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Nonlinear periodic problems with nonsmooth potential restricted in one direction

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Abstract. We study a nonlinear periodic problem driven by the ordinary scalar p-Laplacian and with a nonsmooth locally Lipschitz potential. Imposing on the potential a growth restriction only in one direction, we establish the existence of a solution. Our approach is variational based on the nonsmooth critical point theory for locally Lipschitz functions.

1. Introduction

In this paper we study the following nonlinear periodic problem:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' \in \partial j\big(t, x(t)\big) & \text{for a.a. } t \in T \\ x(0) = x(b),\ x'(0) = x'(t), \end{cases} \tag{1.1}$$

where $T = [0, b]$ is an interval, $p \in (1, +\infty)$. Here the potential function $j(t, \cdot)$ is not in general C^1, it is only locally Lipschitz and $\partial j(t, \cdot)$ is the subdifferential in the sense of Clarke (see Section 2). Recently there has been increasing interest for periodic problems involving the ordinary p-Laplacian. We refer to the works of Del Pino–Manasevich–Murua [3],

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Fabry–Fayyad [6], Guo [11], Fan–Zhao–Huang [7], Gasiński–Papageorgiou [9], [10] (scalar problems) and Kyritsi–Matzakos–Papageorgiou [13], Manasevich–Mawhin [14], Mawhin [15], [16] (vector problems). In all these works the approach is degree theoretical and only Gasiński–Papageorgiou [9, 10] use a variational approach. Moreover, in these works the potential function is nonsmooth (locally Lipschitz), as is the case in the present paper. However, in all the aforementioned works the growth of the potential function is restricted in both directions as $\zeta \to \pm\infty$. In contrast here we impose a growth restriction only in one direction and using a variational approach based on the nonsmooth critical point theory (see Chang [1] and Kourogenis–Papageorgiou [12]), we prove the existence of a solution for problem (1.1).

Semilinear (i.e. $p = 2$) Neumann problems with restriction in one direction, were studied by de Figueiredo–Ruf [8] and Villegas [18], under the assumptions that the right hand side nonlinearity $f(t, \zeta) = \partial j(t, \zeta)$ is jointly continuous (smooth potential) and asymptotically there is no interaction with γ (see section 3; in fact in de Figueiredo–Ruf [8] they assumed hypothesis $H(j)''$(iv)). Finally, we should point out that problems like (1.1) are known in the literature as "hemivariational inequalities" and arise in mechanics and engineering. For concrete applications we refer to the book of Naniewicz–Panagiotopoulos [17].

2. Mathematical background

As we already mentioned our approach is variational based on the nonsmooth critical point theory for locally Lipschitz functions, as this was formulated initially by Chang [1] and extended recently by Kourogenis–Papageorgiou [12]. The basic tool of this theory is the notion of generalized (or Clarke) subdifferential of a locally Lipschitz function. For the convenience of the reader, in this section we recall some basic definitions and facts from the subdifferential theory of locally Lipschitz functions and from the corresponding nonsmooth critical point theory. For more details about locally Lipschitz functions the interested reader can consult the books of Clarke [2] and Denkowski–Migórski–Papageorgiou [4].

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|_X$ we denote the norm of X and by $\langle\cdot,\cdot\rangle_X$ the duality brackets for the pair (X, X^*). A function $\varphi : X \longmapsto \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, we can find an open set $U \subseteq X$ with $x \in U$ and a constant $k_U > 0$ depending on U, such that $|\varphi(z) - \varphi(y)| \leq k_U\|z - y\|_X$ for all $z, y \in U$. From convex analysis we know that a proper, convex and lower semicontinuous function $\psi : X \longmapsto \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi \stackrel{df}{=} \{x \in X : \psi(x) < +\infty\}$ (see Denkowski–Migórski–Papageorgiou [4, Proposition 5.2.10, p. 532]). In particular, an $\mathbb{R}$-valued, convex and lower semicontinuous function is locally Lipschitz. Moreover, if X is finite dimensional, then every convex and $\mathbb{R}$-valued function defined on X is locally Lipschitz.

In analogy with the directional derivative of a convex function, we define the *generalized directional derivative* of a locally Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x; h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x' + th) - \varphi(x')}{t}.$$

The function $X \ni h \longmapsto \varphi^0(x; h) \in \mathbb{R}$ is sublinear, continuous and by the Hahn–Banach theorem it is the support function of a nonempty, convex and w^*-compact subset of X^*, defined by

$$\partial\varphi(x) \stackrel{df}{=} \big\{x^* \in X^* : \langle x^*, h\rangle_X \leq \varphi^0(x; h) \text{ for all } h \in X\big\}.$$

The multifunction $X \ni x \longmapsto \partial\varphi(x) \in 2^{X^*} \setminus \{\emptyset\}$ is known as *the Clarke* (or *generalized*) *subdifferential* of φ at x. This multifunction is upper semicontinuous from X with the norm topology into X^* with the w^*-topology (i.e. for all w^*-open sets $V \subseteq X^*$, we have that $\partial\varphi^+(V)$ is strongly open in X, where $\partial\varphi^+(V) = \{x \in X : \partial\varphi(x) \subseteq V\}$; see Denkowski–Migórski–Papageorgiou [4, p. 407]).

If $\varphi, \psi : X \longmapsto \mathbb{R}$ are two locally Lipschitz functions, then $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(t\varphi)(x) = t\partial\varphi(x)$ for all $t \in \mathbb{R}$ and all $x \in X$.

If $\varphi : X \longmapsto \mathbb{R}$ is continuous, convex (thus locally Lipschitz as well), then for all $x \in X$, the generalized subdifferential introduced above coincides with the subdifferential of φ in the sense of convex analysis, given by

$$\partial\varphi(x) \stackrel{df}{=} \big\{x^* \in X^* : \big\langle x^*, y-x\big\rangle_X \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\big\}.$$

If φ is strictly differentiable at x (in particular if φ is continuously Gâteaux differentiable at x), then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a *critical point* of the locally Lipschitz function φ, if $0 \in \partial\varphi(x)$. If $x \in X$ is a critical point, the value $c = \varphi(x)$ is a *critical value* of φ. It is easy to check that, if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(x)$ (i.e. $x \in X$ is a critical point).

In the classical (smooth) theory, a compactness-type condition, known as the *Palais–Smale condition* plays a central role. In the present nonsmooth setting this condition takes the following form:

> A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the *nonsmooth Palais-Smale condition*, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and
>
> $$m_\varphi(x_n) \longrightarrow 0 \quad \text{as } n \to +\infty,$$
>
> where
>
> $$m_\varphi(x_n) \stackrel{df}{=} \min\big\{\|x^*\|_{X^*} : x^* \in \partial\varphi(x_n)\big\}$$
>
> has a strongly convergent subsequence.

