Publ. Math. Debrecen
68/1-2 (2006), 77–88

On p-nilpotency of finite groups with some c-supplemented subgroups of prime power order

By YANGMING LI (Hangzhou), YANMING WANG (Guangzhou)
and HUAQUAN WEI (Guangzhou)

Abstract. A subgroup H of a group G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is contained in H_G, the core in G of H. In this paper we give some sufficient conditions of p-nilpotency of a finite group under the assumption that some subgroups of prime square order of the group are c-supplemented. These are the duals of some recent results, such as Wang's [14] and Guo and Shum's [9].

1. Introduction

Let G be a finite group. The relationship between the properties of subgroups of the Sylow subgroups of G and the structure of G has been investigated by a number of authors (for example, see [4], [8], [12]–[15]). In particular, Srinivasan [12] proved that a finite group is supersolvable if every maximal subgroup of every Sylow subgroup is normal. Later on, Wall [13] gave a complete classification of finite groups under the assumption of Srinivasan. In [14] and [9], the finite group G in which some maximal subgroups of the Sylow subgroups of G are c-supplemented were

Mathematics Subject Classification: 20D10, 20D20.
Key words and phrases: maximal subgroup, 2-maximal subgroup, subgroup of prime square order, c-supplemented subgroup, p-nilpotent group.
Project of the first author supported in part by China Postdoctoral Science Foundation (20040350708), NSF of Guangdong Province (04300023).
Project of the second author supported in part by NSFC, Fund of Doctoral Program.

investigated. It is known, the concepts of maximal subgroup and minimal subgroup are dual in finite group theory, so in the meanwhile, the structure of a finite group G in which some minimal subgroups of the Sylow subgroups of G are c-supplemented was investigated (see [15]). Furthermore, in [14] Wang showed: Let G be a finite group and p the smallest prime dividing $|G|$. If G is A_4-free and every second maximal subgroup of a Sylow p-subgroup of G is c-supplemented in G, then $G/O_p(G)$ is p-nilpotent ([14, Theorem 4.2]). In [9], this was generalized as follows: Let G be a finite group and p the smallest prime dividing $|G|$. If that G is A_4-free and every second maximal subgroup of a Sylow p-subgroup of G is c-supplemented in G, then G is p-nilpotent ([9], Theorem 3.4). In this paper we first continue the discussion in [14], [9]. Then we investigate the structure of a group G with some subgroups of prime square order of a Sylow subgroup, and the dual concept of 2-maximal subgroups of a Sylow subgroup, of G c-supplemented in G. We get some sufficient conditions for the p-nilpotency of a finite group.

Recall that a formation $\mathcal{F}$ of groups is a class of groups which is closed under homomorphic images such that $G/M \cap N \in \mathcal{F}$ whenever M, N are normal subgroups of a group G with $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [10, Ch VI]). It is easy to see that the class of groups with Sylow tower of supersolvable type is a saturated formation. For a formation $\mathcal{F}$, each group has a smallest normal subgroup N such that G/N is in $\mathcal{F}$. This uniquely determined normal subgroup of G is called the $\mathcal{F}$-residual subgroup of G and is denoted by $G^{\mathcal{F}}$. Usually $\mathcal{N}$ will denote the class of all nilpotent groups.

Throughout this paper all groups are finite. Our notions and notation are standard, see e.g. Robinson [11].

2. Preliminaries

Recall that a subgroup H of a group G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq \mathrm{core}_G(H) = H_G$, or equivalently, $H \cap K = \mathrm{core}_G(H) = H_G$, where H_G is the core in G of H ([4]). We first cite several lemmas that will be useful in the sequel.

