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Some remarks on solutions of the functional equation $f(x + f(x)^n y) = tf(x)f(y)$

By JANUSZ BRZDĘK (Rzeszów)

1. Introduction

Let X be a linear space over a commutative field K and $t \in K \setminus \{0\}$. Let k and n be non-negative integers. The functional equation

$$f(f(y)^k x + f(x)^n y) = tf(x)f(y)\,, \tag{1}$$

where the unknown function f maps X into K, has been studied by many authors in various cases (see e.g. [1]–[16]).

We consider the particular case of (1) where $k = 0,\ n > 0$, and $t \neq 0$, i.e. the functional equation

$$f(x + f(x)^n y) = tf(x)f(y)\,. \tag{2}$$

This case has been investigated for $t \neq 1$ only in [5] and [6] in the class of continuous functions mapping a real linear topological space into the set of all reals.

Equation (2) is a generalization of the well known Gołąb-Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y)\,.$$

We give a description of the general solution of (2) in the class of functions $f : X \to K$. Moreover, we solve (2) under some additional assumptions on f, K, and X. In particular, we determine the continuous solutions $f : X \to K$ of (2) in the case where K is the set of all complex numbers and X is a complex linear topological space.

Throughout the paper $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, integers, rationals, reals, and complex numbers respectively.

2. General solution

First, we present a description of the general solution of (2). Let us start with the following simple observation.

Proposition 1. *A function $f : X \to K$ satisfies equation (2) iff there exists a solution $g : X \to K$ of the equation*

$$g(x + g(x)^n y) = g(x)g(y) \tag{3}$$

such that

$$g(t^n x) = g(x) \quad \text{and} \quad f(x) = t^{-1}g(x) \qquad \text{for } x \in X\,. \tag{4}$$

Proof. Suppose that $f : X \to K$ is a solution of equation (2). The case $f = 0$ (i.e. $f(X) = \{0\}$) is trivial. So, assume that there is $x_0 \in X$ with $f(x_0) \neq 0$. Putting $x = y = 0$ in (2) we get $f(0) \in \{0, t^{-1}\}$. Suppose that $f(0) = 0$. Then $f(x_0) = f(x_0 + f(x_0)^n 0) = tf(x_0)f(0) = 0$. This is a contradiction. Consequently $f(0) = t^{-1}$. Thus, setting $x = 0$ in (2) we obtain

$$f(t^{-n}y) = f(y) \qquad \text{for } y \in X\,. \tag{5}$$

Define a function $g : X \to K$ by the formula: $g(x) = tf(x)$ for $x \in X$. Then, by (5), for every $x, y \in X$, $g(t^n x) = g(x)$ and

$$\begin{aligned} g(x + g(x)^n y) &= tf(x + f(x)^n t^n y) = \\ &= t^2 f(x)f(t^n y) = g(x)g(t^n y) = g(x)g(y)\,. \end{aligned}$$

Now, assume that $g : X \to K$ is a solution of equation (3) such that (4) holds. Then, for every $x, y \in X$,

$$\begin{aligned} f(x + f(x)^n y) &= t^{-1}g(x + g(x)^n t^{-n} y) = \\ &= t^{-1}g(x)g(t^{-n}y) = t^{-1}g(x)g(y) = tf(x)f(y)\,. \end{aligned}$$

This completes the proof.

We also need the following

Proposition 2. *A function $g : X \to K$, $g \neq 0$ (i.e. $g(X) \neq \{0\}$), is a solution of equation (3) iff there exist a multiplicative subgroup W of $K \setminus \{0\}$, an additive subgroup T of X, and a function $w : W \to X$ such that*

$$a^n T = T \qquad \text{for } a \in W\,; \tag{6}$$

$$w(ab) - a^n w(b) - w(a) \in T \qquad \text{for } a, b \in W\,; \tag{7}$$

$$w(a) \in T \quad \text{iff } a = 1\,; \tag{8}$$

$$g(x) = \begin{cases} a & \text{if } x \in w(a) + T \text{ and } a \in W\,; \\ 0 & \text{otherwise}\,, \end{cases} \qquad \text{for } x \in X\,. \tag{9}$$

Moreover $W = g(X) \setminus \{0\}$ *and* $T = g^{-1}(\{1\})$.

A simple modification of the proof of Proposition 2 for $n = 1$ given by P. Javor in [11] (cf. also [1], pp. 316–318, and [16]) supplies a proof of Proposition 2 for every positive integer n. However, we present a new, simpler proof. The next lemma is necessary for this.

