the resulting lattice is $C_w$.

In the lattice $C_w$ the interval $[(b^0_{k_0}, b_0), (b^0_{k_0}, b_{n-1})]$ is isomorphic to $C$. This interval is an ideal of $C_w$. On the other hand $C$ is a filter of $L_Q$. Now, we apply the Hall-Dilworth gluing for the lattices $L_Q$ and $C_w$, we obtain the lattice $T$. Then $(b^0_{k_0}, b_k)$ is identified with $b_k$. (See *Figure* 2.)

Figure 2.

**Lemma 3.** $Con\, T \cong 2^P$.

PROOF. We determine the irreducible congruence relations of $T$. By Lemma 2 every congruence relation of $C_w$ is determined by its projections to $\mathcal{C}$ and $C$. On the other hand every congruence relation of $L_Q$ is determined by its restriction to filter $C = [a_Q)$. Then we have:

(∗) Every congruence relation of $T$ is determined by its restriction to the subchains $\mathcal{C}$ and $C$.

Let $\Theta$ be an irreducible congruence relation of $T$. We distinguish two cases:

*Case 1.* $(1^*, b_0) \not\equiv (c, b_0)(\Theta)$ for all $c \in \mathcal{C}$. If $b_{k-1} \le x < y \le b_k$ for some $k \in \{k_1, k_2, \ldots, k_r\}$ in the chain $C$ and $(x, b_0) \equiv (y, b_0)(\Theta)$ then by Lemma 2 $(b^0_{k_0}, x) \equiv (b^0_{k_0}, y)(\Theta)$ holds, i.e. $\Theta$ is determined by its restriction to $C$. This proves that $\Theta$ is the extension of a congruence relation in the form $\Theta(b_{k-1}, b_k)$ of $L_Q$, i.e. $\Theta = \bar{\Theta}(b_{k-1}, b_k)$.

It is easy to see that every $\Theta = \Theta(b_{k-1}, b_k)$ can be extended to $T$. We describe the $\Theta$ classes.

If $k \in \{k_1, k_2, \ldots, k_r\}$ i.e. $p_k$ is a cover of $p$ in the poset $P$ then the nontrivial $\Theta$–classes on $C_w$ are the intervals: $[(b^i_{k-1}, x), (b^i_k, x)]$, $[(y, b_{k-1}), (y, b_k)]$ and the monochromatic squares $[(b^i_{k-1}, b_{k-1}), (b^i_k, b_k)]$, where $x \in C$, $y \in \mathcal{C}$.

If $k \notin \{k_1, k_2, \ldots, k_r\}$ then the nontrivial $\Theta$–classes on $C_w$ are the intervals $[(y, b_{k-1}), (y, b_k)]$.

It is easy to prove that these relations are indeed congruences.

*Case 2.* $(1^*, b_0) \equiv (c, b_0)(\Theta)$ for some $c \in \mathcal{C}$, $c < 1^*$. Then by the definition of $\mathcal{C}$ there exists a natural number $i$, with the property: $c \in C_{i-1} \subseteq \mathcal{C}$. In this case $(b^i_k, b_0) \equiv (b^i_{k-1}, b_0)(\Theta)$ for $k \in \{k_1, \ldots, k_r\}$, i.e. $(1^*, b_0) \equiv (b^0_{k_0}, b_0) = a_Q(\Theta)$ and $(b^0_{k_0}, b_k) \equiv (b^0_{k_0}, b_{k-1})(\Theta)$. This proves that $\Theta \ge \bar{\Theta}(b_{k-1}, b_k)$ for all $k \in \{k_1, \ldots, k_r\}$, and $\Theta_p = \Theta((b^0_{k_0}, b_0), (1^*, b_0))$. Denote this congruence relation by $\Theta_p$ then we have

$$\Theta \ge \bar{\Theta}(b_{k-1}, b_k) \text{ if and only if } k \in \{k_1, k_2, \ldots, k_r\}.$$

$\varphi$ can be extended on the following way: $\varphi([(b^0_{k_0}, b_0), (1^*, b_0)]) = p$. Now $\varphi$ is a bijection between $J(\operatorname{Con} T)$ and $P^d$, i.e. $\operatorname{Con} T \cong 2^P$.

The lattice $T$ does not satisfy condition (2), therefore we define $L_P$ as an extension of $T$.

The chain $\mathcal{C}$ can be represented in the form $2^{R^d}$ with a well-ordered chain $R$. Let $0_T$ be the zero element of $T$. First we consider the direct product $S_0 \cong [0_T, a_Q] \times 2^{R^d}$. This lattice has the elements $(a_Q, 0)$, $(0_T, 1)$, $(a_Q, 1)$, where 0 is the zero of $2^{R^d}$ and 1 is the unit element of $2^{R^d}$. $\mathcal{M}_3$ has the canonical embedding into $\mathcal{M}_3^{R^d}$, i.e. $\bar{o} < \bar{a}, \bar{b}, \bar{c} < \bar{i}$ is a sublattic of $\mathcal{M}_3^{R^d}$ isomorphic to $\mathcal{M}_3$. The ideal $[\bar{o}, \bar{b}]$ of $\mathcal{M}_3^{R^d}$ is isomorphic to $2^{R^d}$ and the filter $[(a_q, 0))$ of $S$ is isomorphic to $2^{R^d}$. Then we apply the Hall-Dilworth gluing construction for $\mathcal{M}_3^{R^d}$ and $S_0$. The resulting lattice is $S$. Finally we apply the Hall-Dilworth gluing construction for the lattice $S$ and $T$ as follows: the ideal $((1^*, b_0)]$ of $T$ and the filter $[(0_T, 1))$ of $S$ are isomorphic. Then we identify by the gluing the elements $\bar{b} \in S$ and $a_Q \in T$. (see *Figure* 3.). This lattice is $L_P$. By Lemma 2. $\bar{b} \equiv \bar{1}(\Theta)$ iff $\bar{a} \equiv \bar{1}(\Theta)$, i.e.

$\Theta(\bar{o},\bar{1}) = \Theta_p$. Let $a_p$ be the element $\bar{a}$ then the chain required in condition (2) is the following: $a_P = b'_0$, $b'_1 = a_P \vee b_0$, $b'_2 = a_P \vee b_1, \dots, b'_n = a_P \vee b_{n-1}$, i.e. we have a lattice $L_P$ such that $\operatorname{Con} L_P \cong 2^P$ and conditions (1), (2) are satisfied.

Figure 3.

## References

[1] E. T. Schmidt, Every finite distributive lattice is the congruence lattice of some modular lattice, *Algebra Universalis* **4** (1974), 49–57.

[2] E. T. Schmidt, Über die Kongruenzverbände der Verbände, *Publ. Math. Debrecen* **9** (1962), 243–256.

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*(Received December 2, 1991; revised May 5, 1992)*