Lemma 2.1 ([4, Lemma 2.1])**.** *Let H be a subgroup of a group G. Then the following statements hold:*

(1) *Let K be a subgroup of G such that H is contained in K. If H is c-supplemented in G then H is c-supplemented in K;*

(2) *Let N be a normal subgroup of G such that N is contained in H. Then H is c-supplemented in G if and only if H/N is c-supplemented in G/N;*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup of G and H a π-subgroup of G. If H is c-supplemented in G then HN/N is c-supplemented in G/N. Furthermore, if N normalizes H, then the converse statement also holds;*

(4) *Let L be a subgroup of G and $H \leq \Phi(L)$. If H is c-supplemented in G then H is normal in G and $H \leq \Phi(G)$.*

Lemma 2.2 ([14, Lemma 4.1])**.** *Let G be a finite group and p a prime dividing the order of G such that $(|G|, p-1) = 1$. Assume that the order of G is not divisible by p^3 and G is A_4-free. Then G is p-nilpotent.*

Now we give a generalization of the above lemma.

Lemma 2.3. *Let G be a group and p a prime dividing the order of G, such that G is A_4-free and $(|G|, p-1) = 1$. Assume that N is a normal subgroup of G with G/N p-nilpotent and the order of N not divisible by p^3. Then G is p-nilpotent.*

Proof. Applying Lemma 2.2 to the subgroup N of G we have that N is p-nilpotent. Then N has a normal p-complement $N_{p'}$, which is also normal in G. Consider the factor group $G/N_{p'}$. If $N_{p'} \neq 1$, then by induction $G/N_{p'}$ is p-nilpotent, thus G is p-nilpotent. So we can suppose that N is a p-group of order not greater than p^2. Since G/N is p-nilpotent, G/N has a normal p-complement, H/N say. Then we can write $H = NH_{p'}$ by the Schur–Zassenhaus Theorem. By Lemma 2.2, we have that H is p-nilpotent, thus $H_{p'}$ is normal in H and then it is also normal in G. It is easy to see that $H_{p'}$ is the p-complement of G, so G is p-nilpotent, as desired. □

Lemma 2.4 ([10, IV, 5.4, p. 434])**.** *Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then*

G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.5 ([10, III,5.2, p. 281])**.** *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(i) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$;*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;*

(iii) *If P is non-abelian and $p \neq 2$, then the exponent of P is p;*

(iv) *If P is non-abelian and $p = 2$, then the exponent of P is 4;*

(v) *If P is abelian, then P is of exponent p;*

(vi) $\Phi(P) \times \Phi(Q) = Z(G) = \Phi(G)$.

Lemma 2.6 ([9, Theorem 3.4])**.** *Let G be a group and p the smallest prime dividing $|G|$. Assume that G is A_4-free and every second maximal subgroup of a Sylow p-subgroup of G is c-supplemented in G. Then G is p-nilpotent.*

Lemma 2.7 ([15, Lemma 2.3])**.** *Let G be a group. Assume that N is a normal subgroup of G ($N \neq 1$) such that $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G which are contained in $F(N)$. In particular, if $\Phi(G) = 1$, then $F(G)$ is the direct product of the minimal normal subgroups of G which are contained in $F(G)$.*

Lemma 2.8 ([3, Theorem 1 and Proposition 1])**.** *Let $\mathcal{F}$ be a saturated formation. Assume that G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then:*

(1) *$G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G;*

(2) *$G^{\mathcal{F}}$ is a p-subgroup for some prime p;*

(3) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$;*

(4) *$G^{\mathcal{F}}$ is either elementary abelian or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$.*

Lemma 2.9 ([8, Lemma 3.16])**.** *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and P a normal p-subgroup of a group G such that $G/P \in \mathcal{F}$ for some prime p. If G is A_4-free and $|P| \leq p^2$, then G belongs to $\mathcal{F}$.*

3. Main results

We first continue the discussion of WANG's in [14] and GUO and SHUM's in [9], that is, we investigate the structure of a finite group with some c-supplemented 2-maximal subgroups of a Sylow p-subgroup, and generalize some results of GUO and SHUM, such as [9, Corollary 3.5 and 3.6].