Lemma 1. *If a function* $g : X \to K,\ g \neq 0$, *is a solution of equation (3),* $T = g^{-1}(\{1\})$, *and* $W = g(X) \setminus \{0\}$, *then*

(i) $g(g(x)^{-n}(z - x)) = g(z)g(x)^{-1}$ *for* $x, z \in X,\ g(x) \neq 0$;

(ii) T *is an additive subgroup of* X;

(iii) W *is a multiplicative subgroup of* $K \setminus \{0\}$;

(iv) $g(g(x)^{-n}x) = g(x)^{-1}$ *for* $x \in X,\ g(x) \neq 0$;

(v) $a^nT = T$ *for* $a \in W$;

(vi) $T \setminus \{0\}$ *is the set of periods of* g;

(vii) $y - x \in T$ *for every* $x, y \in X$ *with* $f(x) = f(y) \neq 0$.

Proof. (i) It suffices to put $z = x + g(x)^ny$ in (3).

(ii), (iii) Fix $x_0 \in X$ with $g(x_0) \neq 0$ and set $x = z = x_0$ in (i). Then we get $0 \in T$ and $1 \in W$. Further, it results from (i) that $z - x \in T$ for $x, z \in T$ and $ab^{-1} \in W$ for $a, b \in W$. This yields the statements (ii) and (iii).

(iv) Since, by (ii), $g(0) = 1$, setting $z = 0$ in (i) we obtain the assertion (iv).

(v) Fix $x \in X$ with $g(x) \neq 0$ and $z \in T$. Then, by (3), $g(x+g(x)^nz) = g(x)$. Thus, according to (iv),

$$\begin{aligned} g(g(x)^nz) &= g(x + g(x)^nz + g(x)^n(-g(x)^{-n}x)) = \\ &= g(x + g(x)^nz)g(-g(x)^{-n}x) = g(x)g(x)^{-1} = 1\,. \end{aligned}$$

This completes the proof of (v), in virtue of (iii).

(vi) Let P denote the set of periods of g. Then, by (ii), for every $w \in P$,

$$1 = g(0) = g(0 + w) = g(0 + g(0)^nw) = g(0)g(w) = g(w)\,.$$

Moreover, for every $z \in T,\ x \in X$,

$$g(z + x) = g(z + g(z)^nx) = g(z)g(x) = g(x)\,.$$

Consequently $P = T \setminus \{0\}$.

(vii) Fix $x, y \in X$ with $g(x) = g(y) \neq 0$. Then, in virtue of (i), $g(g(x)^{-n}(y - x)) = 1$. Hence, by (iii) and (v), $y - x \in T$. This ends the proof of (vii).

Proof of Proposition 2. Assume that g is a solution of (3) and put $W = g(X) \setminus \{0\}$ and $T = g^{-1}(\{1\})$. By Lemma 1, W is a multiplicative subgroup of $K \setminus \{0\}$, T is an additive subgroup of X, and (6) holds. Let $w : W \to X$ be a function such that $w(a) \in g^{-1}(\{a\})$ for $a \in W$. Then condition (8) is valid and, on account of Lemma 1 (ii), (vi), (vii), $g^{-1}(\{a\}) = w(a) + T$ for $a \in W$. Thus g and w satisfy (9). It remains to show (7).

Fix $a, b \in W$. Then, according to the definition of w,

$$g(w(a) + a^n w(b)) = g(w(a) + g(w(a))^n w(b)) = \\ = g(w(a))g(w(b)) = ab = g(w(ab))\,,$$

which, in view of Lemma 1 (vii), implies (7).

Now, assume that g is given by (9). First, we show that g is well defined.

Fix $a, b \in W$ and suppose that there are $x, y \in T$ with $w(a) + x = w(b) + y$. Put $c = ab^{-1}$. Then $a = bc$ and, by (7),

$$y - x - b^n w(c) = w(b) + y - x - b^n w(c) - w(b) = \\ = w(a) - b^n w(c) - w(b) = w(bc) - b^n w(c) - w(b) \in T\,.$$

Thus $b^n w(c) \in T$, because T is an additive group. Consequently, according to (6), $w(c) \in T$. Hence (8) yields $c = 1$, which means that $a = b$.

In order to complete the proof we must yet show that g satisfies equation (3). Therefore fix $x, y \in X$.

If $g(x) = 0$, then $g(x + g(x)^n y) = g(x) = 0 = g(x)g(y)$.

Next, if $g(x)g(y) \neq 0$, there are $a, b \in W$ with $x \in w(a) + T$ and $y \in w(b) + T$. Since, by (6) and (7), $x + g(x)^n y = x + a^n y \in w(ab) + T$, we get $g(x + g(x)^n y) = ab = g(x)g(y)$.

Finally, suppose that $g(x) \neq 0$, $g(y) = 0$, and $g(x + g(x)^n y) \neq g(x)g(y) = 0$. On account of (9) there are $a, c \in W$ with $x \in w(a) + T$ and $x + g(x)^n y \in w(c) + T$. Put $b = ca^{-1}$. Then $c = ab$ and $a^n y = g(x)^n y \in (T + w(c) - x)$. Further, in virtue of (7), $T + w(c) - x = T + w(ab) - w(a) = T + a^n w(b)$. Thus, by (6), $y \in T + w(b)$, which means that $g(y) = b$. This is a contradiction.