We recall the following geometric notion of linking, which plays a crucial role in critical point theory (classical and nonsmooth alike).

Definition 2.1. Suppose that X is a Hausdorff topological space and E_1 and D are nonempty subsets of X. We say that the sets E_1 and D link (homotopically) in X if and only if

(a) $E_1 \cap D = \emptyset$, and

(b) there exists a set $E \subseteq X$, such that $E_1 \subseteq E$ and for any continuous function $\eta : E \longrightarrow X$, such that $\eta|_{E_1} = id_{E_1}$, we have $\eta(E) \cap D \neq \emptyset$.

Using this notion, Kourogenis–Papageorgiou [12] proved the following abstract minimax principle. In fact the result of Kourogenis–Papageorgiou [12] is more general. However, the formulation that follows suffices for our purposes here.

Theorem 2.2. *If X is a reflexive Banach space, E_1 and D are nonempty subsets of X with D closed, E_1 and D link in X, $\varphi : X \to \mathbb{R}$ is locally Lipschitz, satisfies the nonsmooth Palais–Smale condition, $\sup_{E_1} \varphi < \inf_D \varphi$ and*

$$c \stackrel{df}{=} \inf_{\eta \in \Gamma} \sup_{v \in E} \varphi(\eta(v)),$$

where

$$\Gamma \stackrel{df}{=} \{\eta \in C(E; X) : \eta|_{E_1} = id_{E_1}\}$$

and $E \supseteq E_1$ is as in the definition of linking sets, then $c \geq \inf_D \varphi$ and c is a critical value of φ, i.e. there exists a critical point $x_0 \in X$ of φ such that $\varphi(x_0) = c$. Moreover, if $c = \inf_D \varphi$, then $x_0 \in D$.

3. Existence Theorem

In this section we prove an existence theorem for problem (1.1) by imposing only a unilateral growth restriction on the nonsmooth potential functional $j(t, \zeta)$. To do this we introduce the following quantity. Let

$$W^{1,p}_{\mathrm{per}}(T) \stackrel{df}{=} \big\{x \in W^{1,p}(T) : x(0) = x(b)\big\}$$

(recall that the embedding $W^{1,p}(T) \subseteq C(T)$ is continuous [in fact compact] and so the pointwise evaluations at $t = 0$ and $t = b$ make sense) and let

$$C \stackrel{df}{=} \Big\{x \in W^{1,p}_{\mathrm{per}}(T) : \min_T x = 0\Big\}.$$

We define

$$\gamma \stackrel{df}{=} \inf_{\substack{x \in C \\ x \neq 0}} \frac{\|x'\|_p^p}{\|x\|_p^p}.$$

Concerning this quantity, we have the following result.

Proposition 3.1. *There exists $\overline{x} \in C$, $\overline{x} \neq 0$, such that*

$$\gamma = \frac{\|\overline{x}'\|_p^p}{\|\overline{x}\|_p^p} > 0.$$

Moreover, if any $\overline{x} \in C$ satisfies the above equality, then

$$\overline{x}(t) > 0 \quad \text{for a.a. } t \in T.$$

PROOF. Let $\{x_n\}_{n\geq 1} \subseteq C$ be a sequence such that

$$\|x_n\|_p = 1 \quad \text{and} \quad \|x_n'\|_p^2 \searrow \gamma \text{ as } n \to +\infty.$$

Evidently the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ is bounded and so by passing to a subsequence if necessary, we may assume that

$$x_n \longrightarrow \overline{x} \quad \text{weakly in } W^{1,p}_{\mathrm{per}}(T),$$

for some $\overline{x} \in W^{1,p}_{\mathrm{per}}(T)$. Exploiting the compactness of the embedding $W^{1,p}_{\mathrm{per}}(T) \subseteq C_{\mathrm{per}}(T)$, we have that

$$x_n \longrightarrow \overline{x} \quad \text{in } C_{\mathrm{per}}(T)$$

and so $\|\overline{x}\|_p = 1$, i.e. $\overline{x} \neq 0$.

Let $t_n \in T$ be such that

$$0 = x_n(t_n) = \min_T x_n \quad \forall\, n \geq 1$$

(recall that $x_n \in C$ for $n \geq 1$). We may assume that $t_n \longrightarrow \overline{t}$ for some $\overline{t} \in T$. We have

$$x_n(t_n) \longrightarrow \overline{x}(\overline{t})$$

and so $\overline{x}(\overline{t}) = 0$. Hence $\min_T \overline{x} \leq 0$. If the inequality is strict, we can find $s \in T$, such that $\overline{x}(s) < 0$ and then $x_n(s) < 0$ for all $n \geq 1$ large enough, a contradiction to the fact that $x_n \in C$ for $n \geq 1$. Therefore

$$0 = \overline{x}(\overline{t}) = \min_T \overline{x}$$

and so $\overline{x} \in C$, $\overline{x} \neq 0$. This proves the first part of the proposition.

Now we will prove that $\overline{x}(t) > 0$ for almost all $t \in T$. To this end, for any interval $[a,c] \subseteq T$, we set

$$\widehat{C}[a,c] \stackrel{df}{=} \Big\{ y \in W^{1,p}_{\mathrm{per}}([a,c]) : \min_{[a,c]} y = 0 \Big\}$$

and

$$\widehat{\gamma}[a,c] \stackrel{df}{=} \inf_{\substack{x \in \widehat{C}[a,c] \\ x \neq 0}} \frac{\|x'\|^p_{L^p(a,c)}}{\|x\|^p_{L^p(a,c)}}.$$

Note that the map

$$C \ni y \longmapsto \widehat{y} \in \widehat{C}[a,c],$$

where

$$\widehat{y}(t) \stackrel{df}{=} y\left(\frac{t-a}{c-a}b\right) \quad \forall\, t \in [a,c],$$

is bijection and so it follows that

$$\widehat{\gamma}[a,c] = \left(\frac{b}{c-a}\right)^p \gamma.$$

Using this fact, we shall show that $\big|\{t \in T : \overline{x}(t) = 0\}\big|_1 = 0$ (here by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$). Suppose that this is not true and let

$$U \stackrel{df}{=} \{t \in T : \overline{x}(t) > 0\}.$$

Then we have that U is open and

$$U = \bigcup_{n \geq 1} J_n \quad \text{with} \quad \{J_n = (a_n, c_n)\}_{n \geq 1} \text{ disjoint and } |U|_1 < b.$$

So we have

$$\sum_{n \geq 1} \frac{c_n - a_n}{b} < 1,$$

hence

$$\sum_{n \geq 1} \left(\frac{c_n - a_n}{b}\right)^p < 1.$$

We have

$$\begin{aligned}
\gamma = \frac{\int_0^b |\overline{x}'(t)|^p dt}{\|\overline{x}\|_p^p} &\geq \frac{\int_{a_n}^{c_n} |\overline{x}'(t)|^p dt}{\|\overline{x}\|_p^p} \\
&\geq \frac{\widehat{\gamma}_n[a_n, c_n] \int_{a_n}^{c_n} |\overline{x}(t)|^p dt}{\|\overline{x}\|_p^p} \\
&= \frac{\left(\frac{b}{c_n - a_n}\right)^p \gamma \int_{a_n}^{c_n} |\overline{x}(t)|^p dt}{\|\overline{x}\|_p^p}
\end{aligned}$$

and so

$$\left(\frac{c_n - a_n}{b}\right)^p \geq \frac{\int_{a_n}^{c_n} |\overline{x}(t)|^p dt}{\|\overline{x}\|_p^p} \quad \forall\, n \geq 1,$$

so

$$1 > \sum_{n\geq 1} \left(\frac{c_n - a_n}{b}\right)^p \geq \frac{1}{\|\overline{x}\|_p^p} \sum_{n\geq 1} \int_{a_n}^{c_n} |\overline{x}(t)|^p dt = 1,$$

a contradiction.

