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Polarization constants for products of linear functionals over $\mathbb{R}^2$ and $\mathbb{C}^2$ and Chebyshev constants of the unit sphere

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Abstract. The topic of this work is the estimation of the *linear polarization constant* of normed spaces. In finite dimensional Hilbert spaces we study the linear polarization constant and the *Chebyshev constant.* By constructing certain generalized trigonometric functions, our investigation leads to the connection of the polarization constant on a 2-dimensional complex Hilbert space and the Chebyshev constant of S^2. This provides estimates for the n^{th} polarization constants. Our main result is asymptotically best possible.

1. Introduction

In the theory of analytic functions on Banach spaces polynomials of finite-type play an important role. A *polynomial of finite type* on a Banach space E is a finite linear combination of finite products of functionals in E^*, the dual space of E. It is known that a weakly continuous entire function on E is uniformly approximable on bounded sets by polynomials of finite

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type, see Gamelin [10, Corollary 3.1.2]. Moreover, any n-homogeneous polynomial that is weakly continuous is a polynomial of finite type. An interesting question in this area is the problem of estimating the best possible lower bounds for the norms of the simplest polynomials of finite type which are products of linear functionals, see [1], [4], [22]. For example, if $x_1, \dots, x_n$ are n unit vectors in a finite dimensional Hilbert space H, we form the n-homogeneous polynomial

$$P(x) = \langle x, x_1 \rangle \langle x, x_2 \rangle \dots \langle x, x_n \rangle \qquad (x \in H). \tag{1}$$

The norm of the polynomial P is

$$\|P\| = \max_{\|x\|=1} |P(x)|. \tag{2}$$

Then the problem is to find the optimal choice of x_i's, $1 \leq i \leq n$, for which the minimum value of the norm is achieved. This is actually a min-max problem. Obviously, the maximum value of the norm is 1 and is achieved when all the unit vectors coincide. But what is the minimum value of the norm? If $m = \dim(H) \geq n$, a reasonable guess is that the minimum is achieved when the x_i's form an orthonormal system. Then

$$\|P\| = \frac{1}{n^{n/2}}. \tag{3}$$

If H is an m-dimensional *complex* Hilbert space with $m \geq n$, J. Arias-de-Reyna has proved in [1] that this value is actually the minimum. His proof is an elegant mixture of probability theory and multilinear algebra. An even stronger result was obtained recently by K. M. Ball [2] using linear algebra and geometric arguments combined with complex function theory. However, the real case for $m \geq n$ is still open, unless $n \leq 5$, see [19].

In the present paper we investigate the case when H is 2-dimensional. Trying to tackle this problem we are led to the problem of calculating the geometric Chebyshev constant for the unit balls of $\mathbb{R}^2$ and $\mathbb{R}^3$.

The real case can be solved explicitly and it is reflected in the fact that the Chebyshev constant for the unit disk in $\mathbb{R}^2$ is well known from potential theory and easy to compute. However, the exact value of the Chebyshev constant for the unit ball of $\mathbb{R}^3$ is not known and because of this, the connection of the complex case with this problem is of interest although we do not explicitly compute the exact minimum.

We discuss first the real case and then the complex case because the methods are similar and because a generalization of the well-known trigonometric functions comes into the picture.

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2. Basic notation and definitions

Let E be a normed space. Consider n continuous linear forms $f_1, \dots, f_n$ in the unit sphere S_{E^*} of E^*. We define the n-homogeneous polynomial

$$P(x) := f_1(x) f_2(x) \dots f_n(x), \tag{4}$$

or in tensorial notation (see for instance [21])

$$P := f_1 \odot f_2 \odot \dots \odot f_n. \tag{5}$$

Following for instance Benitez, Sarantopoulos and Tonge [4], we define the (linear) polarization constants of E as follows.

Definition 1. The n^{th} *polarization constant* and the *polarization constant* of E are defined by

$$c_n(E) := 1 \Big/ \inf_{f_1,\dots,f_n \in S_{E^*}} \sup_{\|x\|=1} |f_1(x) f_2(x) \dots f_n(x)| \tag{6}$$

and

$$c(E) := \limsup c_n(E)^{1/n}, \tag{7}$$

respectively. Actually, lim sup exists as a lim, see [20].