In this way we have proved that g is a solution of (3). The equalities $W = g(X) \setminus \{0\}$ and $T = g^{-1}(\{1\})$ result from (8) and (9). This ends the proof.

Now, we have all tools to prove the following

Theorem 1. *A function $f : X \to K$, $f \neq 0$, is a solution of equation (2) iff there exist a multiplicative subgroup W of $K \setminus \{0\}$, an additive subgroup T of X, and a function $w : W \to K$ such that*

$$a^n T = T \quad \text{for} \quad a \in W \bigcup \{t\}\,; \tag{10}$$

(11) $w(ab) - a^n w(b) - w(a),\ (t^n - 1)w(a) \in T$ *for* $a, b \in W$;

(12) $w(a) \in T$ *iff* $a = 1$;

(13) $$f(x) = \begin{cases} t^{-1}a & \text{if } x \in w(a) + T \text{ and } a \in W\,; \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } x \in X\,.$$

Proof. Assume that f is a solution of equation (2). According to Proposition 1 there exists a function $g : X \to K$ satisfying equation (3) and condition (4). Thus, by Proposition 2, there exist a multiplicative subgroup W of $K \setminus \{0\}$, an additive subgroup T of X, and a function $w : W \to X$ such that conditions (6)–(9) are valid. It is easily seen that (12) and (13) follow from (8), (9), and (4). Condition (10) results from (4) and (6), because $T = g^{-1}(\{1\})$. Further, by (4), for every $a \in W$ we have $g(w(a)) = g(t^n w(a))$. This, in virtue of Lemma 1 (vii), yields $(t^n - 1)w(a) \in T$. Consequently (11) holds, too.

For the converse define a function $g : X \to K$ by the formula: $g(x) = tf(x)$ for $x \in X$. It is easy to notice that conditions (6)–(9) are valid. Thus, on account of Proposition 2, g is a solution of equation (3). Moreover, by (10) and (11), for every $a \in W,\ z \in T$

$$t^n(w(a) + z) - w(a) - z = (t^n - 1)w(a) + t^n z - z \in T,$$
$$t^{-n}(w(a) + z) - w(a) - z = t^{-n}(z - t^n z - (t^n - 1)w(a)) \in t^{-n}T = T\,.$$

Thus, in virtue of (13), $g(w(a) + z) = g(t^n(w(a) + z)) = g(t^{-n}(w(a) + z))$ for $a \in W,\ z \in T$. This means that condition (4) holds. Consequently Proposition 1 implies that f satisfies equation (2). This ends the proof.

Using Theorem 1 we can determine all solutions of (2) for many t. Namely, we have the following

Proposition 3. *Suppose that t fulfils the condition*

there are $k \in \mathbb{N} \bigcup \{0\}$ and $a_{-k}, \dots, a_k \in L$ such that

(∗) $$(t^n - 1)\left(\sum_{i=-k}^{k} a_i t^{in}\right) - 1 = 0\,,$$

where L is the simple subfield of K if $\operatorname{char} K \neq 0$ *and $L = \mathbb{Z}$ if* $\operatorname{char} K = 0$. *Then a function $f : X \to K,\ f \neq 0$, is a solution of equation (2) iff there exists an additive subgroup T of X such that $t^n T = T$ and*

$$f(x) = \begin{cases} t^{-1} & \text{if } x \in T\,; \\ 0 & \text{if } x \in X \setminus T\,. \end{cases}$$

Proof. Assume that f is a solution of (2). Then, in view of Theorem 1, there exist a multiplicative subgroup W of $K\setminus\{0\}$, an additive subgroup T of X, and a function $w : W \to X$ such that conditions (10)–(13) are valid. Let S be the ring generated by the set $\{t^{-n}, t^n\}$. Notice that

$$bT \subset T \quad \text{for every} \quad b \in S\,. \tag{14}$$

In fact, let $b \in S$. Then there are $k \in \mathbb{N}\setminus\{0\}$, $a_{-k},\dots,a_k \in L$ such that $b = a_{-k}t^{-kn} + \dots + a_k t^{kn}$. Since T is an additive group and, on account of (10), $t^{nj}T = T$ for $j \in \mathbb{Z}$, we get $bT \subset T$.

Fix $a \in W$. According to (11), $(t^n - 1)w(a) \in T$. Since, by the hypothesis on t, $(t^n - 1) = (a_{-k}t^{-kn} + \dots + a_k t^{kn})^{-1}$, in virtue of (14), we get $w(a) \in T$. Consequently, (12) yields $a = 1$.