This proves that $|U|_1 = b$ and so $\overline{x}(t) > 0$ for almost all $t \in T$. $\square$

In fact we can produce a more precise description of the quantity γ.

Proposition 3.2. $\gamma = \dfrac{p-1}{b^p}\left(2\displaystyle\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right)^p.$

Proof. Extend by periodicity the functions of C on $[0, 2b]$ and denote the set of these extended functions by $\widetilde{C}$. If $\tau \in T$, we set $C_\tau \stackrel{df}{=} \widetilde{C}|_{[\tau,\tau+b]}$. Also, we introduce the space

$$E_\tau \stackrel{df}{=} \left\{W_0^{1,p}[\tau, \tau + b]\right\}$$

and we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = \lambda |x(t)|^{p-2}x(t) \quad \text{for a.a. } t \in [\tau, \tau + b] \\ x(\tau) = 0 = x(\tau + b), \end{cases} \tag{3.1}$$

where $p \in (1, +\infty)$.

Let $\overline{x} \in C$ be the minimizer obtained in Proposition 3.1 and fix $\overline{\tau} \in T$ to be a zero of $\overline{x}$. As described above, we extend $\overline{x}$ on $[0, 2b]$. We have

$$\begin{aligned} \gamma &= \min_{y\in C_{\overline{\tau}}\setminus\{0\}} \frac{\int_{\overline{\tau}}^{\overline{\tau}+b} |y'(t)|^p dt}{\int_{\overline{\tau}}^{\overline{\tau}+b} |y(t)|^p dt} = \min_{y\in (C_{\overline{\tau}}\cap E_{\overline{\tau}})\setminus\{0\}} \frac{\int_{\overline{\tau}}^{\overline{\tau}+b} |y'(t)|^p dt}{\int_{\overline{\tau}}^{\overline{\tau}+b} |y(t)|^p dt} \\ &= \min_{y\in E_{\overline{\tau}}\setminus\{0\}} \frac{\int_{\overline{\tau}}^{\overline{\tau}+b} |y'(t)|^p dt}{\int_{\overline{\tau}}^{\overline{\tau}+b} |y(t)|^p dt} = \frac{p-1}{b^p}\left(2\int_0^1 \frac{dt}{(1-t)^{\frac{1}{p}}}\right)^p, \end{aligned}$$

where the last quantity is the first eigenvalue of problem (3.1) (see Mawhin [15]). $\square$

Now we can introduce our hypotheses on the nonsmooth potential $j(t, \zeta)$.

$\underline{H(j)}$ $j : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) for every $\zeta \in \mathbb{R}$, $j(\cdot, \zeta)$ is measurable and $j(\cdot, 0) \in L^1(T)$;

(ii) for almost all $t \in T$, $j(t, \cdot)$ is locally Lipschitz with L^1-Lipschitz constant;

(iii) for every $r > 0$, there exists $a_r \in L^{p'}(T)_+$, with $\frac{1}{p} + \frac{1}{p'} = 1$, such that for almost all $t \in T$, all $|\zeta| \leq r$ and all $u \in \partial j(t, \zeta)$, we have $|u| \leq a_r(t)$;

(iv) $\limsup_{\zeta \to +\infty} \frac{u}{\zeta^{p-1}} \leq h(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t, \zeta)$, where $h \in L^1(T)$, $h(t) \leq \gamma$ for almost all $t \in T$ with strict inequality on a set of positive measure;

(v) $\limsup_{\zeta \to -\infty} \big[\max_{\partial j(t,\zeta)} u\big] < 0 < \liminf_{\zeta \to +\infty} \big[\min_{\partial j(t,\zeta)} u\big]$ uniformly for almost all $t \in T$.

We introduce the energy functional $\varphi : W^{1,p}_{\mathrm{per}}(T) \longrightarrow \mathbb{R}$, defined by

$$\varphi(x) \stackrel{df}{=} \frac{1}{p}\|x'\|_p^p - \int_0^b j\big(t, x(t)\big)dt \quad \forall\, x \in W^{1,p}_{\mathrm{per}}(T).$$

We know that φ is locally Lipschitz (see Clarke [2, p. 80] and Denkowski–Migórski–Papageorgiou [4, p. 615]).

Proposition 3.3. *If hypotheses $H(j)$ hold, then φ satisfies the nonsmooth Palais–Smale condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ be a sequence, such that

$$|\varphi(x_n)| \leq M \text{ for } n \geq 1 \quad \text{and} \quad m_\varphi(x_n) \longrightarrow 0,$$

with some $M > 0$. Let $x_n^* \in \partial\varphi(x_n)$ be such that $m_\varphi = \|x_n^*\|_*$, $n \geq 1$. This is possible because $\partial\varphi(x_n) \subseteq \big(W^{1,p}_{\mathrm{per}}(T)\big)^*$ is weakly compact, the norm functional is weakly lower semicontinuous and the embedding $W^{1,p}_{\mathrm{per}}(T) \subseteq C_{\mathrm{per}}(T)$ is compact. We have

$$x_n^* = A(x_n) - u_n \quad \forall\, n \geq 1.$$

Here $A : W^{1,p}_{\mathrm{per}}(T) \longrightarrow \big(W^{1,p}_{\mathrm{per}}(T)\big)^*$ is the nonlinear operator, defined by

$$\langle A(x), y\rangle_{W^{1,p}_{\mathrm{per}}(T)} \stackrel{df}{=} \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt \quad \forall\, x, y \in W^{1,p}_{\mathrm{per}}(T)$$

and $u_n \in L^{p'}(T)$ with $u_n(t) \in \partial j\big(t, x_n(t)\big)$ for almost all $t \in T$ (see Clarke [2, p. 80]).

Note that A is monotone, demicontinuous, therefore it is maximal monotone (see Denkowski–Migórski–Papageorgiou [5, p. 37]).

