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On generalized h-recurrent Finsler connection with deflection and torsion

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Summary. In 1934 E. Cartan [1] published his monograph 'Les espaces de Finsler' and fixed his method to determine a notion of connection in the Geometry of Finsler space. Matsumoto [4] determined uniquely the Cartan connection $C\Gamma$ by the following conditions: (1) The connection is metrical; (2) the deflection tensor field vanishes; (3) the torsion tensor field T vanishes; (4) the torsion tensor field S vanishes.

Hojo [3] introduced the connections, which depend on a real parameter p and make the v-covariant derivative $\varphi^{(p)}_{ij\|k}$ of $\varphi^{(p)}_{ij}(=\dot\partial_i\dot\partial_j L^P)$ zero just as $g_{ij}|_k=0$ in case of $C\Gamma$. The Cartan connection is really the case when p takes the value two and so the connection determined by Hojo is a generalization of $C\Gamma$.

Recently B.N. Prasad and Lalji Srivastava ([7]) have investigated the generalized h-recurrent Finsler connection which is deflection and torsion free. In this paper we investigate a generalized h-recurrent Finsler connection with given deflection- and torsion-tensor fields.

1. Introduction

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x,y)$, where $x=(x^i)$ denotes the positional variable of F^n and $y=(y^i)$ denote the components of a tangent vector with respect to (x^i). Throughout the following, L is assumed to be positively homogeneous of degree one with respect to (y^i). The metric tensor of (F^n, L) is given by $g_{ij}=\frac{1}{2}\dot\partial_i\dot\partial_j L^2$ where $\dot\partial_i=\partial/\partial y^i$.

A Finsler connection of (F^n, L) is a triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of an h-connection F^i_{jk}, a non-linear connection N^i_k and a vertical connection C^i_{jk} (Matsumoto [5]). If a Finsler connection is given, the h- and v-covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j|k}=d_kV^i_j+V^m_jF^i_{mk}-V^i_mF^m_{jk} \tag{1.1}$$

$$V^i_j|_k = \dot\partial_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk}, \tag{1.2}$$

where $d_k = \partial_k - N^m_k \dot\partial_m$, $\partial_k = \partial/\partial x^k$.

For any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ the hv-curvature tensor P^i_{hjk} is given by ([6])

$$P^i_{hjk} = \dot\partial_k F^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk} \tag{1.3}$$

2. Generalized h-recurrent Finsler connection

Let $p \neq 1$ be a real number. We define $\phi^{(p)}(x, y)$ as

$$\phi^{(p)} = \frac{1}{p} L^p \ (p \neq 0), \qquad \phi^{(0)} = \log L. \tag{2.1}$$

We denote $\dot\partial_i\, \phi^{(p)}$ and $\dot\partial_i \dot\partial_j\, \phi^{(p)}$ as $\phi^{(p)}_i$ and $\phi^{(p)}_{ij}$ and so on. Thus

$$\phi^{(p)}_i = L^{(p-1)} \ell_i, \quad \phi^{(p)}_{ij} = L^{(p-2)} \ (g_{ij} + (p-2)\ell_i \ell_j). \tag{2.2}$$

In the following, we restrict our considerations to a domain, where the matrix $\|\phi^{(p)}_{ij}\|$ is regular and then its inverse $\phi^{(p)ij}$ is given by

$$\phi^{(p)ij} = L^{-(p-2)} \Big[g^{ij} - \frac{(p-2)}{(p-1)} \ell^i \ell^j \Big]. \tag{2.3}$$

Differentiating (2.2) by y^k, we have

$$\begin{aligned} \phi^{(p)}_{ijk} = L^{(p-2)} \Big[2C_{ijk} + (p-2) L^{-1} \big\{ h_{ij}\ell_k + h_{jk}\ell_i \\ + h_{ki}\ell_j + (p-1)\ell_i \ell_j \ell_k \big\} \Big]. \end{aligned} \tag{2.4}$$

To avoid confusion, we denote h- and v-covariant derivatives with respect to Cartan's connection by $_{|k}$ and $|_k$, while these covariant derivatives with respect to a generalized h-recurrent Finsler connection will be denoted by $_{\|k}$ and $\|_k$ respectively. The quantities corresponding to a generalized h-recurrent Finsler connection will be denoted by putting p on the top of the quantity while the quantities corresponding to Cartan's connection will be denoted as usual.