So, we have proved that $W = \{1\}$. Whence Theorem 1 implies that f has the desired form.

The converse also results from Theorem 1.

The next three examples show that assumption $(*)$ of Proposition 3 is essential.

Example 1. Let $\mathbb{K} = \mathbb{R}$, $x_0 \in X\setminus\{0\}$, and $W = \{q \in \mathbb{Q} : q > 0\}$. Suppose that t is transcendental (over Q). Denote by S the ring generated by the set $W\bigcup\{t, t^{-1}\}$. Put $T = \{a(t^n - 1)x_0 : a \in S\}$ and $w(a) = (a^n - 1)x_0$ for $a \in W$. Then (10) holds. Further, $w(ab) = a^n w(b) + w(a)$ for $a, b \in W$ and $(t^n - 1)w(a) = (a^n - 1)(t^n - 1)x_0 \in T$ for $a \in W$. Thus w fulfils (11).

Next, fix $a \in W$ and suppose that $w(a) \in T$. Then there is $b \in S$ with $(a^n - 1) = (t^n - 1)b$. Since t is a transcendental number and $a > 0$, it is possible only in the case $a = 1$ and $b = 0$. Consequently condition (12) is satisfied, too. Hence, in view of Theorem 1, the function $f : X \to K$ given by (13) is a solution of equation (2).

Example 2. Suppose that $t^n = 1$. Let W be a multiplicative subgroup of $K\setminus\{0\}$ such that $a^n \neq 1$ for $a \in W\setminus\{1\}$. Fix $x_0 \in X\setminus\{0\}$ and put $w(a) = (a^n - 1)x_0$ for $a \in W$. It is easy to check that conditions (10)–(12) are valid with $T = \{0\}$. Thus, in virtue of Theorem 1, formula (13) (with $T = \{0\}$) gives us a solution of (2).

Example 3. Assume that $n = 1$, char $K \neq 0$, and t does not satisfy $(*)$, where L is the simple subfiled of K. Denote by S the ring generated by the set $L\bigcup\{t, t^{-1}\}$ and fix $x_0 \in X\setminus\{0\}$. Put $W = L\setminus\{0\}$, $T = \{a(t-1)x_0 : a \in S\}$, and $w(a) = (a-1)x_0$ for $a \in W$. In the same way like in Example 1 one can prove that conditions (10) and (11) hold. Further, fix $a \in W\setminus\{1\}$ and suppose that $w(a) \in T$. Then there are $k \in \mathbb{N}\bigcup\{0\}$ and $a_{-k},\dots,a_k \in L$ with $(a_{-k}t^{-k} + \dots + a_k t^k)(t - 1)x_0 = (a - 1)x_0$. Hence $(t - 1)((a - 1)^{-1}a_{-k}t^{-k} + \dots + (a - 1)^{-1}a_k t^k) - 1 = 0$. This brings a contradiction, because t does not satisfy $(*)$. Hence (12) holds. Consequently, by Theorem 1, (13) supplies a solution of (2).

Remark. It results from Example 3 and Proposition 3 that in the case where $n = 1$ and char $K \neq 0, 2$ every solution $f : X \to K$ of (2) is of the form described in Proposition 3 iff t satisfies condition $(*)$.

In a similar way, like in Examples 1–3, one can find other numerous examples of functions satisfying equation (2) for many t. We have as well the following

Proposition 4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Then there are 2^K solutions $f : K \to K$ of equation (2).*

Proof. Let T_0 be an additive subgroup of K. Put

$$T = \left\{ \sum_{i=-k}^{k} a_i t^{in} : \ k \in \mathbb{N}, \ a_i \in T_0 \ \text{ for } i = -k, \dots, k \right\},$$

$W = \{1\}$, and $w(1) = 0$. Then it is easy to observe that conditions (10)–(12) are valid. Thus, in virtue of Theorem 1, the function $f : K \to K$ given by (13), with $X = K$, satisfies equation (2).

Since there are 2^K linear subspaces of the linear space K over the simple extension of the field $\mathbb{Q}$ by the element t^n, we obtain in this way 2^K solutions of (2). This ends the proof.

In general, it seems to be difficult to determine all solutions $f : X \to K$ of (2) explicitly. However, like in the case $n = 1$ and $t = 1$ (see [1]–[3], [7], [8], [10], [12], [14], and [16]), this can be done under some additional assumptions and we shall make it in the sequel.