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ is bounded. Suppose that this is not the case. Then by passing to a subsequence if necessary, we may assume that $\|x_n\|_{W^{1,p}(T)} \longrightarrow +\infty$. Let us set

$$y_n \stackrel{df}{=} \frac{x_n}{\|x_n\|_{W^{1,p}(T)}} \quad \forall\, n \geq 1.$$

Exploiting the reflexivity of $W^{1,p}_{\mathrm{per}}(T)$ and the compactness of the embedding $W^{1,p}_{\mathrm{per}}(T) \subseteq C_{\mathrm{per}}(T)$, we may say that

$$\begin{aligned} &y_n \longrightarrow y \quad \text{weakly in } W^{1,p}_{\mathrm{per}}(T),\\ &y_n \longrightarrow y \quad \text{in } C_{\mathrm{per}}(T). \end{aligned}$$

Claim 1. $y_n \longrightarrow y$ in $W^{1,p}_{\mathrm{per}}(T)$ and $y \neq 0$.

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$, we have

$$\big|\langle x_n^*, z\rangle_{W^{1,p}_{\mathrm{per}}(T)}\big| \leq \varepsilon_n \|z\|_{W^{1,p}(T)} \quad \forall\, z \in W^{1,p}_{\mathrm{per}}(T), \tag{3.2}$$

with $\varepsilon_n \searrow 0$. Take $z \equiv 1 \in W^{1,p}_{\mathrm{per}}(T)$. We obtain

$$\left|\int_0^b u_n(t)dt\right| \leq \varepsilon_n', \tag{3.3}$$

with $\varepsilon_n' \searrow 0$. Dividing the last inequality by $\|x_n\|^{p-1}_{W^{1,p}(T)}$, we have

$$\left|\int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt\right| \leq \frac{\varepsilon_n'}{\|x_n\|^{p-1}_{W^{1,p}(T)}}$$

and so

$$\int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}} dt \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.4}$$

By virtue of hypothesis $H(j)$(iv), for a given $\varepsilon > 0$, we can find $M_1 = M_1(\varepsilon) > 0$, such that

$$u \leq \big(h(t) + \varepsilon\big)\zeta^{p-1} \quad \text{for a.a. } t \in T, \text{ all } \zeta \geq M_1 \text{ and all } u \in \partial j(t, \zeta).$$

Moreover, from hypothesis $H(j)$(v), we see that we can find $M_2 > 0$, such that

$$u \geq 0 \quad \text{for a.a. } t \in T, \text{ all } \zeta \geq M_2 \text{ and all } u \in \partial j(t, \zeta).$$

Thus finally, we have

$$|u| = u \leq \big(h(t) + \varepsilon\big)\zeta^{p-1} \quad \text{for a.a. } t \in T, \text{ all } \zeta \geq M_3, \ u \in \partial j(t, \zeta), \tag{3.5}$$

with $M_3 = \max\{M_1, M_2\}$.

On the other hand, again from hypothesis $H(j)$(v) as well as hypothesis $H(j)$(iii), we see that we can find $M_4 \geq M_3$, such that

$$|u| \leq \widehat{a}(t) - u, \quad \text{for a.a. } t \in T, \text{ all } \zeta \leq M_4, \ u \in \partial j(t, \zeta), \tag{3.6}$$

with $\widehat{a} \in L^{p'}(T)_+$.

For a function $z \in W^{1,p}_{\text{per}}(T)$, we define $z^+ \overset{df}{=} \max\{z, 0\}$. We know that $z^+ \in W^{1,p}_{\text{per}}(T)$ (see Denkowski–Migórski–Papageorgiou [4, p. 348]).

Now we have

$$\begin{aligned} \int_0^b \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}} dt &= \int_{\{x_n \geq M_4\}} \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}} dt \\ &\quad + \int_{\{x_n < M_4\}} \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}} dt. \end{aligned} \tag{3.7}$$

Using (3.5), we have

$$\begin{aligned} \int_{\{x_n \geq M_4\}} \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}} dt &\leq \int_{\{x_n \geq M_4\}} \big(h(t) + \varepsilon\big)|y_n(t)|^{p-1} dt \\ &\leq \int_0^b \big(h(t) + \varepsilon\big)|y_n^+(t)|^{p-1} dt. \end{aligned} \tag{3.8}$$

Also because of (3.6) and (3.8), we have

$$\begin{aligned}
&\int_{\{x_n<M_4\}} \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt \\
&\leq \frac{\|\widehat{a}\|_1}{\|x_n\|^{p-1}_{W^{1,p}(T)}} - \int_{\{x_n<M_4\}} \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt \\
&\leq \frac{\|\widehat{a}\|_1}{\|x_n\|^{p-1}_{W^{1,p}(T)}} - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt + \int_{\{x_n\geq M_4\}} \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt \\
&\leq \frac{\|\widehat{a}\|_1}{\|x_n\|^{p-1}_{W^{1,p}(T)}} - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt + \int_0^b \big(h(t)+\varepsilon\big)|y_n^+(t)|^{p-1}dt. \qquad (3.9)
\end{aligned}$$

Using (3.8) and (3.9) in (3.7), we get

$$\begin{aligned}
\int_0^b \frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt \leq \frac{\|\widehat{a}\|_1}{\|x_n\|^{p-1}_{W^{1,p}(T)}} &- \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}dt \\
&+ 2\int_0^b \big(h(t)+\varepsilon\big)\big|y_n^+(t)\big|^{p-1}dt.
\end{aligned} \qquad (3.10)$$

Thus from (3.4), we infer that the sequence $\left\{\frac{u_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}}\right\}_{n\geq 1} \subseteq L^1(T)$ is bounded.

Now we have

$$\big|\langle x_n^*, y_n - y\rangle_{W^{1,p}_{\mathrm{per}}(T)}\big| \leq \varepsilon_n \|y_n - y\|_{W^{1,p}(T)},$$

with $\varepsilon_n \searrow 0$, so

$$\left|\left\langle \frac{x_n^*}{\|x_n\|^{p-1}_{W^{1,p}(T)}}, y_n - y\right\rangle_{W^{1,p}_{\mathrm{per}}(T)}\right| \leq \frac{\varepsilon_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}}\|y_n - y\|_{W^{1,p}(T)}$$

and

$$\left|\big\langle A(y_n), y_n - y\big\rangle_{W^{1,p}_{\mathrm{per}}(T)} - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}(y_n - y)(t)dt\right|$$

$$\leq \frac{\varepsilon_n}{\|x_n\|_{W^{1,p}(T)}^{p-1}} \|y_n - y\|_{W^{1,p}(T)}.$$