Theorem 3.1. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G such that $G/N\in\mathcal{F}$. Suppose G is A_4-free. If for every prime p dividing the order of N and $P \in \mathrm{Syl}_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G belongs to $\mathcal{F}$.*

PROOF. It is easy to see that N is a Sylow tower group of supersolvable type by Lemmas 2.1 and 2.6. Let r be the largest prime in $\pi(N)$ and $R \in \mathrm{Syl}_r(N)$. Then R is normal in G and $(G/R)/(N/R) \simeq G/N$ is a Sylow tower group of supersolvable type. Let $\overline{P} = PR/R$ be a Sylow p-subgroup of N/R with $r \neq p$. We may assume that P is a Sylow p-subgroup of N. If $\overline{P_1}$ is a 2-maximal subgroup of $\overline{P}$, then, without loss of generality, we may assume that $\overline{P_1} = P_1R/R$ with P_1 a 2-maximal subgroup of P. Since P_1 is c-supplemented in G, we know that $\overline{P_1}$ is c-supplemented in G/R by Lemma 2.1 (3). Therefore, G/R satisfies the hypotheses of our theorem for the normal subgroup N/R. Thus, by induction, G/R is a Sylow tower group of supersolvable type, and of course, every 2-maximal subgroup of R is c-supplemented in G.

If r is the largest prime dividing the order of G, then it is clear that G is a Sylow tower group of supersolvable type. In this case, we may assume that q is the largest prime dividing the order of G with $q > r$. Let Q be a Sylow q-subgroup of G. Since G/R is a Sylow tower group, we see that RQ is normal in G. By the Frattini argument we have $G = RN_G(Q)$.

If $RQ < G$, then RQ is a Sylow tower group of supersolvable type by induction on $|G|$, thus $Q \lhd RQ$, then $G = N_G(Q)$, i.e., Q is normal in G. Now we consider the quotient group G/Q and its normal subgroup NQ/Q. For any prime p dividing the order of NQ/Q, then $p < q$. For any 2-maximal subgroup $\overline{P_2}$ of a Sylow p-subgroup $\overline{P}$ of NQ/N, we can write $\overline{P_2} = P_2Q/Q$, where P_2 is a 2-maximal subgroup of some Sylow subgroup P of N. By the hypotheses, P_2 is c-supplemented in G, then

$\overline{P_2}$ is c-supplemented in G/Q by Lemma 2.1(3). So G/Q with its normal subgroup NQ/N satisfies the hypotheses of our theorem. By induction, G/Q is a Sylow tower group and therefore G must be a Sylow tower group of supersolvable type.

So suppose $G = RQ$. Now let L be a minimal normal subgroup of G with $L \leq R$. Then it is easy to see that the quotient group G/L satisfies the hypotheses of our theorem for the normal subgroup of R/L. By induction we see that G/L is a Sylow tower group of supersolvable type. By a trivial argument, we may assume that L is the unique minimal normal subgroup of G which is contained in R. If $L \leq \Phi(G)$, then it follows that G is a Sylow tower group of supersolvable type. Thus, we may further assume that $R \cap \Phi(G) = 1$ and therefore $L = F(R) = R$ is an abelian minimal normal subgroup of G by Lemma 2.7.

If R is a cyclic group of order r, then because $\mathrm{Aut}(R)$ is a cyclic group of order $r-1$ and $G/C_G(P) \leq \mathrm{Aut}(R)$, we see that $|Q| \mid |C_G(R)|$, therefore we may assume that $Q \leq C_G(R)$ and then $G = R \times Q$. Thus G is a Sylow tower group of supersolvable type. If $|R| > r^2$, then let R_1 be a 2-maximal subgroup of R. Now, by our hypotheses there exists a subgroup K of G such that $G = R_1K$ and $R_1 \cap K = 1$ since L is the unique minimal normal subgroup of G contained in R. Thus $R = R_1(R \cap K)$. Since $R \cap K$ is normal in K and R is abelian, $R \cap K$ is a normal subgroup of G. The minimality of $R = L$ implies that $R \cap K = R$, and therefore $R_1 = 1$, a contradiction. Hence R is an elementary abelian group of order r^2. Since R is normal in G, any element g of Q induces an automorphism σ of R. When $r = 2$, if $\sigma \neq 1$, noticing that $|\mathrm{Aut}(R)| = (r+1)r(r-1)^2$, the order of σ must be 3 ($q = r+1 = 3$) as $q > r$. Then $R < \sigma > \cong A_4$, contrary to the hypothesis. So suppose that $r > 2$, noticing that $r+1$ is not a prime, hence we see that $\sigma = 1$ and therefore $G = R \times Q$, so G is a Sylow tower group of supersolvable type. The proof is now complete. □