3. Algebraic assumptions

In this part we determine solutions of equation (2) satisfying some algebraic assumptions. Let us begin with the following

Lemma 2. *Let W be a cyclic multiplicative subgroup of $K \setminus \{0\}$ and let T be an additive subgroup of X such that (10) holds. Suppose that a function $w : W \to X$ satisfies conditions (11) and (12) and there is $a_0 \in W$ with $a_0^n \neq 1$. Then*

$$a^n \neq 1 \quad \textit{for every} \ \ a \in W \setminus \{1\}$$

and there exists $x_0 \in X \setminus \bigcup\{(a^n - 1)^{-1} T : a \in W \setminus \{1\}\}$ such that $(t^n - 1)x_0 \in \bigcap\{(a^n - 1)^{-1} T : a \in W \setminus \{1\}\}$ and

$$w(a) - (a^n - 1)x_0 \in T \qquad \textit{for} \ \ a \in W\,. \tag{15}$$

Proof. First of all notice that (12) and (15) yield $a^n \neq 1$ and $x_0 \notin (a^n-1)^{-1}T$ for $a \in W \setminus \{1\}$. Further, by (10) and (15), we get $(t^n-1)(w(a)-(a^n-1)x_0) \in T$ for $a \in W$, which, in view of (11), implies $(t^n-1)(a^n-1)x_0 \in T$ for $a \in W$. Thus it remains only to show that there is $x_0 \in X$ satisfying (15).

On account of the hypothesis on W, there is $c \in W$ such that $W = \{c^k : k \in \mathbb{Z}\}$. Since $a_0^n \neq 1$, we have $c^n \neq 1$. Put $x_0 = (c^n-1)^{-1}w(c)$. We want to prove that

$$w(c^m) - (c^{mn}-1)x_0 \in T \tag{16}$$

for every $m \in \mathbb{Z}$. First, we shall do this for $m \geq 0$ by induction.

It results from (12) that (16) holds for $m = 0$. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned} w(c^{m+1}) - (c^{(m+1)n}-1)x_0 &= w(c^{m+1}) - c^n w(c^m) - w(c) \\ &+ c^n w(c^m) + (c^n-1)(c^n-1)^{-1}w(c) - (c^{(m+1)n}-1)x_0 \\ &= w(c^{m+1}) - c^n w(c^m) - w(c) + c^n(w(c^m) - (c^{mn}-1)x_0)\,. \end{aligned}$$

Hence, in view of (10), (11), and the induction hypothesis, $w(c^{m+1}) - (c^{(m+1)n}-1)x_0 \in T$. So, we have proved that (16) is valid for every $m \in \mathbb{N} \bigcup \{0\}$.

Next, fix $m \in \mathbb{N}$ and put $a = c^{-m}$ and $b = c^m$ in (11). Then we obtain $w(1) - c^{-mn}w(c^m) - w(c^{-m}) \in T$. Hence (12) implies

$$w(c^{-m}) + c^{-mn}w(c^m) \in T\,. \tag{17}$$

On the other hand

$$\begin{aligned} & w(c^{-m}) - (c^{-mn}-1)x_0 = \\ & \qquad = w(c^{-m}) + c^{-mn}w(c^m) - c^{-mn}(w(c^m) - (c^{mn}-1)x_0)\,. \end{aligned}$$

Consequently, by (10), (16), and (17), $w(c^{-m}) - (c^{-mn}-1)x_0 \in T$. This completes the proof.

Proposition 5. *A function $f : X \to K$ is a solution of equation (2), the set $\{tf(x) : x \in X,\ f(x) \neq 0\}$ is a cyclic multiplicative subgroup of $K \setminus \{0\}$, and*

$$\text{there is } a_0 \in f(X) \setminus \{0\} \text{ with } a_0^n \neq t^{-n} \tag{18}$$

iff there exist a cyclic multiplicative subgroup $W \neq \{1\}$ of $K \setminus \{0\}$ with

$$a^n \neq 1 \quad \text{for } a \in W \setminus \{1\}\,, \tag{19}$$

an additive subgroup T of X satisfying (10), and $x_0 \in X$ such that

$$x_0 \notin \bigcup\{(a^n-1)^{-1}T : a \in W \setminus \{1\}\}\,; \tag{20}$$

$$(21) \qquad (t^n - 1)x_0 \in \bigcap\{(a^n - 1)^{-1}T : a \in W \setminus \{1\}\}\,;$$

$$(22) \qquad f(x) = \begin{cases} t^{-1}a & \textit{if } x \in (a^n - 1)x_0 + T \textit{ and } a \in W\,; \\ 0 & \textit{otherwise,} \end{cases} \quad \textit{for } x \in X\,.$$

Proof. Assume that f is a solution of equation (2). Then, in virtue of Theorem 1, there exist a multiplicative subgroup W of $K \setminus \{0\}$, an additive subgroup T of X, and a function $w : W \to X$ such that conditions (10)–(13) are valid. It is easily seen that $W = t(f(X) \setminus \{0\})$. Thus W is a multiplicative cyclic group and, by (18), there is $a_0 \in W$ with $a_0^n \neq 1$. Hence Lemma 2 and (13) imply that there exists $x_0 \in X$ fulfilling conditions (20)–(22) and (19) holds.