Recently Prasad and L. Srivastava [7] have introduced a generalized h-recurrent Finsler connection $\{F^{(p)i}_{jk},\ N^{(p)i}_k,\ C^{(p)i}_{jk}\}$ which is determined uniquely by the following axioms:

- (C_1) The connection is h-recurrent with respect to the vector field a_k i.e. $g_{ij\|k} = a_k g_{ij}$

(C_2) the v-covariant derivative of $\phi^{(p)}_{ij}$ vanishes i.e. $\phi^{(p)}_{ij}\|_k = 0$

(C_3) the deflection tensor field $D^{(p)i}_{k}$ vanishes i.e.

$$D^{(p)i}_{k} = F^{(p)i}_{jk} y^j - N^{(p)i}_{k} = 0$$

(C_4) the torsion tensor field $T^{(p)i}_{jk}$ vanishes i.e.

$$T^{(p)i}_{jk} = F^{(p)i}_{jk} - F^{(p)i}_{kj} = 0$$

(C_5) the torsion tensor field $S^{(p)i}_{jk}$ vanishes i.e.

$$S^{(p)i}_{jk} = C^{(p)i}_{jk} - C^{(p)i}_{kj} = 0$$

In this paper we omit conditions $(C3)$, $(C4)$ and investigate a generalized h-recurrent Finsler connection with given deflection- and torsion-tensor fields.

3. Generalized h-recurrent Finsler connections with deflection and torsion

1. First we investigate connections where the nonlinear connection and the $(h)h$-torsion are prefixed.

Theorem 3.1. *Given in a Finsler space, a nonlinear connection $N^{(p)i}_{k}$, a skew symmetric (1,2) tensor field $T^{(p)i}_{jk}$ and a covariant vector field a_k, there exists a unique Finsler connection $(F^{(p)i}_{jk}, N^{(p)i}_{k}, C^{(p)i}_{jk})$ satisfying axioms $(C1)$, $(C2)$, $(C5)$ and the new axioms $(C3')$: the nonlinear connection is the given $N^{(p)i}_{k}$; $(C4')$: the (h) h-torsion tensor field is the given $T^{(p)i}_{jk}$.*

Proof. From $(C2)$ it follows that

$$\phi^{(p)}_{ij}\|_k = \phi^{(p)}_{ijk} - \tilde{C}^{(p)}_{ijk} - \tilde{C}^{(p)}_{jik} = 0,$$

where

$$\tilde{C}^{(p)}_{ijk} = \phi^{(p)}_{rj} C^{(p)r}_{ik}.$$

By cyclic permutation of the indices i, j and k, we get

$$\tilde{C}^{(p)}_{ijk} = (1/2)\big[\phi^{(p)}_{ijk} + \phi^{(p)}_{jki} - \phi^{(p)}_{kij}\big] = (1/2)\phi^{(p)}_{ijk},$$

which implies

$$C^{(p)r}_{ik} = (1/2)\phi^{(p)rj} \quad \phi^{(p)}_{ijk} = C^{r}_{ik} + \sigma^{(p)r}_{ik}, \tag{3.1}$$

where $\sigma^{(p)r}_{ik}$ are given as below by (2.3) and (2.4),

$$\sigma^{(p)r}_{ik} = \{(p-2)/2L\}[\delta^r_i \ell_k + \delta^r_k \ell_i + h_{ik}\ell^r/(p-1) - \ell_i \ell_k \ell^r]. \tag{3.2}$$

From the axioms $(C1)$ and $(C3')$ we have

$$\partial_k g_{ij} - N^{(p)m}_k \dot{\partial}_m g_{ij} - g_{mj} F^{(p)m}_{ik} - g_{im} F^{(p)m}_{jk} = a_k g_{ij}.$$