Corollary 3.2. *Let G be a group which is A_4-free, and N a normal subgroup of G such that G/N is supersolvable. If, for every prime p dividing the order of N and $P \in \mathrm{Syl}_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G is supersolvable.*

Corollary 3.3 ([9, Corollary 3.6])**.** *Let G be a group of odd order, and N a normal subgroup of G such that G/N is a Sylow tower group* *of supersolvable type. If, for every prime p dividing the order of N and $P \in \mathrm{Syl}_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G is a Sylow tower group of supersolvable type.*

In the sequel, we discuss the influence of the properties of subgroups of prime square order of a Sylow subgroup, and the dual concept of a 2-maximal subgroup of a Sylow subgroup, on the structure of G.

Theorem 3.4. *Let G be a group and p a prime dividing the order of G. Suppose that $(|G|, p-1) = 1$ and G is A_4-free. If there exists a normal subgroup N of G such that G/N is p-nilpotent and every subgroup of order p^2 of every Sylow p-subgroup of N is c-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then we may make the following claims:

(i) The hypotheses are inherited by all proper subgroups of G, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

In fact, for all $H < G$, $H/H \cap N \cong HN/N$, thus $H/H \cap N$ is p-nilpotent. If $|H \cap N|_p \leq p^2$, then H is p-nilpotent by Lemma 2.3. So suppose that $|H \cap N|_p > p^2$. Then we can take an arbitrary subgroup P_2 of order p^2 of any Sylow p-subgroup of $H \cap N$. Obviously P_2 is also a subgroup of order p^2 of some Sylow p-subgroup of N. Thus it is c-supplemented in G by the hypotheses and then it is c-supplemented in H by Lemma 2.1. Hence H satisfies the hypotheses of the theorem. The minimal choice of G implies that H is p-nilpotent, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

(ii) $G = PQ$, where $P \lhd G$ and Q is not a normal subgroup of G. Furthermore, p^3 divides the order of P.

These can be induced by Lemmas 2.4, 2.5 and 2.2.

(iii) Every subgroup of order p^2 of P is normal in G.

Suppose there exists a subgroup P_2 of order p^2 which is not normal in G. By the hypotheses, P_2 is c-supplemented in G, so there exists a subgroup K of G such that $G = P_2K$ and $P_2 \cap K = (P_2)_G < P_2$. Then K is a proper subgroup of G, thus K is nilpotent. Denote $K = K_p \times K_{p'}$, then $P = P_2K_p$. Now consider $N_G(K_p)$. Since $K \leq N_G(K_p)$, we have that $[G : N_G(K_p)] \leq p$. If $[G : N_G(K_p)] = 1$, then K_p is normal in G. By Lemma 2.5 (ii), $K_p \leq \Phi(P)$ or $K_p = P$. If $K_p \leq \Phi(P)$, then $P = P_2$, contrary to (ii). If $K_p = P$, then $K = G$, a contradiction. Now suppose that $[G : N_G(K_p)] = p$, then we can write $N_G(K_p) = P_1 \times K_{p'}$, where P_1 is a maximal subgroup of P containing K_p. Now $N_G(P_1)$ contains P and $K_{p'}$, so P_1 is normal in G, then, again by Lemma 2.5(ii), $P_1 \leq \Phi(P)$ or $P_1 = P$, which implies that $P = P_2$ or P_1, a contradiction.

(iv) Every subgroup of order p^2 of P is contained in $\Phi(P)$, thus in $Z(G)$.

Suppose P_2 is an arbitrary subgroup of order p^2 of P, then P_2 is normal in G by (iii), therefore $P_2\Phi(G) = P$ or $P_2\Phi(P) = \Phi(P)$ by Lemma 2.5(ii). If $P_2\Phi(G) = P$, then $P = P_2$, contrarily to (ii), so P_2 is contained in $\Phi(P)$, hence it is contained in $Z(G)$ by Lemma 2.5(vi).

(v) $\Phi(P) = 1$.