For the converse it is enough to notice that the function $w : W \to X$, given by: $w(a) = (a^n - 1)x_0$ for $a \in W$, satisfies (11) and (12) and use Theorem 1. This ends the proof.

The next proposition describes the solutions of (2), which do not satisfy condition (18).

Proposition 6. *A function $f : X \to K,\ f \neq 0$, satisfies equation (2) and the condition*

$$(23) \qquad f(x)^n = t^{-n} \quad \textit{for every } \ x \in X \textit{ with } f(x) \neq 0$$

iff there exist $k \in \mathbb{N},\ a_0 \in K \setminus \{0\},\ x_0 \in X$, and an additive subgroup T of X such that $a_0^k = a_0^n = 1,\ kx_0 \in T,\ (t^n - 1)x_0 \in T,\ t^nT = T$,

$$(24) \qquad a_0^i \neq 1 \ \textit{ and } \ ix_0 \notin T \quad \textit{for every } i \in \mathbb{N},\ i < k\,,$$

and

$$(25) \qquad f(x) = \begin{cases} t^{-1}a_0^i & \textit{if } x \in ix_0 + T \textit{ and } i \in N,\ i \leq k\,; \\ 0 & \textit{otherwise}\,, \end{cases} \quad \textit{for } x \in X\,.$$

Proof. Assume that f satisfies equation (2) and condition (23). According to Theorem 1 there are a multiplicative subgroup W of $K \setminus \{0\}$, an additive subgroup T of X, and a function $w : W \to X$ such that conditions (10)–(13) are valid. In view of (23) we have $a^n = 1$ for every $a \in W$, which means that W is a finite cyclic group. Thus there is $a_0 \in K \setminus \{0\}$ and $k \in \mathbb{N}$ such that $W = \{a_0^i : i \in \mathbb{N},\ i \leq k\},\ a_0^k = 1$, and $a_0^i \neq 1$ for $i \in \mathbb{N}, i < k$. It is easily seen that we also must have $a_0^n = 1$. Put $x_0 = w(a_0)$. Then, by induction, we get from (11) $w(a_0^i) - ix_0 = w(a_0^i) - iw(a_0) \in T$ for $i \in \mathbb{N}$. Consequently, in virtue of (10)–(13), $kx_0 \in T,\ (t^n - 1)x_0 \in T$, and (24), (25) hold. The fact $t^nT = T$ result from (10).

For the converse, according to Theorem 1, it suffices to observe that if f is of form (25), then conditions (10)–(13) are valid with $W = \{a_0^i : i \in \mathbb{N},\ i \le k\}$ and $w(a_0^i) = ix_0$ for $i \in \mathbb{N},\ i \le k$. This ends the proof.

From Proposition 6 we obtain, in particular, the following

Corollary 1. *Suppose that* $\mathrm{char}K \neq 0$. *Then a function* $f : X \to K,\ f \neq 0$, *satisfies (23) and equation (2) iff there exists an additive subgroup* T *of* X *such that* $t^n T = T$ *and*

$$f(x) = \begin{cases} t^{-1} & \text{for } x \in T\,; \\ 0 & \text{for } x \in X \setminus T\,. \end{cases} \tag{26}$$

Proof. Assume that f satisfies equation (2) and condition (23). Then, on account of Proposition 6, f is of form (25). Notice that T is a linear subspace of X over the simple subfield of K. Thus from the fact that $kx_0 \in T$ and (24) we deduce $x_0 \in T$ or $\mathrm{char}K = k$.

Suppose that $k = \mathrm{char}K$. Then $(a+b)^k = a^k + b^k$ for every $a, b \in K$ and consequently $a^k \neq 1$ for $a \in K \setminus \{1\}$, because k is a prime number. Hence $a_0 = 1$, which, in view of (24), means that $k = 1$. This brings a contradiction, since $\mathrm{char}K > 1$.

So, we have proved that $x_0 \in T$. Thus $k = 1$ and $a_0 = 1$. Hence (25) yields (26).

For the converse it suffices to put $k = 1,\ a_0 = 1$ and $x_0 = 0$ and use Proposition 6 again. This ends the proof.

Now, we are in a position to give a description of solutions $f : X \to K$ of (2) in the case where K is a finite field.

Theorem 2. *Suppose that* K *is a finite field. Then a function* $f : X \to K,\ f \neq 0$, *is a solution of equation (2) iff there are* $b \in K \setminus \{0\},\ x_0 \in X$, *and an additive subgroup* T *of* X *such that*

(27) *the numbers* n *and* $r(b) := \min\{j \in N : b^j = 1\}$ *are relatively prime*;

$$b^n T = T \text{ and } t^n T = T\,; \tag{28}$$

$$\text{if } b \neq 1, \text{ then } (t^n - 1)x_0 \in \bigcap\{(b^{nj} - 1)^{-1} T : j \in N,\ j < r(b)\}\,; \tag{29}$$

$$x_0 \notin \bigcup\{(b^{jn} - 1)^{-1} T : j \in N,\ j < r(b)\}\,; \tag{30}$$

$$f(x) = \begin{cases} t^{-1} b^j & \text{if } x \in (b^{jn} - 1)x_0 + T \text{ and } j \in N,\ j \le r(b)\,; \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

for $x \in X$.