Applying the Christoffel process to the above equation and using axiom $(C4')$, we get

$$\begin{aligned} F^{(p)i}_{jk} =& \gamma^i_{jk} - (C^i_{km} N^{(p)m}_j + C^i_{jm} N^{(p)m}_k - g^{hi} C_{jkm} N^{(p)m}_h) \\ & - \frac{1}{2}(a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk}) + A^{(p)i}_{jk}, \quad \text{where} \end{aligned} \tag{3.3}$$

$$\gamma^i_{jk} = \frac{1}{2} g^{ih}(\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}), \tag{3.4}$$

$$A^{(p)i}_{jk} = \frac{1}{2}(T^{(p)}_{kjh} g^{hi} + T^{(p)}_{jkh} g^{hi} + T^{(p)i}_{jk}) \tag{3.5}$$

$$a^i = g^{ij} a_j \quad \text{and} \quad T^{(p)}_{kjh} = g_{jr} T^{(p)r}_{kh}.$$

In view of (3.1), (3.3) and axiom $(C3')$ it is clear that the Finsler connection $(F^{(p)i}_{jk}, N^{(p)i}_k, C^{(p)i}_{jk})$ is uniquely determined from the metric function L and from the given vector fields a_k, $T^{(p)i}_{jk}$.

2. For the above connection the deflection tensor field $D^{(p)i}_k$ defined in $(C3)$ is obtained by contracting (3.3) by y^j

$$\begin{aligned} D^{(p)i}_k =& G^i_k + 2C^i_{km} G^m - C^i_{km} N^{(p)m}_o - N^{(p)i}_k \\ & - \frac{1}{2}(a_o \delta^i_k + a_k y^i - a^i y_k) + A^{(p)i}_{ok}, \quad \text{where} \end{aligned} \tag{3.6}$$

$$G^i_k = \dot{\partial}_k G^i = \gamma^i_{ok} - 2C^i_{km} G^m, \tag{3.7}$$

$$G^i = \frac{1}{2}\gamma^i_{oo}. \tag{3.8}$$

The Suffix 'o' denotes contraction with respect to the element of support y^i.

Contracting (3.6) with y^k, we get

$$N^{(p)i}_o = 2G^i - D^{(p)i}_o - a_o y^i + \frac{1}{2} a^i L^2 + A^{(p)i}_{oo}. \tag{3.9}$$

Substituting the value of $N_o^{(p)i}$ in (3.6) and using $C^i_{jk}y^j = 0$, $C_{jhk}y^j = 0$, we get

$$\begin{aligned} N_k^{(p)i} =& G_k^i - C^i_{km}(A_{oo}^{(p)m} - D_o^{(p)m} + \frac{1}{2}a^m L^2) \\ &+ (A_{ok}^{(p)i} - D_k^{(p)i}) - \frac{1}{2}(a_o\delta_k^i + a_k y^i - a^i y_k). \end{aligned}$$

Hence we have the following

Theorem 3.2. *Given in a Finsler space a* (1,1) *tensor field* $D_k^{(p)i}$, *a covariant vector field* a_k *and a skew-symmetric* (1,2) *tensor field* $T_{jk}^{(p)i}$ *there exists a unique Finsler connection* $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ *satisfying the axioms* $(C1)$, $(C2)$, $(C4')$, $(C5)$ *and the new axiom* $(C3'')$: *the deflection tensor field is the given* $D_k^{(p)i}$.

3. The v-connection $F_{jk}^{(p)i}$ is given by (3.3) in which the nonlinear connection is given by

$$N_k^{(p)i} = G_k^i - C^i_{km}B_o^{(p)m} + B_k^{(p)i}, \quad \text{where} \tag{3.10}$$

$$B_k^{(p)i} = A_{ok}^{(p)i} - D_k^{(p)i} - \frac{1}{2}(a_o\delta_k^i + a_k y^i - a^i y_k). \tag{3.11}$$

The vertical connection is given by (3.1).