To find these polarization constants is usually a difficult task. Note that in the special case of a Hilbert space H by the Riesz Representation Theorem the previous definitions can be formulated as follows:

$$c_n(H) = 1 \Big/ \inf_{x_1,\dots,x_n \in S_H} \sup_{\|x\|=1} |\langle x, x_1 \rangle \langle x, x_2 \rangle \dots \langle x, x_n \rangle| \tag{8}$$

and

$$c(H) = \limsup c_n(H)^{1/n}, \tag{9}$$

respectively. This has an obvious geometrical interpretation since it involves vectors of H only. For a finite dimensional Hilbert space of dimension d over the field $\mathbb{K}$ (where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$) we denote the n^{th} polarization constant by $c_n(\mathbb{K}^d)$ and the polarization constant by $c(\mathbb{K}^d)$.

For more information and related results we refer to the book of S. Dineen [7, Chapter 1].

Suppose now that $\mathcal{K}$ is a compact set in E.

Definition 2. The n^{th} *Chebyshev constant* of $\mathcal{K}$ is defined by

$$M_n(\mathcal{K}) := \inf_{y_1,\dots,y_n \in \mathcal{K}} \sup_{y \in F} \|y - y_1\| \|y - y_2\| \dots \|y - y_n\|. \tag{10}$$

With this definition we follow Pólya, Szegő and Carleson [6], but not Zaharjuta [26]. For more on Chebyshev constants see the recent very general account [9].

3. Real case

Given a 2-dimensional real Hilbert space, we try to find the minimum value of $\|P\|$, where P is defined in (1) for all possible configurations of the points $x_1, x_2, \dots, x_n$. We are going to exploit the geometrical nature of this problem rather than the analytical one. Equation (4) takes the form

$$P(x) = \cos\vartheta_1 \cos\vartheta_2 \dots \cos\vartheta_n, \tag{11}$$

where $\angle(x, x_j) = \vartheta_j$, $1 \le j \le n$, are the angles between the unit vectors x and x_j. Searching for the maximum of the polynomial, x runs all over the unit circle. Of course the same happens if we consider a unit vector y, with $\angle(y, x) = \pi/2$. Then $\angle(y, x_j) = \varphi_j = \vartheta_j + \pi/2$ and the polynomial takes the form

$$P(y) = \sin\varphi_1 \sin\varphi_2 \dots \sin\varphi_n. \tag{12}$$

Consider now the complex numbers $z, z_j \in \mathbb{C}$, $|z| = |z_j| = 1$, corresponding to the unit vectors y, x_j respectively. Using the well known identity

$$2|\sin(a-b)| = |e^{i2a} - e^{i2b}| \,, \tag{13}$$

where $a, b \in \mathbb{R}$, we conclude that

$$|P(y)| = \frac{1}{2^n}|z^2 - z_1^2|\,|z^2 - z_2^2| \dots |z^2 - z_n^2|. \tag{14}$$

We also know that the function $w = T(z) = z^2$ defines a group homomorphism of the unit circle onto itself. Thus the original min-max problem can be re-stated as follows. Let

$$Q(w) = (w - w_1)(w - w_2) \dots (w - w_n) \tag{15}$$

be a complex polynomial with the roots on the unit circle. What is the optimal choice of the w_j's such that the maximum of $|Q(w)|$ on the unit circle has the minimal value possible? Thus if $M_n(S^1)$ is the n^{th} Chebyshev constant of $S^1 := \{w \in \mathbb{C} : |w| = 1\}$, we have found a formula relating this constant and $c_n(\mathbb{R}^2)$.

Proposition 1. *For the n^{th} polarization constant $c_n(\mathbb{R}^2)$ we have*

$$c_n(\mathbb{R}^2) = 2^n / M_n(S^1).$$

Using Proposition 1 we are led to the following result.

Theorem 2. *The n^{th} polarization constant of $\mathbb{R}^2$ is equal to 2^{n-1}, that is*

$$c_n(\mathbb{R}^2) = 2^{n-1}.$$

Proof. It suffices to show that $M_n(S^1) = 2$. The trick is to use the following DFT (Discrete Fourier Transform) type calculation

$$\frac{1}{n}\sum_{k=0}^{n-1} Q(e^{i2\pi\frac{k}{n}}w) = w^n + (-1)^n w_1 w_2 \dots w_n = w^n - Q(0), \tag{16}$$

where Q is the polynomial defined by (15). Multiplying the points $w_1, \dots,$ w_n by an appropriate complex unit – that is, rotating the point system – we can assume that $Q(0) = 1$. But then

$$|w^n - Q(0)| = \left|\frac{1}{n}\sum_{k=0}^{n-1} Q(e^{i2\pi\frac{k}{n}}w)\right| \le \frac{1}{n}\sum_{k=0}^{n-1}\left|Q(e^{i2\pi\frac{k}{n}}w)\right| \le \|Q\|.$$