Since the sequence $\left\{ \frac{u_n}{\|x_n\|_{W^{1,p}(T)}^{p-1}} \right\}_{n \geq 1} \subseteq L^1(T)$ is bounded and $y_n \longrightarrow y$ in $C_{\mathrm{per}}(T)$, we see that

$$\int_0^b \frac{u_n(t)}{\|x_n\|_{W^{1,p}(T)}^{p-1}} (y_n - y)(t) dt \longrightarrow 0$$

and so

$$\lim_{n \to +\infty} \left| \left\langle A(y_n), y_n - y \right\rangle_{W_{\mathrm{per}}^{1,p}(T)} \right| = 0.$$

But A being maximal monotone, it is also generalized pseudomonotone (see Denkowski–Migórski–Papageorgiou [5, p. 58]) and so we have

$$\|y_n'\|_p^p = \langle A(y_n), y_n \rangle_{W_{\mathrm{per}}^{1,p}(T)} \longrightarrow \langle A(y), y \rangle_{W_{\mathrm{per}}^{1,p}(T)} = \|y'\|_p^p.$$

Because $y_n' \longrightarrow y'$ weakly in $L^p(T)$ and the space $L^p(T)$ is uniformly convex, from the Kadec–Klee property, we have that

$$y_n' \longrightarrow y' \quad \text{in } L^p(T)$$

(see Denkowski–Migórski–Papageorgiou [4, p. 309]). Hence

$$y_n \longrightarrow y \quad \text{in } W_{\mathrm{per}}^{1,p}(T)$$

and $\|y\|_{W^{1,p}(T)} = 1$, i.e. $y \neq 0$. This proves Claim 1.

Claim 2. y has a root in T, i.e. there exists $\overline{\tau} \in T$, such that $y(\overline{\tau}) = 0$.

From (3.3), we have

$$\int_0^b u_n(t) dt \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Suppose that the claim is not true. Then we have that either $y(t) > 0$ or $y(t) < 0$ for all $t \in T$. Suppose that

$$y(t) > 0 \quad \forall\, t \in T$$

(the analysis is similar if we suppose that $y(t) < 0$ for all $t \in T$). This means that

$$x_n(t) \longrightarrow +\infty \quad \forall\, t \in T \text{ as } n \to +\infty.$$

We claim that this convergence is uniform in $t \in T$. To this end, let $\delta > 0$ be such that $\delta < \min_T y$ (recall that we have assumed that $y(t) > 0$ for all $t \in T$). Since $y_n \longrightarrow y$ in $C_{\text{per}}(T)$, we can find $n_0 = n_0(\delta) \geq 1$, such that

$$\bigl|y_n(t) - y(t)\bigr| \leq \delta \quad \forall\, n \geq n_0,\ t \in T,$$

so

$$\bigl|y_n(t)\bigr| \geq \bigl|y(t)\bigr| - \delta \geq \delta_1 > 0 \quad \forall\, t \in T,$$

with some $\delta_1 > 0$.

Since $\|x_n\|_{W^{1,p}(T)} \longrightarrow +\infty$, for a given $\beta_1 > 0$, we can find $n_1 = n_1(\beta_1) \geq 1$, such that

$$\|x_n\|_{W^{1,p}(T)} \geq \beta_1 > 0 \quad \forall\, n \geq n_1.$$

Let us set $n_2 \stackrel{df}{=} \max\{n_0, n_1\}$. Then, we have

$$\frac{|x_n(t)|}{\beta_1} \geq \frac{|x_n(t)|}{\|x_n\|_{W^{1,p}(T)}} = \bigl|y_n(t)\bigr| \geq \delta_1 > 0 \quad \forall\, n \geq n_2,\ t \in T,$$

so

$$\bigl|x_n(t)\bigr| \geq \beta_1\delta_1 > 0 \quad \forall\, n \geq n_2,\ t \in T.$$

Since $\beta_1 > 0$ was arbitrary, it follows that

$$\bigl|x_n(t)\bigr| = x_n(t) \longrightarrow +\infty \quad \text{uniformly in } t \in T \text{ as } n \to +\infty.$$

Because of this and hypothesis $H(j)(\mathrm{v})$, we have that

$$\liminf_{n \to +\infty} \int_0^b u_n(t)dt > 0,$$

which contradicts (3.11). So we conclude that y has a zero in T. This proves Claim 2.

Now consider $y^+ = \max\{y, 0\} \in W^{1,p}_{\text{per}}(T)$. Let us consider two disjoint cases. In both of them we will get a contradiction, which will finish the proof of the boundedness of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$.

Case I. If $y^+ \equiv 0$, this means that $\max_T y = 0$. Passing to the limit in (3.10) as $n \to +\infty$ and using (3.4) and the fact that $y_n^+ \longrightarrow y^+ \equiv 0$ in $C(T)$, we infer that

$$\frac{u_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}} \longrightarrow 0 \quad \text{in } L^1(T). \tag{3.12}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$, we have

$$\left| \big\langle A(x_n), y \big\rangle_{W^{1,p}_{\text{per}}(T)} - \int_0^b u_n(t) y(t) dt \right| \leq \varepsilon_n \|y\|_{W^{1,p}(T)},$$

so

$$\left| \big\langle A(y_n), y \big\rangle_{W^{1,p}_{\text{per}}(T)} - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}} y(t) dt \right| \leq \frac{\varepsilon_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}} \|y\|_{W^{1,p}(T)}.$$

Since $A(y_n) \longrightarrow A(y)$ weakly in $W^{1,p}_{\text{per}}(T)^*$ and using also (3.12), in the limit we have

$$\big\langle A(y), y \big\rangle_{W^{1,p}_{\text{per}}(T)} = \|y'\|_p^p = 0,$$

so $y \equiv \xi \in \mathbb{R}$, hence $\xi = 0$ (because $\max_T y = 0$), a contradiction since $\|y\|_{W^{1,p}(T)} = 1$).

Case II. Therefore $y^+ \neq 0$ and clearly $y^+ \in C$. Using as a test function $z = y_n^+ \in W^{1,p}_{\text{per}}(T)$, from the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$ (see (3.2)), we have

$$\left| \big\|(y_n^+)'\big\|_p^p - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} y_n^+(t) dt \right| \leq \varepsilon_n'' \|y_n^+\|_{W^{1,p}(T)},$$

with $\varepsilon_n'' \stackrel{df}{=} \frac{\varepsilon_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}} \searrow 0$, so

$$\big\|(y_n^+)'\big\|_p^p \leq \varepsilon_n'' \|y_n^+\|_{W^{1,p}(T)} + \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}} \chi_{\{y_n>0\}}(t) y_n(t) dt. \tag{3.13}$$

By virtue of hypotheses $H(j)$(iii) and $H(j)$(iv), for almost all $t \in T$, all $\zeta \geq 0$ and all $u \in \partial j(t,\zeta)$, we have that

$$|u| \leq a_1(t) + c_1(t)|\zeta|^{p-1},$$

with $a_1, c_1 \in L^1(T)_+$, so

$$\frac{|u_n(t)|}{\|x_n\|^{p-1}_{W^{1,p}(T)}}\chi_{\{y_n>0\}}(t) \leq \frac{a_1(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}} + c_1(t)\big|y_n(t)\big|^{p-1} \quad \text{for a.a. } t \in T.$$