As a special case of the above theorem, if we impose the axiom $(C3)$ instead of $(C3'')$, the $B_k^{(p)i}$ in (3.11) becomes

$$B_k^{(p)i} = A_{ok}^{(p)i} - \frac{1}{2}(a_o\delta_k^i + a_k y^i - a^i y_k), \tag{3.12}$$

and we have the following:

Theorem 3.3. *Given in a Finsler space a skew-symmetric* (1,2) *tensor field* $T_{jk}^{(p)i}$ *and a covariant vector field* a_k *there exists a unique Finsler connection* $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ *satisfying the axioms* $(C1)$, $(C2)$, $(C3)$, $(C4')$ *and* $(C5)$.

These coefficients are given by (3.3), (3.1) and

$$\begin{aligned} N_k^{(p)i} =& G_k^i - C^i_{km}(A_{oo}^{(p)m} - \frac{1}{2}a^m L^2) + A_{ok}^{(p)i} \\ &- \frac{1}{2}(a_o\delta_k^i + a_k y^i - a^i y_k). \end{aligned} \tag{3.13}$$

4. If we assume that $B_k^{(p)i} = 0$, equation (3.10) reduces to $N_k^{(p)i} = G_k^i$, and we have the following results which gives the Finsler connection with deflection and torsion:

Theorem 3.4. *Given in a Finsler space a skew-symmetric* (1.2) *tensor field* $T^{(p)i}_{jk}$ *and a covariant vector field* a_k, *there exists a unique Finsler connection* ($F^{(p)i}_{jk}$, $N^{(p)i}_{k}$, $C^{(p)i}_{jk}$) *satisfying the axioms* ($C1$), ($C2$), ($C4'$), ($C5$) *and the new axiom* ($C3''$)*: the nonlinear connection is the one given by* E. Cartan.

The coefficients $F^{(p)i}_{jk}$ are given in this case by

$$F^{(p)i}_{jk} = \gamma^i_{jk} - (C^i_{km}G^m_j + C^i_{jm}G^m_k - g^{hi}C_{jkm}G^m_h) - \frac{1}{2}(a_j\delta^i_k + a_k\delta^i_j - a^i g_{jk}) + A^{(p)i}_{jk} \tag{3.14}$$

The deflection tensor field $D^{(p)i}_{k}$ is expressed as

$$D^{(p)i}_{k} = A^{(p)i}_{ok} - \frac{1}{2}(a_o\delta^i_k + a_k y^i - a^i y_k). \tag{3.15}$$

5. Now we investigate a connection which bears resemblance to the Wagner connection.

Theorem 3.5. *Given in a Finsler space the covariant vector field* $s_j \neq 0$ *and the recurrence vector* $a_j \neq 0$, *there exists a unique Finsler connection* ($F^{(p)i}_{jk}$, $N^{(p)i}_{k}$, $C^{(p)i}_{jk}$) *satisfying the axioms* ($C1$), ($C2$), ($C3$), ($C5$) *and* ($C4''$)*: the* (h)h*-torsion field is the given* $T^{(p)i}_{jk} = \delta^i_j s_k - \delta^i_k s_j$.

Proof. From the axiom ($C2$) it follows that the vertical connection $C^{(p)i}_{jk}$ is given by (3.1).

From axiom ($C1$) we have

$$\partial_k g_{ij} - N^{(p)m}_k \dot{\partial}_m g_{ij} - g_{mj}F^{(p)m}_{ik} - g_{im}F^{(p)m}_{jk} = a_k g_{ij}.$$

Applying the Christoffel process to the above equation and using axiom ($C4''$) we get

$$F^{(p)i}_{jk} = \gamma^i_{jk} - (C^i_{km}N^{(p)m}_j + C^i_{jm}N^{(p)m}_k - g^{hi}C_{jkm}N^{(p)m}_h) - \frac{1}{2}(a_j\delta^i_k + a_k\delta^i_j - a^i g_{jk}) + g_{jk}s^i - \delta^i_k s_j. \tag{3.16}$$

Contracting (3.16) with y^j, using axiom ($C3$) and the fact that C^i_{jk} is the indicatory tensor, we get

$$N^{(p)i}_{k} = \gamma^i_{ok} - C^i_{km}N^{(p)m}_o - \frac{1}{2}(a_k y^i + a_o\delta^i_k - a^i y_k) + y_k s^i - \delta^i_k s_o. \tag{3.17}$$