If $\Phi(P) \neq 1$, we can pick an element a of order p in $\Phi(P)$. If $\exp(P) = p$, then for any element b of P not in $\langle a\rangle$, $\langle a\rangle\langle b\rangle$ is a group of order p^2, so $\langle a\rangle\langle b\rangle$ is contained in $Z(G)$, thus $P \leq Z(G)$. Therefore $G = P \times Q$, a contradiction. So we may suppose that $p = 2$ and $\exp(P) = 4$. For any element b of P not in $\langle a\rangle$, if b is of order 2, then $\langle a\rangle\langle b\rangle$ is a group of order 4, hence $\langle a\rangle\langle b\rangle \leq \Phi(P)$ by (iv), so $b \in \Phi(P)$; if b is of order 4, then $\langle b\rangle$ is contained in $\Phi(P)$ by (iv), which again implies $b \in \Phi(P)$. Hence $P = \Phi(P) \leq Z(G)$ and from here $G = P \times Q$, a contradiction.

(vi) The final contradiction.

Take $a \in P$, then a is of order p. Now we can find an element b of order p which is not in $< a >$ by (ii), then the order of the subgroup $< a >< b >$ is p^2, thus it is contained in $\Phi(P)$ by (iv), which is contrary to (v), the final contradiction.

If we choose the subgroup N in Theorem 3.4 as $G^{\mathcal{N}}$, the nilpotent residual of G, then we can see that the following is an equivalent form of Theorem 3.4. □

Corollary 3.5. *Let p be a prime number dividing the order of a group G such that $(|G|, p-1) = 1$ and let G be A_4-free. Suppose P is a Sylow p-subgroup of G. If every subgroup of order p^2 of $P \cap G^{\mathcal{N}}$ is c-supplemented in G, then G is p-nilpotent.*

Remark 3.6. We observe that the assumption $(|G|, p-1) = 1$ cannot be removed in Corollary 3.5. In fact, assume G is a non-cyclic group of order 21 and $p = 7$. Then G is A_4-free and there does not exist a subgroup of order 7^2 in G, but G is not 7-nilpotent. It is easy to see that the assumption that G is A_4-free cannot be removed either in our result, because A_4 is a counterexample.

Corollary 3.7. *Let G be a group. If, for every prime p dividing the order of G and $P \in \mathrm{Syl}_p(G)$, every subgroup of order p^2 of P is c-supplemented in G and G is A_4-free, then G is a Sylow tower group of supersolvable type.*

Proof. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and therefore Corollary 3.7 follows immediately from Theorem 3.4. □

Now we generalize Corollary 3.7 as follows.

Theorem 3.8. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and H a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If G is A_4-free and all subgroups of prime square order of every Sylow subgroup of H are c-supplemented in G, then G is in $\mathcal{F}$.*

Proof. Suppose the result is false and let G be a counterexample of minimal order. Then by Corollary 3.7, we can see that H has a Sylow tower of supersolvable type. Let p be the largest prime in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Then P is a normal subgroup of G. Now consider the factor group G/P. It is easy to see that all subgroups of prime square order of every Sylow subgroup of H/P are c-supplemented in G/P and G/P is A_4-free. Thus, by the minimal choice of G, we have $G/P \in \mathcal{F}$ and every subgroup of order p^2 of P is c-supplemented in G.

So $G^{\mathcal{F}}$ is a p-subgroup. By [1, Theorem 3.5], there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \mathrm{Soc}\,(G \bmod \Phi(G))$ and $G/M_G \notin \mathcal{F}$. Hence $G = MG^{\mathcal{F}} = MF(G)$ (since $G^{\mathcal{F}}$ is a p-group). It is obvious that M satisfies the hypotheses of the theorem on its normal subgroup $M \cap P$. By the minimal choice of G, we have that M lies in $\mathcal{F}$.

Now, by Lemma 2.8, $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^{\mathcal{F}}$ or $(G^{\mathcal{F}})' = \Phi(G^{\mathcal{F}}) = Z(G^{\mathcal{F}})$ is elementary abelian.

Now we distinguish two cases:

Case 1 $\Phi(G^{\mathcal{F}}) = 1$.