Proof. Assume that f is a solution of (2). In the case where (23) holds, by Corollary 1, it is enough to put $b = 1$ and $x_0 = 0$. Therefore suppose that there is $x \in X$ with $f(x)^n \notin \{0, t^{-n}\}$. Then, according to Proposition 1 and lemma 1 (iii), the set $t(f(X) \setminus \{0\})$ is a finite multiplicative group. Thus it is a cyclic group and consequently, on account of Proposition 5, f is of form (22). Hence there is $b \in \mathbb{K} \setminus \{0\}$ with $W = \{b^j : j \in N\}$. It is easily seen that (28)–(31) result from (10) and (20)–(22). It remains to show (27).

For the proof by contradiction suppose that (27) is not valid. Then there are $k, m, j \in N$ with $k > 1$, $n = kj$, and $r(b) = km$. Thus $b^m \neq 1$ and $(b^m)^n = (b^{r(b)})^j = 1$. This brings a contradiction with (19).

For the converse, note that the case where $b = 1$ is trivial (we may use Corollary 1). Thus, in view of Proposition 5, it suffices to show that (19)–(22) and (10) hold with $b \neq 1$ and $W = \{b^j : j \in N,\ j \leq r(b)\}$. It is easy to see that (28)–(31) imply (20)–(22) and (10). Next, suppose that $a \in W$ and $a^n = 1$. Then there exists $j \in N$ with $j \leq r(b)$ and $a = b^j$. Thus $b^{jn} = 1$, which means that $r(b)$ is a divisor of jn. Since $r(b)$ and n are relatively prime and $j \leq r(b)$, this yields $j = r(b)$.

In this way we have proved that (19) holds, which completes the proof.

Corollary 2. *Suppose that K is a simple finite field. Then a function $f : X \to K$, $f \neq 0, t^{-1}$, is a solution of equation (2) iff,*

1° in the case $t^n \neq 1$, there is an additive subgroup T of X such that $T \neq X$ and (26) holds;

2° in the case $t^n = 1$, there are an additive subgroup T of X, $b \in K \setminus \{0\}$, and $x_0 \in X \setminus T$ such that conditions (27) and (31) are valid.

Proof. Since every additive subgroup T of X is a linear subspace of X, in the case $t^n \neq 1$ conditions (29) and (30) can be fulfilled only for $b = 1$. Further, if $t^n = 1$, then (29) is valid for every $b \in K \setminus \{0\}$, $x_0 \in X$, and every additive subgroup T of X. Moreover, in the case $b \neq 1$, (30) holds iff $x_0 \notin T$. Hence Theorem 2 yields the assertion.

Using Theorem 2 and some results from [8] and [12] we obtain as well the next two corollaries.

Corollary 3. *Suppose that K is a finite field and $g \neq 0$ is a function mapping X into K. Then the binary operation $\circ : X \times X \to X$ given by the formula*

$$x \circ y = x + g(x)y \qquad \text{for } x, y \in X \tag{32}$$

is associtative if and only if there exist an additive subgroup T of X, $b \in K \setminus \{0\}$, and $x_0 \in X \setminus \bigcup\{(b^j - 1)^{-1}T :\ j \in N,\ j < r(b)\}$ such that

$bT = T$ and

$$g(x) = \begin{cases} b^j & \text{if } x \in (b^j - 1)x_0 + T \text{ and } j \in \mathbb{N},\ j \le r(b)\,; \\ 0 & \text{otherwise}\,, \end{cases} \quad \text{for } x \in X\,.$$

Proof. It is easy to check that the operation is associative iff g satisfies functional equation (3) with $n = 1$ (cf. [12], Lemma 1). Thus Theorem 2 implies the statement.

The problem of characterization of binary operations of form (32) has been already studied in [12] and solved there in the case where $K = \mathbb{R}$, X is a real linear topological space, and f is continuous.

Next, let us introduce in the set $A = (K \setminus \{0\}) \times X$ a binary operation $\circ : A \times A \to A$ as follows:

$$(a, x) \circ (b, y) = (ab, x + a^n y) \quad \text{for } (a, x), (b, y) \in A\,.$$

It is easy to verify that $(A, \circ)$ is a group (cf. [8]).