Again contracting (3.17) with y^k, we get

$$N_o^{(p)i} = \gamma^i_{oo} - a_o y^i + \frac{1}{2}L^2 a^i + L^2 s^i - y^i s_o. \tag{3.18}$$

Substituting (3.18) in (3.17) and using (3.7), we get

$$N_k^{(P)i} = G^i_k + B^{ir}_k(s_r + \frac{1}{2}a_r) + s_k y^i, \quad \text{where} \tag{3.19}$$

$$B^{ir}_k = (y_k g^{ir} - \delta^i_k y^r - \delta^r_k y^i - L^2 C^{ir}_k) \quad \text{and} \tag{3.20}$$

$$C^{ir}_k = C^i_{kh} g^{hr}. \tag{3.21}$$

Substituting (3.19) in (3.16), we get

$$F_{jk}^{(p)i} = \Gamma^{*i}_{jk} + U^{ir}_{jk}(s_r + \frac{1}{2}a_r) + \delta^i_j s_k, \quad \text{where} \tag{3.22}$$

$$\Gamma^{*i}_{jk} = \frac{1}{2}g^{ih}[d_k g_{jh} + d_j g_{kh} - d_h g_{jk}] \quad \text{and} \tag{3.23}$$

$$\begin{aligned} U^{ir}_{jk} = g_{jk}g^{ir} &- \delta^i_j \delta^r_k - C^{ir}_j y_k - C^{ir}_k y_j \\ &+ C^r_{jk} y^i + C^i_{jk} y^r - \delta^i_k \delta^r_j \\ &+ L^2(C^{mr}_j C^i_{mk} + C^{im}_j C^r_{mk} - C^{ir}_m C^m_{jk}). \end{aligned} \tag{3.24}$$

From (3.22), (3.19) and (3.1) it is clear that the connection ($F_{jk}^{(p)i}$, $N_k^{(p)i}$, $C_{jk}^{(p)i}$) is uniquely determined from the metric function L and from the given vector fields s_j and a_j.

The connection defined in the above theorem will be called generalized h-recurrent Wagner connection with respect to the vector field s_j and the recurrence vector a_j.

Theorem 3.6. *Given the covariant vector field s_j and the recurrence vector a_j in a Finsler space, there exists a unique Finsler connection ($F_{jk}^{(p)i}$, $N_k^{(p)i}$, $C_{jk}^{(p)i}$) satisfying the axioms ($C1$), ($C2$), ($C4''$), ($C5$) and ($C3''$): the nonlinear connection $N_k^{(p)i}$ is the one given by* Cartan.

Proof. Putting $N_k^{(p)i} = G^i_k$ in (3.16) and using (3.23) we get

$$F_{jk}^{(p)i} = \Gamma^{*i}_{jk} - \frac{1}{2}(a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk}) + g_{jk}s^i - \delta^i_k s_j. \tag{3.25}$$

Thus $C_{jk}^{(p)i}$ is determined uniquely from axiom ($C2$), $N_k^{(p)i}$ is determined from axiom ($C3''$) and $F_{jk}^{(p)i}$ is determined from axioms ($C1$) and ($C3''$).

6. For simplicity we shall use the following terminology. A generalized h-recurrent Finsler connection $(F^{(p)i}_{jk}, N^{(p)i}_{k}, C^{(p)i}_{jk})$ means, if nothing else is said, such a connection with vanishing deflection and $(h)h$-torsion tensor fields. — Omitting of the term "h-recurrent" means that $g_{ij\|k} = 0$.

Definition 3.1. A Finsler space is said to be a generalized h-recurrent Berwald space resp. such a space with torsion if it is possible to introduce a generalized h-recurrent Finsler connection without torsion (resp. with torsion) in such a way that the connection coefficient $F^{(p)i}_{jk}$ depends on position only.

Definition 3.2. A Finsler space is called a generalized h-recurrent Wagner space if it is possible to introduce a generalized h-recurrent Wagner connection in such a way that the connection coefficient $F^{(p)i}_{jk}$ depends on the position alone.