In this case, $G^{\mathcal{F}}$ is a minimal normal subgroup of G. If $|G^{\mathcal{F}}| \leq p^2$, then $G \in \mathcal{F}$ by Lemma 2.9. So suppose $|G^{\mathcal{F}}| \geq p^3$, then we can take a subgroup P_2 of order p^2 of $G^{\mathcal{F}}$ and P_2 is c-supplemented in G by the hypotheses. So there exists a subgroup K of G such that $G = P_2K$ and $P_2 \cap K = (P_2)_G = 1$. Therefore $G^{\mathcal{F}} = P_2(K \cap G^{\mathcal{F}})$. Since $G^{\mathcal{F}}$ is elementary abelian, it is easy to see that $G^{\mathcal{F}} \cap K$ is normal in G, thus $G^{\mathcal{F}} \cap K = 1$ or $G^{\mathcal{F}}$ by the minimality of $G^{\mathcal{F}}$. If $G^{\mathcal{F}} \cap K = 1$, then $G^{\mathcal{F}} = P_2$ and it is of order p^2, while $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $G^{\mathcal{F}} \cap K = G^{\mathcal{F}}$, then $P_2 = P_2 \cap K = (P_2)_G = 1$, a contradiction too.

Case 2 $\Phi(G^{\mathcal{F}}) \neq 1$.

We consider two subcases.

Subcase 2.1. $p = 2$ and $\exp(G^{\mathcal{F}}) = 4$.

Now we can take an element x of order 4 in $G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$. Then there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = \langle x \rangle_G$. Hence $G^{\mathcal{F}} = \langle x \rangle (K \cap G^{\mathcal{F}})$. Note that $x^2 \in \Phi(G^{\mathcal{F}})$, thus $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$ is a maximal subgroup of $G^{\mathcal{F}}$, so x normalizes $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$. Since $[x^2, K] \leq \Phi(G^{\mathcal{F}}) \leq \langle x^2 \rangle (K \cap G^{\mathcal{F}})$, we get that $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$ is a normal subgroup of G. Hence $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}}$ by the minimality of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. If $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ then $G^{\mathcal{F}} = \langle x \rangle$, so $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ then $G^{\mathcal{F}} = K \cap G^{\mathcal{F}}$, so $\langle x \rangle = \langle x \rangle \cap K = \langle x \rangle_G$ is normal in G, and then $\langle x \rangle = G^{\mathcal{F}}$. By Lemma 2.9 we have $G \in \mathcal{F}$, another contradiction.

Subcase 2.2 $\exp(P) = p$.

Since $G^{\mathcal{F}} \neq \Phi(G^{\mathcal{F}}) \neq 1$, we can take two elements a and b of order p such that $a \in \Phi(G^{\mathcal{F}})$ and $b \in G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$. Then $\langle a \rangle \langle b \rangle$ is a subgroup of $G^{\mathcal{F}}$ of order p^2, so there exists a subgroup K of G such that $G = \langle a \rangle \langle b \rangle K$ and $\langle a \rangle \langle b \rangle \cap K = (\langle a \rangle \langle b \rangle)_G$. Hence $G^{\mathcal{F}} = \langle a \rangle \langle b \rangle (K \cap G^{\mathcal{F}})$. Note that $\langle a \rangle (K \cap G^{\mathcal{F}})$ is a maximal subgroup of $G^{\mathcal{F}}$, so b normalizes $\langle a \rangle (K \cap G^{\mathcal{F}})$. Since $[a, K] \leq \Phi(G^{\mathcal{F}}) \leq \langle a \rangle (K \cap G^{\mathcal{F}})$, we get that $\langle a \rangle (K \cap G^{\mathcal{F}})$ is a normal subgroup of G. Hence $\langle a \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $< c > (K \cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ by the minimality of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. If $\langle a\rangle(K\cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$, then $G^{\mathcal{F}} = \langle b\rangle$, so $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $\langle a\rangle(K\cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ then $G^{\mathcal{F}} = K\cap G^{\mathcal{F}}$, so $\langle a\rangle\langle b\rangle = \langle a\rangle\langle b\rangle \cap K = (\langle a\rangle\langle b\rangle)_G$ is normal in G which implies $G^{\mathcal{F}} = \langle a\rangle\langle b\rangle = \langle b\rangle$. By Lemma 2.9 we have $G \in \mathcal{F}$, a contradiction. □

The following are immediate corollaries of Theorem 3.8.