Since the maximum of the left hand side can be easily calculated as

$$\sup_{|w|=1} |w^n - Q(0)| = 2,$$

we finally get

$$2 \leq \|Q\|.$$

Moreover, $\|Q\| = 2$ if and only if the w_j's are the n^{th} roots of unity. $\square$

Corollary 3. *The polarization constant of* $\mathbb{R}^2$ *is equal to* 2*, that is*

$$c(\mathbb{R}^2) = 2.$$

Remarks. A closely related topic is the exact constant in Mahler's inequality. Y. Sarantopoulos, based on investigations of the polarization constants, conjectured that the original estimate 2^n of Mahler [15] could be improved to 2^{n-1}. In 1997 this conjecture was proved by Kroó and Pritsker [12]. The simple argument in the proof of Theorem 2 is in fact a special case of this sharp Mahler-type inequality which has already been presented e.g. in [18].

In line to the above comments, just after finishing our work, a paper of Bojanov, Haussman and Nikolov [5] appeared, which, in a different setting, contains a result stronger than the above Theorem 2. In this paper the authors consider products of linear two-variable polynomials and they estimate the relation between the norm of the product and the product of the norms. The interested reader may verify easily that indeed their Theorem 6 implies Theorem 2.

However, we have decided to present the above detailed proof of Theorem 2 because the two basic steps of the real case translate (in the complex case) to the extension of the trigonometric relations of the real plane to $\mathbb{C}^2$, and deduction of equivalence to the corresponding Chebyshev constant. To the best of our knowledge, these were not investigated in the complex case before; therefore our presentation of the real case serves as an instructive model to the complex version.

4. Complex case

Now we will imitate the proof of Theorem 2 in order to transform the two-dimensional complex case into a Chebyshev problem. For $z \in \mathbb{C}^2$, let

$$P(z) = \langle z, z_1\rangle\langle z, z_2\rangle \dots \langle z, z_n\rangle.$$

The original problem is to find the value of

$$c_n(\mathbb{C}^2) = 1/\min_{z_1,\dots,z_n \in S} \max_{z\in S} |P(z)|, \tag{17}$$

where, as usual, $S = \partial B_{\mathbb{C}^2}$. We will proceed in this situation the same way as we did for the real case. First of all we define the cosine of a point $z \in S$. We can define a multitude of these cosines with respect to any given unit vector w by the following standard way

$$\cos_w(z) := \langle z, w\rangle. \tag{18}$$

In calculus we learn that the sine function can be defined to satisfy $\sin^2 a + \cos^2 a = 1$. On the other hand the sine function can be defined as $\sin x = \cos(x - \frac{\pi}{2})$. These two identities motivate the following definition. First we define the transformation

$$T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} \overline{z}_2 \\ -\overline{z}_1 \end{pmatrix}. \tag{19}$$

If z_1 and z_2 are reals this corresponds to a rotation by $-\pi/2$. Having defined the rotation, the definition of the sine is straightforward. Namely, we define

$$\sin_w(z) := \cos_w(T(z)). \tag{20}$$

With definitions (18), (19) and (20) in hand, a simple calculation yields

$$|\sin_w(z)|^2 + |\cos_w(z)|^2 = \|z\|^2 \cdot \|w\|^2 = 1, \tag{21}$$

for any unit vectors $z, w \in S$. Moreover, it can be seen that for any unit vectors $z, w \in S$ the above defined sine and cosine functions inherit some of the well-known trigonometric identities with minor modifications.

If we want to generalize identity (13), we need to find a way to "double" the angles of vectors of S. Note that S does not have a group structure. However, we can view this process geometrically in the plane and generalize to S.

For this, consider the unit vector $e_1 = (1, 0)$ in $\mathbb{R}^2$. If we are given a unit vector $x = (x_1, x_2) \in \mathbb{R}^2$ and we want to double its angle from e_1, then the new vector x' is

$$x' = 2\langle x, e_1\rangle x - e_1 = \begin{pmatrix} 2x_1^2 - 1 \\ 2x_1x_2 \end{pmatrix}. \tag{22}$$

With the double angle vectors defined above, equation (13) reads

$$2|\sin_y x| = \|x' - y'\|. \tag{23}$$

This is exactly the property we wish to extend to vectors $z, w \in \mathbb{C}^2$. The key idea is to define a proper "doubling function" $D : S \to S$ with similar properties. We cannot blindly substitute in the real definition (22) complex vectors because an ambiguity comes up as far as the inner product is concerned. Among $\langle e_1, z\rangle$ and $\langle z, e_1\rangle$, which is the right choice? The resulting "double angle" vector must have norm equal to 1 in order to lie again on the sphere. Therefore the right choice is

$$D(z) := 2\langle e_1, z\rangle z - e_1 = \begin{pmatrix} 2|z_1|^2 - 1 \\ 2\overline{z}_1 z_2 \end{pmatrix}, \tag{24}$$

where now for unit $z = (z_1, z_2)$ also $D(z)$ is a unit vector in $\mathbb{C}^2$.