Theorem 3.7. *If the generalized h-recurrent Finsler connection $(F^{(p)i}_{jk}, N^{(p)i}_{k}, C^{(p)i}_{jk})$ with torsion satisfies the condition $\dot\partial_\ell F^{(p)i}_{jk} = 0$ then $\dot\partial_\ell a_k = 0$.*

Proof. From (1.3) it follows that the condition $\dot\partial_\ell F^{(p)i}_{jk} = 0$ is equivalent to

$$P^{(p)i}_{jk\ell} = -C^{(p)i}_{j\ell\|k} + C^{(p)i}_{jm} P^{(p)m}_{k\ell}. \tag{3.26}$$

Applying the Ricci identity ([6]) for the metric tensor g_{ij} we get

$$\begin{aligned} g_{ij}\|_{\ell\|k} - g_{ij\|k}\|_\ell = {} & g_{ij\|h} C^{(p)h}_{k\ell} + g_{ij}\|_h P^{(p)h}_{k\ell} \\ & + g_{hj} P^{(p)h}_{ik\ell} + g_{ih} P^{(p)h}_{jk\ell} \end{aligned}$$

which in view of $g_{ij\|k} = a_k g_{ij}$, $g_{ij}\|_\ell = -g_{im}\sigma^{(p)m}_{j\ell} - g_{mj}\sigma^{(p)m}_{i\ell}$ and (3.26) gives

$$(\dot\partial_\ell a_k) g_{ij} + 2C_{jim} P^{(p)m}_{k\ell} + 2a_k C_{ij\ell} - 2C_{ij\ell\|k} = 0. \tag{3.27}$$

Contracting this equation with y^i, we get

$$(\dot\partial_\ell a_k) y_j + 2C_{ij\ell} D^{(p)i}_{k} = 0.$$

Again contracting with y^j and using $C_{ij\ell} y^j = 0$, we get

$$(\dot\partial_\ell a_k) L^2 = 0 \quad \text{which implies that} \quad \dot\partial_\ell a_k = 0.$$

4. Conformal transformations of generalized h-recurrent Wagner spaces

1. Let L be the metric function of a Berwald space and let us consider whether this Berwald space may become a generalized h-recurrent Wagner space by a conformal transformation σ:

$$\bar{L} = e^{\sigma} L \tag{4.1}$$

In the Finsler space with metric $\bar{L}$, a generalized h-recurrent Wagner connection $(\bar{F}^{(p)i}_{jk}, \bar{N}^{(p)i}_{k}, \bar{C}^{(p)i}_{jk})$ is given by

$$\bar{F}^{(p)i}_{jk} = \bar{\Gamma}^{*i}_{jk} + \bar{U}^{ir}_{jk}(s_r + \frac{1}{2}a_r) + \delta^i_j s_k \tag{4.2}$$

$$\bar{N}^{(p)i}_{k} = \bar{G}^i_k + \bar{B}^{ir}_k(s_r + \frac{1}{2}a_r) + y^i s_k \tag{4.3}$$

$$\bar{C}^{(p)i}_{jk} = \bar{C}^i_{jk} + \bar{\sigma}^{(p)i}_{jk} \tag{4.4}$$

Since U^{ir}_{jk}, B^{ir}_k and C^i_{jk} are conformally invariant we can express these in terms of L.

We know that

$$\bar{\Gamma}^{*i}_{jk} = \Gamma^{*i}_{jk} - U^{ir}_{jk}\sigma_r, \tag{4.5}$$

$$\bar{G}^i_k = G^i_k - B^{ir}_k\sigma_r, \tag{4.6}$$

$$\bar{C}^i_{jk} = C^i_{jk} \tag{4.7}$$

where $\sigma_r = \partial_r\sigma$. Also from (3.2) and (4.1), we have

$$\bar{\sigma}^{(p)i}_{jk} = \sigma^{(p)i}_{jk}, \tag{4.8}$$

which shows that $\sigma^{(p)i}_{jk}$ is also conformally invariant.