Therefore by the Dunford–Pettis Theorem (see Denkowski–Migórski–Papageorgiou [4, p. 333]), we see that

$$\frac{u_n}{\|x_n\|^{p-1}_{W^{1,p}(T)}} \longrightarrow g \quad \text{weakly in } L^1(T).$$

For a given $\varepsilon > 0$, let us define

$$C_n \stackrel{df}{=} \Big\{t \in T : x_n(t) > 0,\ \frac{u_n(t)}{x_n(t)^{p-1}} \leq h(t) + \varepsilon\Big\}$$

$$\chi_n(t) \stackrel{df}{=} \chi_{C_n}(t).$$

Note that

$$\chi_n(t) \longrightarrow 1 \quad \text{for a.a. } t \in \{y > 0\} \text{ as } n \to +\infty.$$

We have

$$\begin{aligned}\frac{u_n(t)}{\|x_n\|^{p-1}_{W^{1,p}(T)}}\chi_n(t) &= \frac{u_n(t)}{x_n(t)^{p-1}}y_n(t)^{p-1}\chi_n(t)\\ &\leq \big(h(t)+\varepsilon\big)y_n(t)^{p-1}\chi_n(t).\end{aligned}$$

Passing to the weak limit in $L^1(\{y > 0\})$, we obtain

$$g(t) \leq \big(h(t)+\varepsilon\big)y^+(t)^{p-1} \quad \text{for a.a. } t \in \{y > 0\}.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$g(t) \leq h(t)y^+(t)^{p-1} \quad \text{for a.a. } t \in \{y > 0\}.$$

Therefore from (3.13) and hypothesis $H(j)$(iv), it follows that

$$\|(y^+)'\|_p^p \leq \int_0^b h(t)y^+(t)^p dt \leq \gamma\|y^+\|_p^p. \tag{3.14}$$

Since $y^+ \in C$, from Proposition 3.1, we have that

$$\|(y^+)'\|_p^p = \gamma\|y^+\|_p^p \quad \text{and} \quad y^+(t) > 0 \quad \text{for a.a. } t \in T.$$

But then from (3.14) and our hypothesis about h (see hypothesis $H(j)$(iv)), we have that

$$\|(y^+)'\|_p^p < \gamma\|y^+\|_p^p,$$

a contradiction to the fact that $y^+ \in C \setminus \{0\}$.

In both cases we have obtained a contradiction. This proves the boundedness of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$. So, passing to a subsequence if necessary, we may assume that

$$x_n \longrightarrow x \quad \text{weakly in } W^{1,p}_{\text{per}}(T),$$

$$x_n \longrightarrow x \quad \text{in } C_{\text{per}}(T).$$

Again from the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$, for all $n \geq 1$, we have

$$\left|\langle A(x_n), x_n - x\rangle_{W^{1,p}_{\text{per}}(T)} - \int_0^b u_n(t)(x_n - x)(t)dt\right| \leq \varepsilon_n\|x_n - x\|_{W^{1,p}(T)}$$

and note that from hypothesis $H(j)$(iii), we have that

$$\int_0^b u_n(t)(x_n - x)(t)dt \longrightarrow 0.$$

So we obtain

$$\lim_{n\to+\infty} \langle A(x_n), x_n - x\rangle_{W^{1,p}_{\text{per}}(T)} = 0$$

and as earlier in the proof, by virtue of the maximal monotonicity of A and using the Kadec–Klee property of $L^p(T)$, we conclude that

$$x_n \longrightarrow x \quad \text{in } W^{1,p}_{\text{per}}(T). \qquad \square$$

Next we shall show that $\varphi|_C$ is bounded. To this end we need some preparations.

Proposition 3.4. *There exists a constant $c > 0$, such that*

$$\|x\|_p \leq c\|x'\|_p \quad \forall\, x \in C.$$

PROOF. Suppose that the result of the Proposition is not true. Then for every $n \geq 1$, we can find $x_n \in C$, such that

$$\|x_n\|_p > n\|x_n'\|_p.$$

Let us set

$$y_n \stackrel{df}{=} \frac{x_n}{\|x_n\|_p} \quad \forall\, n \geq 1.$$

Evidently

$$y_n \in C, \quad \|y_n\|_p = 1, \quad \|y_n'\|_p < \frac{1}{n} \quad \forall\, n \geq 1.$$

Hence

$$y_n' \longrightarrow 0 \quad \text{in } L^p(T).$$

Therefore the sequence $\{y_n\}_{n\geq 1} \subseteq C$ is bounded in $W^{1,p}_{\text{per}}(T)$ and so we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W^{1,p}_{\text{per}}(T),$$

$$y_n \longrightarrow y \quad \text{in } C_{\text{per}}(T).$$

It follows that $\|y\|_p = 1$, hence $y \neq 0$ and $y' = 0$, so that $y \equiv \xi \in \mathbb{R} \setminus \{0\}$, a contradiction to the fact that $y \in C$. $\square$

Proposition 3.5. *If hypotheses $H(j)$ hold, then there exists a constant $\beta > 0$, such that*

$$\|x'\|_p^p - \int_0^b h(t)x(t)^p dt \geq \beta\|x'\|_p^p \quad \forall\, x \in C.$$

PROOF. Let $\psi : C \longrightarrow \mathbb{R}$ be the functional defined by

$$\psi(x) \stackrel{df}{=} \|x'\|_p^p - \int_0^b h(t)x(t)^p dt \quad \forall\, x \in C.$$

From our hypothesis on h (see hypothesis $H(j)$(iv)) and Proposition 3.1, we have that $\psi \geq 0$. If the result of the Proposition is not true, we can find a sequence $\{x_n\}_{n\geq 1} \subseteq C$ with $\|x_n'\|_p = 1$, such that $\psi(x_n) \searrow 0$. From Proposition 3.4, we know that

$$\|x_n\|_p \leq c\|x_n'\|_p \quad \forall\, n \geq 1.$$

From this it follows that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$ is bounded and so, after passing to a subsequence if necessary, we may assume that

$$x_n \longrightarrow x \quad \text{weakly in } W^{1,p}_{\text{per}}(T),$$

$$x_n \longrightarrow x \quad \text{in } C_{\text{per}}(T).$$

We have

$$\|x'\|_p^p - \int_0^b h(t)x(t)^p dt \leq \lim_{n\to+\infty} \psi(x_n) = 0,$$

so from hypothesis $H(j)$(iv), we get

$$\|x'\|_p^p \leq \int_0^b h(t)x(t)^p dt \leq \gamma\|x\|_p^p. \tag{3.15}$$

If $x = 0$, then $\|x_n'\|_p \longrightarrow 0$ and so

$$x_n \longrightarrow 0 \quad \text{in } W^{1,p}_{\text{per}}(T),$$

a contradiction to the fact that $\|x_n'\|_p = 1$ for $n \geq 1$. So $x \in C \setminus \{0\}$ and by virtue of Proposition 3.1, we have

$$\|x'\|_p^p = \gamma\|x\|_p^p \quad \text{and} \quad x(t) > 0 \quad \text{for a.a. } t \in T.$$

Using the second fact in (3.15), we obtain that $\|x'\|_p^p < \gamma\|x\|_p^p$, a contradiction. $\square$

Now we are ready to show that $\varphi|_C$ is bounded below.