A generalization of (13), i.e. (23) is formulated in the following result.

Proposition 4. *If $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are in S, then*

$$2|\sin_w(z)| = \|D(z) - D(w)\|. \tag{25}$$

Proof. We have

$$\begin{aligned} \|D(z) - D(w)\| &= \left\| \begin{pmatrix} 2|z_1|^2 - 1 \\ 2\overline{z}_1 z_2 \end{pmatrix} - \begin{pmatrix} 2|w_1|^2 - 1 \\ 2\overline{w}_1 w_2 \end{pmatrix} \right\| \\ &= 2 \left\| \begin{pmatrix} |z_1|^2 - |w_1|^2 \\ \overline{z}_1 z_2 - \overline{w}_1 w_2 \end{pmatrix} \right\|. \end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{4}\|D(z)-D(w)\|^2 &= (|z_1|^2-|w_1|^2)^2+|\overline{z}_1z_2-\overline{w}_1w_2|^2\\
&= |z_1|^4+|w_1|^4-2|z_1|^2|w_1|^2+|z_1|^2|z_2|^2+|w_1|^2|w_2|^2-2\,\mathrm{Re}(\overline{z}_1z_2w_1\overline{w}_2)\\
&= |z_1|^2+|w_1|^2-2|z_1|^2|w_1|^2-2\,\mathrm{Re}(\overline{z}_1z_2w_1\overline{w}_2)\\
&= |z_1|^2|w_2|^2+|w_1|^2|z_2|^2-2\,\mathrm{Re}(\overline{z}_1z_2w_1\overline{w}_2)\\
&= |\overline{z}_2\overline{w}_1-\overline{z}_1\overline{w}_2|^2=|\langle T(z),w\rangle|^2\\
&= |\sin_w(z)|^2. \qquad \square
\end{aligned}$$

Let us make some comments regarding the functions T and D.

Remarks. (i) It is easy to observe that T is an isometric homeomorphism from S onto S. So $T(S)=S$ and $S=T^{-1}(S)$.

(ii) By standard calculations it is easy to see that the "doubling function" D is a continuous mapping from S onto $\Omega=(\mathbb{R}\times\mathbb{C})\cap S$. Actually, Ω can be identified with the unit sphere of $\mathbb{R}^3$.

Utilizing all the previous discussion we arrive at the following result.

Proposition 5. *For the n^{th} polarization constant $c_n(\mathbb{C}^2)$ we have*

$$c_n(\mathbb{C}^2)=2^n \Big/ \min_{z_1,\dots,z_n\in\Omega}\ \max_{z\in\Omega}\|z-z_1\|\,\|z-z_2\|\dots\|z-z_n\|.$$

Proof. Since

$$\|P\|=\max_{z\in S}|\langle z,z_1\rangle\langle z,z_2\rangle\dots\langle z,z_n\rangle|,$$

by using Proposition 4 and the previous Remark (i) we have

$$\begin{aligned}
\|P\| &= \max_{z\in S}|\langle T(z),z_1\rangle\langle T(z),z_2\rangle\dots\langle T(z),z_n\rangle|\\
&= \max_{z\in S}|\sin_{z_1}(z)\sin_{z_2}(z)\dots\sin_{z_n}(z)|\\
&= \frac{1}{2^n}\max_{z\in S}\|D(z)-D(z_1)\|\,\|D(z)-D(z_2)\|\dots\|D(z)-D(z_n)\|.
\end{aligned}$$

Finally, by Remark (ii)

$$c_n(\mathbb{C}^2)=2^n/\min_{w_1,\dots,w_n\in\Omega}\ \max_{w\in\Omega}\|w-w_1\|\,\|w-w_2\|\dots\|w-w_n\|. \qquad \square$$

Hence, after identifying Ω with the unit sphere of $\mathbb{R}^3$ we obtain.