Corollary 4. *Suppose that K is a finite field and $f \neq 0$ is a function mapping X into K. Then the set $D = \{(f(x), x) : x \in X,\ f(x) \neq 0\}$ is a subgroup of the group $(A, \circ)$ if and only if f is of the form described in Theorem 2 with $t = 1$.*

Proof. According to Theorem 1 from [8], D is a subgroup of $(A, \circ)$ iff f satisfies equation (2) with $t = 1$, i.e. equation (3). Thus Theorem 2 yields the statement.

For further details concerning the group $(A, \circ)$ and that way of finding subgroups refer to [2], [7], [8], [13], and [16].

4. Continuous solutions

Finally, we shall give the continuous solutions $f : X \to K$ of equation (2) in the case where K is either $\mathbb{R}$ or $\mathbb{C}$ (with the usual topology) and X is a linear topological space over K.

The continuous solutions of (2), with $t = 1$ and $n = 1$, have been already determined, in the case $K = \mathbb{R}$, in [7] and [12] and, in the case $K = \mathbb{C}$, in [3] (see also [14]). N. Brillouët–Belluot [5] has found the continuous solutions $f : X \to K$ of (2) in the case $K = \mathbb{R}$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. The case where $t = 1$, K is either $\mathbb{R}$ or $\mathbb{C}$, and n is any positive integer is solved in [8].

In the sequel we will need a result from [8]. Let us recall it.

Lemma 3. (see [8], Corollary 4). *Suppose that K is either $\mathbb{R}$ or $\mathbb{C}$ and X is a topological linear space over K. Then a function $g : X \to K,\ g \neq 0$, is continuous and satisfies equation (3) iff*

$$g(X) \subset R \quad \text{or} \quad n = 1 \tag{33}$$

and the following two conditions hold:

(i) *if $g(X) \subset \mathbb{R}$, then there exists a continuous $\mathbb{R}$–linear functional $h : X \to \mathbb{R}$ such that,*

1° *in the case where n is odd,*

$$g(x) = \sqrt[n]{h(x) + 1} \quad \text{for} \quad x \in X \tag{34}$$

or

$$g(x) = \sqrt[n]{\sup(h(x) + 1, 0)} \quad \text{for } x \in X\,; \tag{35}$$

2° *in the case where n is even, g is of form (35);*

(ii) if $g(X) \not\subset \mathbb{R}$ and $n = 1$, then there exists a continuous $\mathbb{C}$–linear functional $h : X \to \mathbb{C}$ such that $g(x) = h(x) + 1$ for $x \in X$.

Now we are in a position to prove the following

Theorem 3. *Suppose that K is either $\mathbb{R}$ or $\mathbb{C}$ and X is a linear topological space over K. Then a function $f : X \to K,\ f \neq 0$, is continuous and satisfies equation (2) iff,*

(i) *in the case $t^n \neq 1,\ f = t^{-1}$;*

(ii) *in the case $t^n = 1$, there exists a function $g : X \to K$ fulfilling (33) and conditions* (i), (ii) *of Lemma 3 such that $f(x) = t^{-1}g(x)$ for $x \in X$.*

Proof. Assume that f is continuous and satisfies equation (2). Then, according to Proposition 1 and Lemma 3, there exists a function $g : X \to K$ fulfilling (4), (33), and conditions (i), (ii) of Lemma 3. The case $t^n = 1$ does not demand any comment. Further, $h(ax) = ah(x)$ for $a \in \mathbb{R},\ x \in X$. Whence, in the case $t^n \in \mathbb{R} \setminus \{1\}$, (4) implies $h = 0$. So does it in the case where $g(X) \not\subset \mathbb{R}$ and $t^n \in \mathbb{C} \setminus \mathbb{R}$. Consequently, it remains to consider only the case where $g(X) \subset \mathbb{R}$ and $t^n \in \mathbb{C} \setminus \mathbb{R}$.

It is easily seen that in this case $K = \mathbb{C}$. On account of (4) and Lemma 1 (vii), $(t^n - 1)x \in T := g^{-1}(\{1\})$ for every $x \in X$ with $g(x) \neq 0$. Notice that $g^{-1}(\{1\}) = h^{-1}(\{0\})$ and $h^{-1}((-1, +\infty)) \subset g^{-1}((0, +\infty))$. Thus $(t^n - 1)h^{-1}((-1, +\infty)) \subset h^{-1}(\{0\})$. Since $h^{-1}((-\infty, 1)) = -h^{-1}((-1, +\infty))$ and $X = h^{-1}((-\infty, 1)) \bigcup h^{-1}((-1, +\infty))$, we obtain $X = (t^n - 1)X \subset h^{-1}(\{0\})$, which means that $h = 0$ an consequently $f = t^{-1}$. This completes the proof.

Some other results from [8] can be generalized for equation (2) in a similar way.

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JANUSZ BRZDĘK
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
REJTANA 16 A
35–310 RZESZÓW
POLAND

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