Proposition 3.6. *If hypotheses $H(j)$ hold, then $\varphi|_C$ is bounded below.*

Proof. By hypothesis $H(j)$(ii) for all $t \in T \setminus N$, with $|N|_1 = 0$, the function $\zeta \longmapsto j(t,\zeta)$ is locally Lipschitz. So it is differentiable at all $\zeta \in \mathbb{R} \setminus D(t)$, with $|D(t)|_1 = 0$ and for all $\zeta \geq 0$, we have

$$j(t,\zeta) - j(t,0) = \int_0^\zeta j_r'(t,r)dr.$$

For all $t \in T \setminus N$ and all $r \in \mathbb{R} \setminus D(t)$, we have

$$j_r'(t,r) \in \partial j(t,r)$$

(see Clarke [2, p. 32] or Denkowski–Migórski–Papageorgiou [4, p. 606]). So by virtue of hypotheses $H(j)$(iii) and (iv), for a given $\varepsilon > 0$, we can find $\widehat{a}_\varepsilon \in L^{p'}(T)_+$, such that

$$\begin{aligned} j(t,\zeta) - j(t,0) &\leq \int_0^\zeta \big(h(t)+\varepsilon\big) r^{p-1} dr + \widehat{a}_\varepsilon(t)\zeta \\ &= \frac{1}{p}\big(h(t)+\varepsilon\big)\zeta^p + \widehat{a}_\varepsilon(t)\zeta \\ &\leq \frac{1}{p} h(t)\zeta^p + \frac{2\varepsilon}{p}\zeta^p + \widehat{\beta}_\varepsilon(t), \end{aligned} \tag{3.16}$$

with $\widehat{\beta}_\varepsilon \in L^1(T)_+$ (the last inequality follows from the Young inequality). So, using also Propositions 3.5 and 3.4, for $x \in C$, we have

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))dt \\ &\geq \frac{1}{p}\|x'\|_p^p - \frac{1}{p}\int_0^b h(t)x(t)^p dt - \frac{2\varepsilon}{p}\|x\|_p^p - M_5 \\ &\geq \frac{\beta}{p}\|x'\|_p^p - \frac{2\varepsilon}{p}\|x\|_p^p - M_5 \\ &\geq \frac{1}{p}\big(\beta - 2\varepsilon c\big)\|x'\|_p^p - M_5, \end{aligned}$$

where $M_5 \stackrel{df}{=} \|\widehat{\beta}_\varepsilon\|_1 + \|j(\cdot,0)\|_1 > 0$. If we choose $\varepsilon < \frac{\beta}{2c}$, it follows that $\varphi|_C$ is coercive, thus bounded from below. $\square$

Proposition 3.7. *If hypotheses $H(j)$ hold, then $\varphi(\xi) \longrightarrow -\infty$ as $|\xi| \to +\infty$, $\xi \in \mathbb{R}$.*

Proof. From the mean value theorem for locally Lipschitz functions (see Clarke [2, p. 41] and Denkowski–Migórski–Papageorgiou [4, p. 609]), for almost all $t \in T$ and for all $\zeta < \zeta'$, we have

$$j(t,\zeta) - j(t,\zeta') = u(\zeta - \zeta'),$$

with $u \in \partial j\big(t, \lambda\zeta + (1-\lambda)\zeta'\big)$, $\lambda \in (0,1)$ (both u and λ depending on t).

First let $\zeta < \zeta' < 0$. By virtue of hypothesis $H(j)$(v), we can find $\vartheta_1 > 0$ and $M_6 > 0$, such that if $\zeta' < M_6$, we have $u \leq -\vartheta_1 < 0$. Hence

$$j(t,\zeta) - j(t,\zeta') = u(\zeta - \zeta') \geq \vartheta_1 |\zeta - \zeta'|,$$

so

$$\int_0^b j(t,\zeta)dt \geq \vartheta_1 |\zeta - \zeta'| b + \int_0^b j(t,\zeta')dt$$

and thus

$$\lim_{\xi \to -\infty} \int_0^b j(t,\xi)dt = +\infty. \tag{3.17}$$

Next if $0 < \zeta < \zeta'$, then by virtue of hypothesis $H(j)$(v), we can find $\vartheta_2 > 0$ and $M_7 > 0$, such that if $\zeta > M_7$, we have $u \geq \vartheta_2 > 0$. Hence

$$j(t,\zeta) - j(t,\zeta') = u(\zeta - \zeta') \leq -\vartheta_2 |\zeta - \zeta'|,$$

so

$$\int_0^b j(t,\zeta)dt + \vartheta_2 |\zeta - \zeta'| b \leq \int_0^b j(t,\zeta')dt$$

and thus

$$\lim_{\xi \to +\infty} \int_0^b j(t,\xi)dt = +\infty. \tag{3.18}$$

Because for $\xi \in \mathbb{R}$, we have that

$$\varphi(\xi) = -\int_0^b j(t,\xi)dt,$$

from (3.17) and (3.18), we conclude that $\varphi(\xi) \longrightarrow -\infty$ as $|\xi| \to +\infty$, $\xi \in \mathbb{R}$. $\square$

No we are ready for an existence theorem concerning problem (1.1).