Corollary 3.9. *Let G be a group which is A_4-free and N a normal subgroup of G such that G/N is supersolvable. If for every prime p dividing the order of H and $P \in \mathrm{Syl}_p(H)$, every subgroup of order p^2 of P is c-supplemented in G, then G is supersolvable.*

Corollary 3.10. *Let G be a group of odd order and N a normal subgroup of G such that G/N is a Sylow tower group of supersolvable type. If all subgroups of prime square order of every Sylow subgroup of N are c-supplemented in G, then G is a Sylow tower group of supersolvable type.*

ACKNOWLEDGEMENTS. We wish to thank the referee(s) and Prof. L. TAMÁSSY who read the paper carefully and helped us to polish this paper to this final version.

References

[1] A. BALLESTER-BOLINCHES, $\mathcal{H}$-normalizers and local definitions of saturated formations of finite groups, *Israel J. Math.* **67** (1989), 312–326.

[2] A. BALLESTER-BOLINCHES and XIUYUN GUO, Some results on p-nilpotence and solubility of finite groups, *J. Algebra* **228** (2000), 491–496.

[3] A. BALLESTER-BOLINCHES and M. C. PEDRAZA AGUILERA, On minimal subgroups of finite groups, *Acta Math. Hungar.* **73** (1996), 335–342.

[4] A. BALLESTER-BOLINCHES, YANMING WANG and XIUYUN GUO, c-supplemented subgroups of finite groups, *Glasgow Math. Journal* **42** (2000), 383–389.

[5] K. DOERK and T. HAWKES, Finite Soluble Groups, *Walter de Gruyter, Berlin – New York*, 1992.

[6] W. FEIT and J. G. THOMPSON, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.

[7] F. GROSS, Conjugacy of odd order Hall subgroups, *Bull. London Math. Soc.* **19** (1987), 311–319.

[8] GUO XIUYUN and K. P. SHUM, Cover-avoidance properties and the structure of finite groups, *J. Pure and Applied Algebra* **181** (2003), 297–308.

[9] Guo Xiuyun and K. P. Shum, On p-nilpotency of finite groups with some subgroups c-supplemented, *Algebra Coll.* **10**:3 (2003).
[10] B. Huppert, Endliche Gruppen I, *Springer-Verlag*, *Berlin*, 1968.
[11] D. J. S. Robinson, A Course in the Theory of Groups, *Springer-Verlag*, *New York – Berlin*, 1993.
[12] S. Srinivasan, Two sufficient conditions for supersolvability of finite groups, *Israel J. Math.* **35** (1980), 210–214.
[13] G. Wall, Groups with maximal subgroups of Sylow subgroups normal, *Israel J. Math.* **43** (1982), 166–168.
[14] Yanming Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *J. Algebra* **224** (2000), 464–478.
[15] Yanming Wang, Yangming Li and Jingtong Wang, Finite groups with c-supplemented minimal subgroups, *Algebra Coll.* **10**:3 (2003), 413–425.

YANGMING LI
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
HANGZHOU, ZHEJIANG, 310027, CHINA
AND
DEPARTMENT OF MATHEMATICS
GUANGDONG INSTITUTE OF EDUCATION
GUANGZHOU, 510310
CHINA

E-mail: liyangming@gdei.edu.cn

YANMING WANG
LINGNAN COLLEGE AND SCHOOL OF MATHEMATICS
ZHONGSHAN UNIVERSITY
GUANGZHOU, 510275
CHINA

E-mail: stswym@zsu.edu.cn

HUAQUAN WEI
DEPARTMENT OF MATHEMATICS
GUANGXI TEACHER'S COLLEGE
NANNING, 530001, CHINA
AND
DEPARTMENT OF MATHEMATICS
ZHONGSHAN UNIVERSITY
GUANGZHOU, 510275
CHINA

E-mail: weihuaquan@163.com

(Received December 17, 2003; 2nd revised November 22, 2004)