Theorem 6. *We have*

$$c_n(\mathbb{C}^2) = 2^n / M_n(S^2),$$

where $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

5. On the Chebyshev constant of S^2

In spite of the elementary definition of the Chebyshev constants given in Section 2, surprisingly enough little is known about them for dimension $\neq 2$ (or complex dimension $\neq 1$). Let us note that one approach could be to follow the basic methods of the general theory of potentials and capacities, which would give that $M_n^{1/n}(S^2) \to \mathrm{cap}_0(S^2)$. Here for any fixed $x_0 \in S^2$ and the surface Lebesgue measure $d\sigma(x)$

$$\log(\mathrm{cap}_0(S^2)) = \frac{1}{4\pi} \int_{S^2} \log|x - x_0|\, d\sigma(x) = \frac{1}{2} \log\left(\frac{4}{e}\right). \tag{26}$$

However, to estimate the n^{th} Chebyshev constant and its deviation from this logarithmic capacity is of a kind of a nontrivial geometrical discrepancy problem. Fortunately there are some estimates that help us in estimating the polarization constant of $\mathbb{C}^2$. The best known estimates for the n^{th} Chebychev constants are due to G. Wagner [24], namely

$$0 < C_1 \leq \log M_n(S^2) - \frac{n}{2} \log\left(\frac{4}{e}\right) \leq C_2 < \infty. \tag{27}$$

In view of this result, Theorem 6 yields

Theorem 7. *There exist absolute constants c and C, with $0 < c < C < \infty$, such that for all $n \in \mathbb{N}$ we have*

$$c(\sqrt{e}\,)^n \leq c_n(\mathbb{C}^2) \leq C(\sqrt{e}\,)^n.$$

Corollary 8. *The polarization constant of $\mathbb{C}^2$ is equal to $\sqrt{e}$, that is*

$$c(\mathbb{C}^2) = \sqrt{e}.$$

6. Concluding remarks

The exact value of $c_n(\mathbb{R}^2)$ could be obtained only due to the existence of regular n-gons on S^1. However, for $\mathbb{R}^d$, $d > 2$ or $\mathbb{C}^d$, $d > 1$, no obvious n-symmetric point set exists to minimize the Chebyshev constant, i.e. to maximize the polarization constants.

We have natural candidates for the extremal point set only in some exceptional cases. Say in case $n = d$ any orthonormal set of vectors $x_1, \dots, x_n \in \mathbb{R}^d$ (or $\mathbb{C}^d$) for the polarization constant, or in case $n = 2d$ and with the above orthonormal system the set $\pm x_1, \dots, \pm x_n$ for the Chebyshev constant can be expected to be extremal. However, even for these exceptionally symmetric cases to prove extremality is a highly nontrivial problem. In particular, $c_n(\mathbb{C}^n) = n^{n/2}$ was obtained only recently, see Arias-de-Reyna [1] and also Ball [2], while $c_n(\mathbb{R}^n)$ is still unknown.

Our work, originally aimed at the natural guess of $c_n(\mathbb{C}^d) = d^{n/2}$, has shown that there is a connection between Chebyshev constants and polarization constants, at least for $d = 2$. In view of the equivalence to Chebyshev constants $M_n(S^2)$ for $d = 2$, to make out the exact value of $c_n(\mathbb{C}^d)$ seems rather unlikely, at least in the foreseeable future. Moreover, the exact value of $c_n(\mathbb{R}^2)$, and the relatively precise estimate on $c_n(\mathbb{C}^2)$ suggests that there is no easy formula to guess for the values of the general polarization constants $c_n(\mathbb{R}^d)$ and $c(\mathbb{R}^d)$ or $c_n(\mathbb{C}^d)$ and $c(\mathbb{C}^d)$.

However, if we relax the demand for the exact or at least relatively precise values of $c_n(\mathbb{R}^d)$ and $c_n(\mathbb{C}^d)$, and settle with determination of the limit quantities $c(\mathbb{R}^d)$ and $c(\mathbb{C}^d)$ (with $d < \infty$ fixed), then there are nice results available. J. García-Vázquez and R. Villa [11] determined the values of the real polarization constants $c(\mathbb{R}^d)$ using a general method relying on existence of the so-called "rendezvous numbers" in compact metric spaces, while A. Pappas, Sz. Révész and Y. Sarantopoulos [19], [20] applied basically potential-theory flavored techniques to compute both $c(\mathbb{R}^d)$ and $c(\mathbb{C}^d)$. For some further estimates on the most intriguing problem of the diagonal case see also [16], [17] and the references therein.

It seems that if we depart from the Hilbert space case, the polarization constants on more general Banach spaces are even harder to compute. In this field only a few results are known, see [1], [2], [3], [4], [20], [22].

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