Theorem 3.8. *If hypothesis $H(j)$ hold, then problem* (1.1) *has a solution*

$$x_0 \in C^1_{\mathrm{per}}(T), \quad \textit{such that} \quad |x_0'(\cdot)|^{p-2}x_0'(\cdot) \in W^{1,p'}_{\mathrm{per}}(T).$$

PROOF. By virtue of Propositions 3.6 and 3.7, we can find $\xi \in \mathbb{R}$, $\xi > 0$, such that

$$\varphi(\pm\xi) < \inf_C \varphi.$$

Let $E_1 \stackrel{df}{=} \{\pm\xi\}$ and

$$E \stackrel{df}{=} \big\{y \in W^{1,p}_{\mathrm{per}}(T) : -\xi \leq y(t) \leq \xi \text{ for all } t \in T\big\}.$$

We claim that E_1 and C link in $W^{1,p}_{\mathrm{per}}(T)$. Indeed let

$$\Gamma \stackrel{df}{=} \Big\{\eta \in C\big(E; W^{1,p}_{\mathrm{per}}(T)\big) : \eta|_{E_1} = id_{E_1}\Big\}$$

and take $\eta \in \Gamma$. We have that $\eta(-\xi) = -\xi$ and $\eta(\xi) = \xi$. Because the function $E \ni y \longmapsto \inf_T \eta(y) \in \mathbb{R}$ is continuous (see DENKOWSKI–MIGÓRSKI–PAPAGEORGIOU [5, p. 464]), from the intermediate value theorem, we conclude that

$$\eta(E) \cap C \neq \emptyset \quad \forall\, \eta \in \Gamma$$

and so the sets E_1 and C link in $W^{1,p}_{\mathrm{per}}(T)$. This fact combined with Proposition 3.3 permits the application of Theorem 2.2. So we can find $x_0 \in W^{1,p}_{\mathrm{per}}(T)$, such that

$$\varphi(x_0) \geq \inf_C \varphi \quad \text{and} \quad 0 \in \partial\varphi(x_0).$$

From the last inclusion, we have that

$$A(x_0) = u_0^*,$$

with $u_0^* \in L^{p'}(T)$, $u_0^*(t) \in \partial j\big(t, x_0(t)\big)$ for almost all $t \in T$ and so

$$\big\langle A(x_0), \vartheta \big\rangle_{W^{1,p}_{\mathrm{per}}(T)} = \int_0^b u_0^*(t)\vartheta(t)dt \quad \forall\, \vartheta \in C_0^1(0,b). \tag{3.19}$$

By integration by parts and since

$$\big(|x_0'(\cdot)|^{p-2}x_0'(\cdot)\big)' \in W^{-1,p'}(T)$$

(see Denkowski–Migórski–Papageorgiou [4, p. 362]), we have

$$\big\langle -\big(|x_0'(\cdot)|^{p-2}x_0'(\cdot)\big)', \vartheta \big\rangle_{W_0^{1,p}(T)} = \int_0^b u_0^*(t)\vartheta(t)dt. \tag{3.20}$$

Because the embedding $C_0^1(0,b) \subseteq W_0^{1,p}(T)$ is dense, from (3.20), it follows that

$$-\big(|x_0'(t)|^{p-2}x_0'(t)\big)' = u_0^*(t) \in \partial j\big(t, x_0(t)\big) \quad \text{for a.a. } t \in T. \tag{3.21}$$

Also from the Green identity, for all $v \in W^{1,p}_{\mathrm{per}}(T)$, we have

$$\begin{aligned}\big\langle A(x_0), v \big\rangle_{W^{1,p}_{\mathrm{per}}(T)} &= \big|x_0'(b)\big|^{p-2}x_0'(b)v(b) - \big|x_0'(0)\big|^{p-2}x_0'(0)v(0) \\ &\quad - \int_0^b \big(|x_0'(t)|^{p-2}x_0'(t)\big)'v(t)dt,\end{aligned}$$

so from (3.19), we have

$$\begin{aligned}\int_0^b u_0^*(t)v(t)dt &= \big|x_0'(b)\big|^{p-2}x_0'(b)v(b) - \big|x_0'(0)\big|^{p-2}x_0'(0)v(0) \\ &\quad - \int_0^b \big(|x_0'(t)|^{p-2}x_0'(t)\big)'v(t)dt.\end{aligned}$$

From (3.21), we have

$$\big|x_0'(0)\big|^{p-2}x_0'(0)v(0) = \big|x_0'(b)\big|^{p-2}x_0'(b)v(b) \quad \forall\, v \in W^{1,p}_{\mathrm{per}}(T)$$

and so

$$\big|x_0'(0)\big|^{p-2}x_0'(0) = \big|x_0'(b)\big|^{p-2}x_0'(b).$$

Because the function $\kappa(r) \stackrel{df}{=} |r|^{p-2}r$ is a homeomorphism on $\mathbb{R}$, we infer that $x_0'(0) = x_0'(b)$. From (3.21), we have that $|x_0'(\cdot)|^{p-2}x_0'(\cdot) \in W_{\mathrm{per}}^{1,p'}(T)$ and so $|x_0'(\cdot)|^{p-2}x_0'(\cdot) \in C_{\mathrm{per}}(T)$, which implies that $x_0' \in C_{\mathrm{per}}(T)$. Therefore

$$x_0' \in C_{\mathrm{per}}^1(T), \quad \text{with} \quad |x_0'(\cdot)|^{p-2}x_0'(\cdot) \in W_{\mathrm{per}}^{1,p'}(T),$$
$$x_0(0) = x_0(b), \quad x_0'(0) = x_0'(b)$$

and it solves (1.1). □

A careful reading of the proof, reveals that we can have a corresponding existence result if the restriction on $j(t,\cdot)$ is imposed in the $-\infty$ direction. More precisely, we impose the following conditions on the nonsmooth potential j.

$\underline{H(j)'}$ $j : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying hypothesis $H(j)$(i), (ii), (iii), (v) and

(iv) $\limsup_{\zeta \to -\infty} \frac{u}{|\zeta|^{p-1}\zeta} \leq h(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,\zeta)$, where $h \in L^1(T)_+$, $h(t) \leq \gamma$ for almost all $t \in T$ with strict inequality on a set of positive measure.

Theorem 3.9. *If hypothesis $H(j)'$ hold, then problem* (1.1) *has a solution*

$$x_0 \in C_{\mathrm{per}}^1(T), \quad \textit{such that} \quad |x_0'(\cdot)|^{p-2}x_0'(\cdot) \in W_{\mathrm{per}}^{1,p'}(T).$$

Now impose the following hypotheses on j.

$\underline{H(j)''}$ $j : T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying hypothesis $H(j)$(i), (ii), (iii) and

(iv) $\limsup_{\zeta \to +\infty} \frac{u}{\zeta^{p-1}} \leq 0$ or $\limsup_{\zeta \to -\infty} \frac{u}{|\zeta|^{p-2}\zeta} \leq 0$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,\zeta)$;

(v) $\limsup_{\zeta \to -\infty} \left[\max_{\partial j(t,\zeta)} u\right] = -\infty$ and $\liminf\limits_{\zeta \to +\infty} \left[\min_{\partial j(t,\zeta)} u\right] = +\infty$ uniformly for almost all $t \in T$.

As a consequence of Theorems 3.8 and 3.9, we obtain the following corollary.

Corollary 3.10. *If hypothesis* $H(j)''$ *hold, then problem* (1.1) *has a solution*

$$x_0 \in C^1_{\mathrm{per}}(T), \quad \textit{such that } |x_0'(\cdot)|^{p-2}x_0'(\cdot) \in W^{1,p'}_{\mathrm{per}}(T).$$

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