Using equations (4.5), (4.6), (4.7) and (4.8), equations (4.2), (4.3) and (4.4) become

$$\bar{F}^{(p)i}_{jk} = \Gamma^{*i}_{jk} + U^{ir}_{jk}(s_r + \frac{1}{2}a_r - \sigma_r) + \delta^i{}_j s_k, \tag{4.9}$$

$$\bar{N}^{(p)i}_{k} = G^i_k + B^{ir}_k(s_r + \frac{1}{2}a_r - \sigma_r) + y^i s_k, \tag{4.10}$$

$$\bar{C}^{(p)i}_{jk} = C^{(p)i}_{jk}. \tag{4.11}$$

If we put $s_r = \sigma_r - \frac{1}{2}a_r$ then (4.9) and (4.10) become

$$\bar{F}^{(p)i}_{jk} = \Gamma^{*i}_{jk} + \delta^i_j s_k \tag{4.12}$$

$$\bar{N}^{(p)i}_{k} = G^i_k + y^i s_k. \tag{4.13}$$

From these observations we have the following

Theorem 4.1. *By any conformal transformation σ, a Berwald space becomes a generalized h-recurrent Wagner space with respect to the vector $(\sigma_r - \frac{1}{2}a_r)$ and the recurrence vector $a_j(x)$.*

2. In the Finsler space with metric $\bar{L}$ a generalized h-recurrent Finsler connection $(\bar{F}^{(p)i}_{jk}, \bar{N}^{(p)i}_{k}, \bar{C}^{(p)i}_{jk})$ is obtained from (4.2), (4.3) and (4.4) by putting $s_j = 0$ in them. Thus

$$\bar{F}^{(p)i}_{jk} = \Gamma^{*i}_{jk} + \frac{1}{2}\bar{U}^{ir}_{jk} a_r\,, \tag{4.14}$$

$$\bar{N}^{(p)i}_{k} = \bar{G}^{i}_{k} + \frac{1}{2}\bar{B}^{ir}_{k} a_r\,, \tag{4.15}$$

$$\bar{C}^{(p)i}_{jk} = \bar{C}^{i}_{jk} + \bar{\sigma}^{(p)i}_{jk}. \tag{4.16}$$

Substituting (4.5), (4.6), (4.7) and (4.8) in the above we have for $a_r = 2\sigma_r$

$$\bar{F}^{(p)i}_{jk} = \Gamma^{*i}_{jk}\,, \tag{4.17}$$

$$\bar{N}^{(p)i}_{k} = G^{i}_{k}\,, \tag{4.18}$$

$$\bar{C}^{(p)i}_{jk} = C^{(p)i}_{jk}. \tag{4.19}$$

Hence we have the following

Theorem 4.2. *By any conformal transformation σ, a Berwald space becomes a generalized h-recurrent Berwald space with respect to the recurrence gradient vector $2\sigma_r$.*

3. The proof of the following theorem can be obtained by checking the axioms $(C2)$ and $(C5)$.

Theorem 4.3. *Let a generalized h-recurrent Finsler connection $(F^{(p)i}_{jk}, N^{(p)i}_{k}, C^{(p)i}_{jk})$ with torsion be given in a Finsler space (F^n, L). If for a conformal transformation $\bar{L} = e^{\sigma} L$ we put*

$$\bar{F}^{(p)i}_{jk} = F^{(p)i}_{jk} + \delta^i_j(\sigma_k + \frac{1}{2}a_k), \tag{4.20}$$

$$\bar{N}^{(p)i}_{k} = N^{(p)i}_{k} + y^i(\sigma_k + \frac{1}{2}a_k), \tag{4.21}$$

$$\bar{C}^{(p)i}_{jk} = C^{(p)i}_{jk}, \tag{4.22}$$

then the coefficients $(\bar{F}^{(p)i}_{jk}, \bar{N}^{(p)i}_{k}, \bar{C}^{(p)i}_{jk})$ *define a generalized Finsler connection with torsion in a Finsler space* $(F^n, \bar{L})$.

From the above theorem and theorem (3.7) it follows that if $F^{(p)i}_{jk}$ depends on the position alone, then $\bar{F}^{(p)i}_{jk}$ also depends on the position alone. Thus we have the following

Theorem 4.4. *A generalized h-recurrent Berwald space with torsion with respect to the recurrence vector a_j transforms to a generalized Berwald space with torsion by any conformal transformation.*

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